Abstract

In this paper we introduce a new class of sets, called $P_p$-open sets, also using this set, we define and investigate some properties of the concept of $P_p$-continuity. In particular, $P_p$-open sets and $P_p$-continuity are defined to extend known results for preopen sets and pre-continuity.

Keywords: $P_p$-open, preopen, $P_p$-g.closed sets, pre-continuous and $P_p$-continuous.

1 Introduction

In 1982, Mashhour et al [15] defined a new class of sets called preopen sets. He proved that the union of any family of preopen sets is also preopen set and introduced two types of continuity called precontinuous and weak precontinuous functions. In 1987, Popa [16] defined pre-neighbourhood of a point $x$ in a space $X$. El-Deeb et al [9] defined the preclosure of a subset $A$ as the intersection of all preclosed sets containing $A$ and the preinterior of $A$ is the union of all preopen sets contained in $A$. In the present paper we introduce a new type of preopen sets called $P_p$-open, this type of sets lies strictly between the pre-$\theta$-open sets and preopen sets. We also study its fundamental properties and then we define further topological properties such as, $P_p$-neighborhood, $P_p$-interior, $P_p$-closure, $P_p$-derived set and $P_p$-boundary of a set. Mashhour et
al [15] defined a function $f : X \rightarrow Y$ to be pre-continuous if $f^{-1}(V)$ is preopen set in $X$ for every open set $V$ of $Y$. Long and Herrington [14] have introduced a new class of functions called strongly $\theta$-continuous function. We also introduce and investigate the concept of $P_p$-continuous functions. It will be shown that $P_p$-continuity is weaker than quasi $\theta$-continuity mentioned in [20], but it is stronger than pre-continuity [15].

2 Preliminaries

Throughout the present paper, a space $x$ always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let $A$ be a subset of a space $X$. The closure and interior of $A$ with respect to $X$ are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset $A$ of a space $X$ is said to be preopen [15] (resp., semi-open [13], $\alpha$-open [18], $\beta$-open [1] and regular open [23]), if $A \subseteq Int(Cl(A))$ (resp., $A \subseteq Cl(\text{Int}(A))$, $A \subseteq Int(Cl(\text{Int}(A)))$, $A \subseteq Cl(\text{Int}(Cl(A)))$ and $A = \text{Int}(Cl(A))$). The complement of a preopen (resp., regular open) set is said to be preclosed [9] (resp., regular closed [23]). The family of all preopen (resp., semi-open, $\alpha$-open, $\beta$-open and regular open) subsets of $X$ is denoted by $PO(X)$ (resp., $SO(X)$, $\alpha O(X)$, $\beta O(X)$ and $RO(X)$). The intersection of all preclosed sets of $X$ containing $A$ is called the preclosure [9] of $A$. The union of all preopen sets of $X$ contained in $A$ is called the preinterior. A subset $A$ of a space $X$ is called preclopen [11], if $A$ is both preopen and preclosed while it is called pre-regular open [16], if $PIntPCl(A) = A$. In 1968, Velicko [24] defined the concepts of $\delta$-open and $\theta$-open sets in $X$ denoted by ($\delta O(X)$ and $\theta O(X)$ respectively). A subset $A$ of a space $X$ is called $\delta$-open (resp., $\theta$-open) set if for each $x \in A$, there exists an open set $G$ such that $x \in G \subseteq Int(Cl(G)) \subseteq A$ (resp., $x \in G \subseteq Cl(G) \subseteq A$). Joseph and Kwack [10] (resp., Di Maio and T. Noiri [12]), defined a subset $A$ of a space $X$ to be $\theta$-semi-open (resp., semi-$\theta$-open), if for each $x \in A$, there exists a semi-open set $G$ such that $x \in G \subseteq Cl(G) \subseteq A$ (resp., $x \in G \subseteq SCl(G) \subseteq A$). The family of all $\theta$-semi-open (resp., semi-$\theta$-open) subsets of $X$ is denoted by $\theta SO(X)$ (resp., $S\theta O(X)$). We recall that a topological $X$ locally indiscrete [8] if every open subset of $X$ is closed.

Definition 2.1 [22] A space $(X , \tau)$ is said to have the property $P$ if the closure is preserved under finite intersection or equivalently, if the closure of intersection of any two subsets equals the intersection of their closures.

From the above definition Paul and Bhattacharyya [22] pointed out the following remark:
Remark 2.2 If a space $X$ has the property $P$, then the intersection of any two preopen sets is preopen, as a consequence of this, $PO(X, \tau)$ is a topology for $X$ and it is finer than $\tau$.

Definition 2.3 The point $x \in X$ is said to be a pre-$\theta$-cluster [19] point of a subset $A$, if $PCl(U) \cap A \neq \emptyset$ for every $U \in PO(X)$.

The set of all pre-$\theta$-cluster points of $A$ is called the pre-$\theta$-closure of $A$ and is denoted by $PCl_\theta(A)$.

A subset $A$ of a topological space $(X, \tau)$ is said to be pre-$\theta$-closed [5] if $PCl_\theta(A) = A$. The complement of a pre-$\theta$-closed set is called pre-$\theta$-open and it is denoted by $P\theta O(X)$.

Lemma 2.4 [6] A subset $U$ of a space $X$ is pre-$\theta$-open in $X$ if and only if for each $x \in U$, there exists a preopen set $V$ with $x \in V$ such that $PCl(V) \subseteq U$.

Lemma 2.5 [8] Let $(X, \tau)$ be a topological space. If $A \in \alpha O(X)$ and $B \in PO(X)$, then $A \cap B \in PO(X)$.

The following results also can be found in [2].

Theorem 2.6 1. Let $(X, \tau)$ be a topological space. If $G \in \tau$ and $Y \in PO(X)$, then $G \cap Y \in PO(X)$.

2. Let $(Y, \tau_Y)$ be a subspace of a space $(X, \tau)$. If $A \in PO(X, \tau)$ and $A \subseteq Y$, then $A \in PO(Y, \tau_Y)$. Moreover, if $Y$ is an $\alpha$-open subspace of $X$, $F \in PC(X, \tau)$ and $F \subseteq Y$, then $F \in PC(Y, \tau_Y)$.

3. Let $(Y, \tau_Y)$ be a subspace of a space $(X, \tau)$. If $A \in PO(Y, \tau_Y)$ and $Y \in PO(X, \tau)$, then $A \in PO(X, \tau)$.

Theorem 2.7 [7] For any subset $A$ of a space $(X, \tau)$. The following statements are equivalent:

1. $A$ is clopen.

2. $A$ is $\alpha$-open and closed.

3. $A$ is preopen and closed.

Theorem 2.8 [2] For any spaces $X$ and $Y$. If $A \subseteq X$ and $B \subseteq Y$ then,

1. $PInt_{X \times Y}(A \times B) = PInt_X(A) \times PInt_Y(B)$.

2. $PCl_{X \times Y}(A \times B) = PCl_X(A) \times PCl_Y(B)$.

Theorem 2.9 [3] Let $(X, \tau)$ be any space, then $PO(X, \tau) = PO(X, \tau_\alpha)$.
Theorem 2.10 [8] A topological space \((X, \tau)\) is locally indiscrete if and only if every subset of \(X\) is preopen.

Theorem 2.11 [2] A topological space \((X, \tau)\) is \(s^*\)-normal if and only if for every semi-closed set \(F\) and every semi-open set \(G\) containing \(F\), there exists an open set \(H\) such that \(F \subseteq H \subseteq \text{Cl}(H) \subseteq G\).

Theorem 2.12 [25] If \(X\) is \(s^*\)-normal, then \(S\theta O(X) = \theta O(X)\).

Definition 2.13 [9] A topological space \(X\) is said to be \(P\)-regular if for each closed subset \(F\) of \(X\) and each point \(x \notin F\) there exist \(U, V \in PO(X)\) such that \(x \in U, F \subseteq V\) and \(U \cap V = \emptyset\).

Definition 2.14 [21] A space \(X\) is said to be pre-regular if for each preclosed set \(F\) and each point \(x \notin F\), there exist disjoint preopen sets \(U\) and \(V\) such that \(x \in U\) and \(F \subseteq V\).

Lemma 2.15 [21] A space \(X\) is pre-regular if and only if for each \(x \in X\) and each \(H \in PO(X)\) there exists \(G \in PO(X)\) such that \(x \in G \subseteq \text{PCl}(G) \subseteq H\).

Theorem 2.16 [4] A space \(X\) is pre-regular if and only if \(\text{PCl}(A) = \text{PCl}_\theta(A)\) for each subset \(A\) of \(X\).

Theorem 2.17 [17] A space \(X\) is pre-\(T_1\) if and only if the singleton set \(\{x\}\) is preclosed for each point \(x \in X\).

3 \(P_p\)-Open Sets

Definition 3.1 A subset \(A\) of a space \(X\) is called \(P_p\)-open, if for each \(x \in A \in PO(X)\), there exists a preclosed set \(F\) such that \(x \in F \subseteq A\).

A subset \(B\) of a space \(X\) is called \(P_p\)-closed, if \(X \setminus B\) is \(P_p\)-open. The family of all \(P_p\)-open (\(P_p\)-closed) subsets of a topological space \((X, \tau)\) is denoted by \(P_pO(X, \tau)\) (or \(P_pO(X)\) (\(P_pC(X, \tau)\) or \(P_pC(X)\)).

Proposition 3.2 A subset \(A\) of a space \(X\) is \(P_p\)-open if and only if \(A\) is preopen set and is a union of preclosed sets.

proof. Obvious.

It is clear from the definition that every \(P_p\)-open subset of a space \(X\) is preopen, but the converse is not true in general as shown in the following example:

Example 3.3 Consider \(X = \{a, b, c\}\) with the topology \(\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}\), then \(\{a\} \in PO(X)\) but \(\{a\} \notin P_pO(X)\).
The following result shows that any union of $P_p$-open sets in a topological space $(X, \tau)$ is $P_p$-open.

**Proposition 3.4** Let $\{A_\lambda : \lambda \in \Delta\}$ be a collection of $P_p$-open sets in a topological space $X$, then $\bigcup\{A_\lambda \in \Delta\}$ is $P_p$-open.

**proof.** Let $A_\lambda$ is $P_p$-open set for each $\lambda$, then $A_\lambda$ is preopen and hence $\bigcup\{A_\lambda : \lambda \in \Delta\}$ is preopen. Let $X \in \bigcup\{A_\lambda : \lambda \in \Delta\}$, there exist $\lambda \in \Delta$ such that $X \in A_\lambda$. Since $A_\lambda$ is $P_p$-open for each $\lambda$, there exists a preclosed set $F$ such that $x \in F \subseteq A_\lambda \subseteq \bigcup\{A_\lambda : \lambda \in \Delta\}$. Therefore, $\bigcup\{A_\lambda : \lambda \in \Delta\}$ is $P_p$-open set.

The following example shows that the intersection of two $P_p$-open sets need not be $P_p$-open.

**Example 3.5** Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$. Here $\{a, b, c\} \in P_pO(X)$ and $\{b, c, d\} \in P_pO(X)$, but $\{a, b, c\} \cap \{b, c, d\} = \{b, c\} \notin P_pO(X)$.

From the above example we notice that the family of all $P_p$-open sets need not be a topology on $X$.

**Proposition 3.6** If the family of all preopen sets of a space $X$ is a topology on $X$, then the family of $P_p$-open is also a topology on $X$.

**proof.** It is enough to show that the finite intersection of $P_p$-open sets is also $P_p$-open set. Let $A$ and $B$ be two $P_p$-open sets, then $A$ and $B$ are preopen sets. Since $PO(X)$ is a topology on $X$. Then $A \cap B \in PO(X)$. Let $x \in A \cap B$, then $x \in A$ and $x \in B$, there exists preclosed sets $E$ and $F$ such that $x \in E \subseteq A$ and $x \in F \subseteq B$, this implies that $x \in E \cap F \subseteq A \cap B$. Since any intersection of preclosed sets is preclosed, then $A \cap B$ is $P_p$-open set. This completes the proof.

**Corollary 3.7** Let $(X, \tau)$ be a topological space. If $X$ has a property $P$, then $P_pO(X)$ forms a topology on $X$.

**proof.** Follows from Proposition 3.6.

**Proposition 3.8** If a space $X$ is pre-$T_1$-space, then $PO(X) = P_pO(X)$.

**proof.** Since the space $X$ is pre-$T_1$, then by Theorem 2.17 every singleton is preclosed set and hence $x \in \{x\} \subseteq A$. Therefore, $A \in P_pO(X)$. Thus $PO(X) = P_pO(X)$.

**Proposition 3.9** For any subset $A$ of a space $X$. If $A \in P\thetaO(X)$, then $A \in P_pO(X)$. 

proof. Let $A \in P\theta O(X)$. If $A = \phi$, then $A \in P_p O(X)$. If $A \neq \phi$, then for each $x \in A$, there exists a preopen set $G$ such that $X \in G \subseteq PCl(G) \subseteq A$ implies that $x \in PCl(G) \subseteq A$. Since $A \in P\theta O(X)$ and $P\theta O(X) \subseteq PO(X)$ in general, then $A \in PO(X)$. Therefore, $A \in P_p O(X)$.

Remark 3.10 It clear that each preregular (preclopen or $\theta$-open) set are $P_p$-open.

The following example shows that a $P_p$-open set need not be $\theta$-open.

Example 3.11 Consider $X = \{a, b, c, d\}$, with the topology $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ Then the set $\{a, b, c\} \in P_pO(X)$, but $\{a, b, c\} \notin \theta O(X)$

Theorem 3.12 For any space $X$, $PCl(PInt(\{x\})) = \{x\}$ if and only if $\{x\}$ is $P_p$-open.

proof. Let $PCl(PInt(\{x\})) = \{x\}$, implies that $\{x\}$ is preopen and preclosed, then $\{x\}$ is preregular open, therefore $\{x\} \in P_p O(X)$.

Conversely, let $\{x\}$ be $P_p$-open, this implies that $x \in \{x\} \subseteq \{x\}$. Since $\{x\} \in P_p O(X)$, then $\{x\} \in PC(X)$ and hence $\{x\}$ is pre-open and pre-closed. Therefore, $PCl(PInt(\{x\})) = \{x\}$.

Proposition 3.13 Let $(X, \tau)$ be a topological space, then $\{x\}$ is $P_p O(X)$ if and only if it is preclopen for every $x \in X$.

proof. Obvious.

Proposition 3.14 A subset $A$ of a space $(X, \tau)$ is $P_p$-open if and only if for each $x \in A$, there exists a $P_p$-open set $B$ such that $x \in B \subseteq A$.

proof. Suppose that for each $x \in A$, there exists a $P_p$-open set $B$ such that $x \in B \subseteq A$. Thus $A = \cup B_\lambda$ where $B_\lambda \in P_p O(X)$ for each $\lambda$, and by Proposition 3.4, $A$ is $P_p$-open.

The other part is obvious.

Proposition 3.15 Let $(X, \tau)$ be a pre-regular space, then $\tau \subseteq P_p(X)$.

proof. Let $A$ be any open subset of a space $X$. This implies that $A$ is preopen. If $A = \phi$, then $A \in P_p O(X)$. If $A \neq \phi$, since $X$ is pre-regular, by Lemma 2.15, for each $x \in A \subseteq X$, there exists a pre-open set $G$ such that $x \in G \subseteq PCl(G) \subseteq A$. Thus we have $x \in PCl(G) \subseteq A$. Therefore, $\tau \subseteq P_p O(X)$. 

Proposition 3.16 Let \((X, \tau)\) be a topological space, and \(A, B \subseteq X\). If \(A \in P_pO(X)\) and \(B\) is both \(\alpha\)-open and preclosed, then \(A \cap B \in P_pO(X)\).

proof. Let \(A \in P_pO(X)\) and \(B\) is \(\alpha\)-open, then \(A\) is preopen set, and by Lemma 2.5, \(A \cap B \in PO(X)\). Let \(x \in A \cap B\), then \(x \in A\) and \(x \in B\), there exists a preclosed set \(F\) such that \(x \in F \subseteq A\). Since \(B\) is preclosed, implies that \(F \cap B\) is preclosed, then \(x \in F \cap B \subseteq A \cap B\). Thus \(A \cap B\) is \(P_p\)-open set in \(X\).

Corollary 3.17 If a space \(X\) is locally indiscrete, then \(PO(X) = P_pO(X)\).

proof. Follows from Theorem 2.10.

Proposition 3.18 If a topological space \((X, \tau)\) is locally indiscrete, then \(\tau \subseteq P_pO(X)\).

proof. Since \(X\) is locally indiscrete, then by Corollary 3.17, \(PO(X) = P_pO(X)\). Therefore, \(\tau \subseteq P_pO(X)\).

The following example shows that the converse of Proposition 3.18 is not true.

Example 3.19 Consider \(X = \{a, b, c, d\}\), with the topology \(\tau = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, X\}\), then \(\tau \subseteq P_pO(X)\), but \(X\) is not locally indiscrete.

Proposition 3.20 If \((X, \tau)\) is indiscrete topology, then \(P_pO(X)\) is discrete topology in \(X\).

proof. Let \(X\) be indiscrete topology, then every subset of \(X\) is preopen, then \(PO(X) = P_pO(X)\). Therefore, \(P_pO(X)\) is discrete topology on \(X\).

The following example shows that the converse of Proposition 3.20 is not true.

Example 3.21 Consider \(X = \{a, b, c\}\), with the topology \(\tau = \{\phi, \{a\}, \{b, c\}, X\}\), then \(P_pO(X)\) is discrete topology in \(X\), but \((X, \tau)\) is not indiscrete topology.

Proposition 3.22 For any topological space \((X, \tau)\), we have:

1. \(P_pO(X)\) is discrete if and only if \(PO(X)\) is discrete.
2. \(\tau\) is discrete if and only if \(P_pO(X)\) is discrete.

proof. Obvious.

Proposition 3.23 For any subset \(A\) of a space \((X, \tau)\). The following statements are equivalent:
1. $A$ is clopen.
2. $A$ is $P_p$-open and closed.
3. $A$ is preopen and closed.

**proof.** Follows from Theorem 2.7.

**Proposition 3.24** For a topological space $(X, \tau)$, the following conditions are equivalent:

1. $X$ is locally indiscrete.
2. Every subset of $X$ is $P_p$-open.
3. Every singleton in $X$ is $P_p$-open.
4. Every closed subset of $X$ is $P_p$-open.

**proof.** Follows from Theorem 2.10.

**Proposition 3.25** Let $X$ and $Y$ be two topological spaces and $X \times Y$ be the product topology. If $A \in P_pO(X)$ and $B \in P_pO(Y)$, then $A \times B \in P_pO(X \times Y)$.

**proof.** Let $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. Since $A \in P_pO(X)$ and $B \in P_pO(Y)$, there exist preclosed sets $F$ and $E$ in $X$ and $Y$ respectively, such that $x \in F \subseteq A$ and $y \in E \subseteq B$. Therefore, $(x, y) \in F \times E \subseteq A \times B$. Since $A \in PO(X)$ and $B \in PO(Y)$. Then by Theorem 2.8(1), $A \times B \in PO(X \times Y)$. Since $F$ is preclosed in $X$ and $E$ is preclosed in $Y$ and by Theorem 2.8 (2), $F \times E$ is preclosed in $(X \times Y)$. Therefore, $A \times B \in P_pO(X \times Y)$.

**Proposition 3.26** Topological spaces $(X, \tau)$ and $(X, \tau_\alpha)$ have the same class of $P_p$-open sets.

**proof.** Let $A$ be any subset of a space $X$ and $A \in P_pO(X, \tau)$. If $A=\phi$, then $A \in P_pO(X, \tau_\alpha)$. If $A \neq \phi$, since $A \in P_pO(X, \tau)$, then $A \in PO(X, \tau)$ and $A=\cup F_\lambda$, where $F_\lambda$ is preclosed for each $\lambda$ Since $A \in PO(X, \tau)$, then by Theorem 2.9, $A \in PO(X, \tau_\alpha)$. Again since $F_\lambda \in PC(X, \tau)$ for each $\lambda$, then by Theorem 2.9, $F_\lambda \in PC(X, \tau_\alpha)$ for each $\lambda$. Therefore, by Proposition 3.2, $A \in P_pO(X, \tau_\alpha)$. Hence $P_pO(X, \tau) \subseteq P_pO(X, \tau_\alpha)$. On the other hand, we can prove similarly $P_pO(X, \tau_\alpha) \subseteq P_pO(X, \tau)$. Therefore, we get $P_pO(X, \tau_\alpha) = P_pO(X, \tau)$.

**Proposition 3.27** Let $(X, \tau)$ be any $s^*\ast$-normal space. If $A \in S\theta O(X)$, then $A \in P_pO(X)$. 

proof. Let \( A \in S\theta O(X) \), if \( A = \phi \), then \( A \in P_pO(X) \). If \( A \neq \phi \). Since the space \( X \) is \( s^* \)-normal, then by Theorem 2.12, \( S\theta O(X) = \theta O(X) \). Hence \( A \in \theta O(X) \). But \( \theta O(X) \subseteq P_pO(X) \) in general. Therefore, \( A \in P_pO(X) \)

Corollary 3.28 Let \((X, \tau)\) be any \( s^* \)-normal space. If \( A \in \theta SO(X) \), then \( A \in P_pO(X) \).

proof. Follows from Proposition 3.27, and the fact that \( \theta SO(X) \subseteq S\theta O(X) \).

Proposition 3.29 Let \( Y \) be an \( \alpha \)-open subspace of a space \((X, \tau)\). If \( A \in P_pO(X, \tau) \) and \( A \subseteq Y \), then \( A \in P_pO(Y, \tau_Y) \).

proof. Let \( A \in P_pO(X, \tau) \), then \( A \in PO(X, \tau) \) and for each \( x \in A \), there exists a preclosed set \( F \) in \( X \) such that \( x \in F \subseteq A \). Since \( A \in PO(X, \tau) \) and \( A \subseteq Y \). Then by Theorem 2.6, \( A \in PO(Y, \tau_Y) \). Since \( F \) preclosed set in \( X \) and \( A \subseteq Y \). Then by Theorem 2.6, \( F \) preclosed set in \( Y \). Hence \( A \in P_pO(Y, \tau_Y) \).

Proposition 3.30 Let \((Y, \tau_Y)\) be a subspace of a space \((X, \tau)\) and \( A \subseteq Y \). If \( A \in P_pO(Y, \tau_Y) \) and \( Y \) is preclopen, then \( A \in P_pO(X, \tau) \).

proof. Let \( A \in P_pO(X, \tau_Y) \), then \( A \in PO(X, \tau_Y) \) and for each \( x \in A \), there exists a preclosed set \( F \) in \( Y \) such that \( x \in F \subseteq A \). Since \( Y \) is preclopen, then \( Y \in PO(X, \tau) \) and since \( A \in PO(X, \tau_Y) \), then by Theorem 2.6, \( A \in PO(X, \tau) \). Again since \( Y \) is preclopen implies \( Y \) is preclosed set in \( X \) and since \( F \) is preclosed set in \( Y \), therefore by Theorem 2.6, \( F \) is preclosed set in \( X \). Hence \( A \in P_pO(X, \tau) \).

From Propositions 3.29 and 3.30, we obtain the following result:

Corollary 3.31 Let \((X, \tau)\) be a topological space and \( A, Y \) subsets of \( X \) such that \( A \subseteq Y \subseteq X \) and \( Y \) is preclopen. Then \( A \in P_pO(Y) \) if and only if \( A \in P_pO(X) \).

Proposition 3.32 Let \((Y, \tau_Y)\) be a subspace of a space \((X, \tau)\). If \( A \in P_pO(Y, \tau_Y) \) and \( Y \in PR(X, \tau) \), then \( A \in P_pO(X, \tau) \).

proof. Obvious.

Corollary 3.33 Let \((X, \tau)\) be a topological space and \( A, Y \) subsets of \( X \) such that \( A \subseteq Y \subseteq X \) and \( Y \in PR(X) \). Then \( A \in P_pO(Y) \) if and only if \( A \in P_pO(X) \).

Corollary 3.34 Let \( A \) and \( Y \) be any subsets of a space \( X \). If \( A \in P_pO(X) \) and \( Y \) is both \( \alpha \)-open preclosed subset of \( X \), then \( A \cap Y \in P_pO(Y) \).
**proof.** Follows from Proposition 3.16 and Proposition 3.29. The following diagram shows that the relations among \( P_{\theta}O(X) \), \( \theta O(X) \), \( P\theta O(X) \), \( \delta O(X) \), \( \tau \), \( \alpha O(X) \) and \( PO(X) \).

\[
\begin{align*}
\theta O(X) & \longrightarrow P\theta O(X) \longrightarrow P_{\theta}O(X) \\
\downarrow & & \downarrow \\
\delta O(X) & \longrightarrow \tau \longrightarrow \alpha O(X) \longrightarrow PO(X)
\end{align*}
\]

Diagram 1

**Remark 3.35** In Diagram 1, we notice the following statements:

1. \( \tau \) is incomparable with \( P_{\theta}O(X) \).
2. \( \delta O(X) \) is incomparable with \( P_{\theta}O(X) \).
3. \( \alpha O(X) \) is incomparable with \( P_{\theta}O(X) \).

**Definition 3.36** Let \( A \) be a subset of a space \( X \) and \( x \in X \), then:

1. A subset \( N \) of \( X \) is said to be \( P_{\theta} \)-neighborhood of \( x \), if there exists a \( P_{\theta} \)-open set \( U \) in \( X \) such that \( x \in U \subseteq N \).
2. \( P_{\theta} \)-interior of a set \( A \) (briefly, \( P_{\theta}Int(A) \)) is the union of all \( P_{\theta} \)-open sets which are contained in \( A \).
3. A point \( x \in X \) is said to be \( P_{\theta} \)-limit point of \( A \) if for each \( P_{\theta} \)-open set \( U \) containing \( x \), \( U \cap (A \setminus \{x\}) \neq \phi \). The set of all \( P_{\theta} \)-limit points of \( A \) is called a \( P_{\theta} \)-derived set of \( A \) and is denoted by \( P_{\theta}D(A) \).
4. A point \( x \in X \) is said to be in \( P_{\theta} \)-closure of \( A \) if for each \( P_{\theta} \)-open set \( U \) containing \( x \) such that \( U \cap A \neq \phi \).
5. \( P_{\theta} \)-closure of a set \( A \) (briefly, \( P_{\theta}Cl(A) \)) is the intersection of all \( P_{\theta} \)-closed sets containing \( A \).
6. \( P_{\theta} \)-boundary of \( A \) is defined as \( P_{\theta}Cl(A) \setminus P_{\theta}Int(A) \) and is denoted by \( P_{\theta}Bd(A) \).

The topological properties of \( P_{\theta} \)-neighborhood, \( P_{\theta} \)-interior, \( P_{\theta} \)-closure, \( P_{\theta} \)-derived and \( P_{\theta} \)-boundary are the same as in the supratopology.
4  \(P_p\)-g. Closed Sets

**Definition 4.1** A subset \(A\) of \(X\) is said to be a \(P_p\)-generalized closed (briefly, \(P_p\)-g.closed) set, if \(P_p\text{Cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(P_p\)-open set in \((X, \tau)\). The family of all \(P_p\)-g.closed sets of a topological space \((X, \tau)\) is denoted by \(P_pGC(X, \tau)\) or \(P_pGC(X)\).

It is clear that every \(P_p\)-closed set is \(P_p\)-g.closed set, but the converse is not true in general as it is shown in the following example.

**Example 4.2** Considering the space \((X, \tau)\) as defined in Example 3.5. Then we have \(\{c\} \in P_pGC(X)\), but \(\{c\} \notin P_pC(X)\).

**Proposition 4.3** The intersection of a \(P_p\)-g.closed set and a \(P_p\)-closed set is always \(P_p\)-g.closed.

**proof.** Let \(A\) be \(P_p\)-g.closed set and \(F\) be \(P_p\)-closed set. Assume that \(U\) be a \(P_p\)-open set such that \(A \cap F \subseteq U\). Set \(G = X \setminus F\), then \(A \subseteq U \cup G\). Since \(G\) is \(P_p\)-open, \(U \cup G\) is \(P_p\)-open and since \(A\) is \(P_p\)-g.closed, then \(P_p\text{Cl}(A) \subseteq U \cup G\). Now, \(P_p\text{Cl}(A \cap F) \subseteq P_p\text{Cl}(A) \cap P_p\text{Cl}(F) = P_p\text{Cl}(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup (G \cap F) = (U \cap F) \cup \phi \subseteq U\).

**Proposition 4.4** If \(A\) is both \(P_p\)-open and \(P_p\)-g.closed set in \(X\), then \(A\) is \(P_p\)-closed set.

**proof.** Since \(A\) is \(P_p\)-open and \(P_p\)-g.closed set in \(X\), \(P_p\text{Cl}(A) \subseteq A\), but \(A \subseteq P_p\text{Cl}(A)\). Therefore \(A = P_p\text{Cl}(A)\), and hence \(A\) is \(P_p\)-closed set.

The union of two \(P_p\)-g.closed sets need not be \(P_p\)-g.closed set in general. It is shown by the following example:

**Example 4.5** In Example 3.5, \(\{a\} \in P_pGC(X)\) and \(\{d\} \in P_pGC(X)\), but \(\{a\} \cup \{d\} = \{a, d\} \notin P_pGC(X)\).

The intersection of two \(P_p\)-g.closed sets need not be \(P_p\)-g.closed set in general. It is shown by the following example:

**Example 4.6** In Example 3.5, \(\{a, c, d\} \in P_pGC(X)\) and \(\{a, b, d\} \in P_pGC(X)\), but \(\{a, c, d\} \cup \{a, b, d\} = \{a, d\} \notin P_pGC(X)\).

**Proposition 4.7** If a subset \(A\) of \(X\) is \(P_p\)-g.closed set and \(A \subseteq B \subseteq P_p\text{Cl}(A)\), then \(B\) is a \(P_p\)-g.closed set in \(X\).
proof. Let $A$ be a $p$-g.closed set such that $A \subseteq B \subseteq P_p\text{Cl}(A)$. Let $U$ be a $P_p$-open set of $X$ such that $B \subseteq U$. Since $A$ is $p$-g.closed, we have $P_p\text{Cl}(A) \subseteq U$. Now $P_p\text{Cl}(A) \subseteq P_p\text{Cl}(B) \subseteq P_p\text{Cl}[P_p\text{Cl}(A)] = P_p\text{Cl}(A) \subseteq U$. That is $P_p\text{Cl}(B) \subseteq U$, where $U$ is $P_p$-open. Therefore, $B$ is a $P_p$-g.closed set in $X$.

The converse of Proposition 4.7 is not true in general as it can be seen from the following example:

Example 4.8 In Example 3.5. Let $A = \{a\}$ and $B = \{a, b\}$, then $A$ and $B$ are $P_p$-g.closed sets in $(X, \tau)$, but $A \subseteq B \not\subseteq P_p\text{Cl}(A)$.

Proposition 4.9 For each $x \in X$, \{x\} is $P_p$-closed or $X \setminus \{x\}$ is $P_p$-g.closed in $(X, \tau)$.

proof. Suppose that \{x\} is not $P_p$-closed, then $X \setminus \{x\}$ is not $P_p$-open. Let $U$ be any $P_p$-open set such that $X \setminus \{x\} \subseteq U$, implies $U = X$. Therefore $P_p\text{Cl}(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is $P_p$-g.closed.

Proposition 4.10 A subset $A$ of $X$ is $P_p$-g.closed if and only if $P_p\text{Cl}(\{x\}) \cap A \neq \phi$, holds for every $x \in P_p\text{Cl}(A)$.

proof. Let $U$ be a $P_p$-open set such that $A \subseteq U$ and let $x \in P_p\text{Cl}(A)$. By assumption, there exists a point $z \in P_p\text{Cl}(\{x\})$ and $z \in A \subseteq U$. Then $U \cap \{x\} \neq \phi$, hence $x \in U$, this implies that $P_p\text{Cl}(A) \subseteq U$. Therefore, $A$ is $P_p$-g.closed.

Conversely, suppose that $x \in P_p\text{Cl}(A)$ such that $P_p\text{Cl}(\{x\}) \cap A = \phi$. Since $P_p\text{Cl}(\{x\})$ is $P_p$-closed. Therefore, $X \setminus P_p\text{Cl}(\{x\})$ is a $P_p$-open set in $X$. Since $A \subseteq X \setminus P_p\text{Cl}(\{x\})$ and $A$ is $P_p$-g.closed implies that $P_p\text{Cl}(A) \subseteq X \setminus P_p\text{Cl}(\{x\})$ holds, and hence $x \not\in P_p\text{Cl}(A)$. This is a contradiction. Therefore, $P_p\text{Cl}(\{x\}) \cap A \neq \phi$.

Proposition 4.11 A subset $A$ of a space $X$ is $P_p$-g.closed if and only if $P_p\text{Cl}(A) \setminus A$ does not contain any non-empty $P_p$-g.closed set.

proof. Necessity. Suppose that $A$ is a $P_p$-g.closed set in $X$. We prove the result by contradiction. Let $F$ be a $P_p$-closed set such that $F \subseteq P_p\text{Cl}(A) \setminus A$ and $F \neq \phi$. Then $F \subseteq X \setminus A$ which implies $A \subseteq X \setminus F$. Since $A$ is $P_p$-g.closed and $X \setminus F$ is $P_p$-open, therefore, $P_p\text{Cl}(A) \subseteq X \setminus F$, that is $F \subseteq X \setminus P_p\text{Cl}(A)$. Hence $F \subseteq P_p\text{Cl}(A) \cap (X \setminus P_p\text{Cl}(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $P_p\text{Cl}(A) \setminus A$ does not contain any non-empty $P_p$-g.closed set in $X$.

Sufficiency. Let $A \subseteq U$, where $U$ is $P_p$-open in $X$. If $P_p\text{Cl}(A)$ is not contained in $U$, then $P_p\text{Cl}(A) \cap X \setminus U \neq \phi$. Now, since $P_p\text{Cl}(A) \cap X \setminus U \subseteq P_p\text{Cl}(A) \setminus A$ and $P_p\text{Cl}(A) \cap X \setminus U$ is a non-empty $P_p$-g.closed set, then we obtain a contradiction. Therefore, $A$ is $P_p$-g.closed.
Proposition 4.12 If $A$ is a $P_p$-g.closed set of a space $X$, then $A$ is $P_p$-closed if and only if $P_p\text{Cl}(A) \setminus A$ is $P_p$-closed.

proof. Necessity. If $A$ is a $P_p$-g.closed set which is also $P_p$-closed, then by Proposition 4.11, $P_p\text{Cl}(A) \setminus A = \emptyset$, which is $P_p$-closed.

Sufficiency. Let $P_p\text{Cl}(A) \setminus A$ be a $P_p$-closed set and $A$ be $P_p$-g.closed. Then by Proposition 4.11, $P_p\text{Cl}(A) \setminus A$ does not contain any non-empty $P_p$-closed subset. Since $P_p\text{Cl}(A) \setminus A$ is $P_p$-closed and $P_p\text{Cl}(A) \setminus A = \emptyset$, this shows that $A$ is $P_p$-closed.

Proposition 4.13 Every subset of a space $X$ is $P_p$-g.closed if and only if $P_pO(X, \tau) = P_p\text{C}(X, \tau)$.

proof. Let $U \in P_pO(X, \tau)$. Then by hypothesis, $U$ is $P_p$-g.closed which implies that $P_p\text{Cl}(U) \subseteq U$, then $P_p\text{Cl}(U) = U$, therefore $U \in P_p\text{C}(X, \tau)$. Also let $V \in P_p\text{C}(X, \tau)$. Then $X \setminus V \in P_pO(X, \tau)$, hence by hypothesis $X \setminus V$ is $P_p$-g.closed and then $X \setminus V \in P_p\text{C}(X, \tau)$, thus $V \in P_pO(X, \tau)$ according to the above we have $P_pO(X, \tau) = P_p\text{C}(X, \tau)$.

Conversely, if $A$ is a subset of a space $X$ such that $A \subseteq U$ where $U \in P_pO(X, \tau)$, then $U \in P_p\text{C}(X, \tau)$ and therefore, $P_p\text{Cl}(U) \subseteq U$ which shows that $A$ is $P_p$-g.closed.

5 $P_p$-Continuous Functions

Definition 5.1 A function $f : X \to Y$ is called $P_p$-continuous at a point $x \in X$, if for each open set $V$ of $Y$ containing $f(x)$, there exists a $P_p$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$. If $f$ is $P_p$-continuous at every point $x$ of $X$, then it is called $P_p$-continuous.

We recall the following definitions.

Definition 5.2 A function $f : X \to Y$ is called:

1. precontinuous [15], if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in P\text{O}(X, x)$ such that $f(U) \subseteq V$.

2. strongly $\theta$-continuous [14], if the inverse image of each open subset of $Y$ is $\theta$-open in $X$.

3. quasi $\theta$-continuous [20] at a point $x \in X$, if for each $\theta$-open set $V$ of $Y$ containing $f(x)$, there exists a $\theta$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$.

Proposition 5.3 A function $f : X \to Y$ is $P_p$-continuous if and only if the inverse image of every open set in $Y$ is a $P_p$-open in $X$.
proof. It is clear.

The proof of the following corollaries follows directly from their definitions and are thus omitted.

**Corollary 5.4** Every $P_p$-continuous function is precontinuous.

**Corollary 5.5** Every quasi $\theta$-continuous is $P_p$-continuous.

The examples are given below demonstrate that the converses of the previous corollaries are false.

**Example 5.6** Consider $X = \{a, b, c\}$ with the topology $\tau = \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then $f$ is precontinuous, but it is not $P_p$-continuous, because $\{a\}$ is an open set in $(X, \sigma)$ containing $f(a) = a$, there exists no $P_p$-open set $U$ in $(X, \tau)$ containing $a$ such that $a \in f(U) \subseteq \{a\}$.

**Example 5.7** Consider $X = \{a, b, c, d\}$ with the two topologies $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then $f$ is $P_p$-continuous, but it is not quasi $\theta$-continuous.

**Proposition 5.8** A function $f : X \to Y$ is $P_p$-continuous if and only if $f$ is pre-continuous and for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists a preclosed set $F$ of $X$ containing $x$ such that $f(F) \subseteq V$.

**proof.** Let $f : X \to Y$ be a $P_p$-continuous and also let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. By hypothesis, there exists a $P_p$-open set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$. Since $U$ is $P_p$-open set. Then for each $x \in U$, there exists a preclosed set $F$ of $X$ such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$. Hence $P_p$-continuous always implies pre-continuous.

**Conversely,** let $V$ be any open set of $Y$. We have to show that $f^{-1}(V)$ is $P_p$-open set in $X$. Since $f$ is pre-continuous, then $f^{-1}(V)$ is preopen set in $X$. Let $x \in f^{-1}(V)$, then $f(x) \in V$. By hypothesis, there exists a preclosed set $F$ of $X$ containing $x$ such that $f(F) \subseteq V$, which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is $P_p$-open set in $X$. Hence by Proposition 5.3, $f$ is $P_p$-continuous.

Here, we begin with the following characterizations of $P_p$-continuous functions.

**Proposition 5.9** For a function $f : X \to Y$, the following statements are equivalent:

1. $f$ is $P_p$-continuous.
2. $f^{-1}(V)$ is a $P_p$-open set in $X$, for each open set $V$ of $Y$.

3. $f^{-1}(F)$ is a $P_p$-closed set in $X$, for each closed set $F$ of $Y$.

4. $f(P_pCl(A)) \subseteq Cl(f(A))$, for each subset $A$ of $X$.

5. $P_pCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for each subset $B$ of $Y$.

6. $f^{-1}(Int(B)) \subseteq P_pInt(f^{-1}(B))$, for each subset $B$ of $Y$.

7. $Int(f(A)) \subseteq f(P_pInt(A))$, for each subset $A$ of $X$.

**proof.** Straightforward.

**Proposition 5.10** Let $f : X \to Y$ be a function and $X$ is locally indiscrete space. Then $f$ is $P_p$-continuous if and only if $f$ is Pre-continuous.

**proof.** Follows from Corollary 3.17.

**Proposition 5.11** Let $f : X \to Y$ be a function and $X$ is pre-$T_1$ space. Then $f$ is $P_p$-continuous if and only if $f$ is Pre-continuous.

**proof.** Follows from Proposition 3.8.

**Proposition 5.12** Let $f : X \to Y$ be a $P_p$-continuous function. If $Y$ is any subset of a topological space $Z$, then $f : X \to Z$ is $P_p$-continuous.

**proof.** Let $x \in X$ and $V$ be any open set of $Z$ containing $f(x)$, then $V \cap Y$ is open in $Y$. But $f(x) \in Y$ for each $x \in X$, then $f(x) \in V \cap Y$. Since $f : X \to Y$ is $P_p$-continuous, then there exists a $P_p$-open set $U$ containing $x$ such that $f(U) \subseteq V \cap Y \subseteq V$. Therefore, $f : X \to Z$ is $P_p$-continuous.

**Proposition 5.13** Let $f : X \to Y$ be $P_p$-continuous function. If $A$ is $\alpha$-open and preclosed subset of $X$, then $f|A : A \to Y$ is $P_p$-continuous in the subspace $A$.

**proof.** Let $V$ be any open set of $Y$. Since $f$ is $P_p$-continuous. Then by Proposition 5.3, $f^{-1}(V)$ is $P_p$-open set in $X$. Since $A$ is $\alpha$-open and preclosed subset of $X$. By Corollary 3.34, $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is a $P_p$-open subset of $A$. This shows that $f|A : A \to Y$ is $P_p$-continuous.

**Proposition 5.14** A function $f : X \to Y$ is $P_p$-continuous, if for each $x \in X$, there exists a preclopen set $A$ of $X$ containing $x$ such that $f|A : A \to Y$ is $P_p$-continuous.
Let \( P \in X \) exist open sets functions. If \( Y \) is Hausdorff, then the set \( E = \{ (x_1, x_2) \in X_1 \times X_2 : f(x_1) = g(x_2) \} \) is \( P \)-open in the product space \( X_1 \times X_2 \).

**proof.** Let \( U = U_1 \times U_2 \), then \( (x_1, x_2) \in U \) and \( U \) is a \( P \)-open set in \( X_1 \times X_2 \), by Proposition 3.25, and \( U \cap E = \emptyset \). Therefore, we obtain \( (x_1, x_2) \notin P_{\alpha}Cl(E) \). Hence \( E \) is \( P \)-closed in the product space \( X_1 \times X_2 \).

**Proposition 5.16** Let \( f : X \to Y \) and \( g : Y \to Z \) be two \( P \)-continuous functions. If \( f \) is \( P \)-continuous and \( g \) is continuous. Then the composition function \( g \circ f : X \to Z \) is \( P \)-continuous.

**proof.** Let \( V \) be any open subset of \( Z \). Since \( g \) is continuous, \( g^{-1}(V) \) is open subset of \( Y \). Since \( f \) is \( P \)-continuous, then by Proposition 5.3, \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is \( P \)-open subset in \( X \). Therefore, by Proposition 5.3, \( g \circ f \) is \( P \)-continuous.

**Proposition 5.17** Let \( f : X \to Y \) be a \( P \)-continuous function and let \( g : Y \to Z \) be a strongly \( \theta \)-continuous function, then \( g \circ f : X \to Z \) is \( P \)-continuous.

**proof.** Let \( V \) be an open subset of \( Z \). In view of strong \( \theta \)-continuity of \( g \), \( g^{-1}(V) \) is a \( \theta \)-open subset of \( Y \). Again, since \( f \) is \( P \)-continuous, \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is a \( P \)-open set in \( X \). Hence \( g \circ f \) is \( P \)-continuous.

**Corollary 5.18** Let \( f : X \to Y \) be a \( P \)-continuous function and let \( g : Y \to Z \) be a quasi \( \theta \)-continuous function, then \( g \circ f : X \to Z \) is \( P \)-continuous.

**Proposition 5.19** If \( f_i : X_i \to Y_i \) is \( P \)-continuous functions for \( i = 1, 2 \). Let \( f : X_1 \times X_2 \to Y_1 \times Y_2 \) be a function defined as follows: \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). Then \( f \) is \( P \)-continuous.

**proof.** Let \( R_1 \times R_2 \subseteq Y_1 \times Y_2 \), where \( R_i \) is open set in \( Y_i \) for \( i = 1, 2 \). Then \( f^{-1}(R_1 \times R_2) = f_1^{-1}(R_1) \times f_2^{-1}(R_2) \). Since \( f_i \) is \( P \)-continuous for \( i = 1, 2 \). By Proposition 5.3, and Proposition 3.25, \( f^{-1}(R_1 \times R_2) \) is \( P \)-open set in \( X_1 \times X_2 \).
Proposition 5.20 Let $X, Y_1, Y_2$ be topological spaces and $f_i : X \to Y_i$, for $i = 1, 2$, be functions. If a functions $g : X \to Y_1 \times Y_2$ defined as: $g(x) = (x_1, x_2)$, where $f_i(x) = x_i$, for $i = 1, 2$ is $P_p$-continuous, then $f_i$ is $P_p$-continuous for $i = 1, 2$.

proof. Let $x \in X$ and $V_1$ be any open set in $Y_1$ containing $f_1(x) = x_1$, then $V_1 \times Y_2$ is open in $Y_1 \times Y_2$, which contain $(x_1, x_2)$. Since $g$ is $P_p$-continuous. Then by Proposition 5.3, $g^{-1}(V_1 \setminus Y_2)$ is $P_p$-open set in $X$. However, $f^{-1}(V_1) = g^{-1}(V_1 \setminus Y_2) (f^{-1}Cl(V_1) = g^{-1}Cl(V_1 \setminus Y_2)$. Thus $f_1$ is $P_p$-continuous. Similarly, we can prove that $f_2$ is $P_p$-continuous. This completes the proof.

References


