Semiderivations and Commutativity
In Semiprime Rings

H. Nabiël

Department of Mathematics, Faculty of Science, Al-Azhar University
11884, Nasr City, Cairo, Egypt
E-mail: hnbieel@yahoo.com

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Abstract

Let $R$ be a semiprime ring. An additive mapping $f : R \to R$ is called a semiderivation if there exists a function $g : R \to R$ such that $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$ for all $x, y \in R$. In the present paper we investigate commutativity of $R$ satisfying any one of the properties (i) $[f(x), f(y)] = 0$, (ii) $[f(x), f(y)] = [x, y]$, (iii) $[f(x), d(y)] = [x, y]$, $d$ is a derivation on $R$, or (iv) $f([x, y]) = \pm [x, y]$, for all $x, y$ in some appropriate subset of $R$. Also we extend two results of Bell and Martindale from prime rings to semiprime rings.

Keywords: prime ring, semiprime ring, essential ideal, derivation, semiderivation, commuting mapping, strong commutativity-preserving mapping.

1 Introduction

Throughout, $R$ will be an associative ring. $R$ is said to be 2-torsion-free, if $2x = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ for $x, y \in R$ will be denoted by $[x, y]$. We shall use basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$, for $x, y, z \in R$. Recall that $R$ is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$ for every $a, b \in R$, and
is semiprime if \( aRa = (0) \) implies \( a = 0 \), for every \( a \in R \). An ideal \( U \) of \( R \) is essential if for every nonzero ideal \( K \) of \( R \) we have \( U \cap K \neq (0) \). If \( R \) is a ring with center \( Z \), a mapping \( f \) from \( R \) to \( R \) is called centralizing on \( S \subseteq R \) if \( [x, f(x)] \in Z \) for all \( x \in S \); in the special case where \( [x, f(x)] = 0 \) for all \( x \in S \), the mapping \( f \) is said to be commuting on \( S \). A mapping \( f : R \to R \) is called strong commutativity-preserving (scp) on \( S \subseteq R \) if \( [f(x), f(y)] = [x, y] \) for all \( x, y \in S \). A derivation \( d : R \to R \) is an additive map which satisfies \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \).

The present paper has been motivated by the works of Chang [7], Daif [9], Bell and Daif [3], Daif and Bell [8], and Bell and Martindale [5]. Bergen [6] has introduced the following notion. An additive mapping \( f \) of a ring \( R \) itself is called a semiderivation if there exists a function \( g : R \to R \) such that \( f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y) \) and \( f(g(x)) = g(f(x)) \) for all \( x, y \in R \). For \( g = 1 \) a semiderivation is of course a derivation. The other main motivating examples are of the form \( f(x) = x - g(x) \) where \( g \) is any ring endomorphism of \( R \). Then \( f \) is a semiderivation of \( R \) with associated map \( g \) which is not a derivation. In [11], Herstein has shown that if \( R \) is a prime ring admitting a nonzero derivation \( d \) such that \( [d(x), d(y)] = 0 \) for all \( x, y \in R \), then \( R \) is commutative whenever \( \text{char} R \neq 2 \), and if \( \text{char} R = 2 \), then either \( R \) is commutative or is an order in a simple algebra which is 4-dimensional over its center. In [7], Chang has given an extension of the above mentioned result of Herstein in the following way. Let \( f \neq 0 \) be a semiderivation of a prime ring \( R \) associated with an epimorphism \( g \) of \( R \) such that \( [f(R), f(R)] = \{0\} \). Then, if \( \text{char} (R) \neq 2 \), \( R \) is a commutative, and if \( \text{char} (R) = 2 \), \( R \) is commutative or is an order in a simple algebra which is 4-dimensional over its center. In [9], Daif has generalized the previously mentioned result of Herstein in the following way. Let \( R \) be a two-torsion-free semiprime ring and \( U \) a nonzero ideal of \( R \). If \( R \) admits a derivation \( d \) which is nonzero on \( U \) and \( [d(x), d(y)] = 0 \) for all \( x, y \in U \), then \( R \) contains a nonzero central ideal. In [8], Daif and Bell have proved that a semiprime ring \( R \) is commutative if it admits a derivation \( d \) for which either \( d([x, y]) = [y, x] \) for all \( x, y \in R \) or \( d([x, y]) = [x, y] \) for all \( x, y \in R \). In [3], Bell and Daif have shown that if a semiprime ring \( R \) admits a strong-commutativity preserving derivation on a nonzero right ideal \( U \) of \( R \), then \( U \subseteq Z \), the center of \( R \). In [5], Bell and Martindale have proved the following three results.

(i) Let \( f \neq 0 \) be a semiderivation of a prime ring \( R \) of characteristic not 2 with associated endomorphism \( g \) of \( R \) and \( U \neq 0 \) be an ideal of \( R \). Suppose that \( a \in R \) such that \( af(U) = 0 \). Then \( a = 0 \).

(ii) Let \( f \) be a semiderivation of a prime ring \( R \) of characteristic not 2 with associated endomorphism \( g \) of \( R \). If there exists a nonzero ideal \( U \) of \( R \) for which \( U \cap g(R) = 0 \), then there exists \( \lambda \in C \) (the extended centroid of \( R \)) such
that \( f(x) = \lambda(x - g(x)) \) for all \( x \in R \).

(iii) Let \( f \) be a semiderivation of a prime ring \( R \) of characteristic not 2 with associated endomorphism \( g \) of \( R \). If \( g \) is not one-one and \( V \neq 0 \) is an ideal of \( R \), then \( f(V) \) is a nonzero ideal of \( R \), and there exists \( \lambda \in C \) such that \( f(x) = \lambda(x - g(x)) \) for all \( x \in R \).

In [1], Ali and Huang have proved the following theorem. Let \( R \) be a 2−torsion free semiprime ring and \( I \) a nonzero ideal of \( R \). Let \( d \) be a derivation of \( R \). If one of the following conditions holds:

(i) \([d(x), d(y)] = [x, y]\) for all \( x, y \in I \),

(ii) \([d(x), d(y)] = -[x, y]\) for all \( x, y \in I \),

(iii) for all \( x, y \in I \), either \([d(x), d(y)] = [x, y]\) or \([d(x), d(y)] = -[x, y]\),

then \( d \) is commuting on \( I \). Further, if \( d(I) \neq 0 \), then \( R \) has a nonzero central ideal.

In [10], De Filippis, Mamouni and Oukhtite have showed the following result. Let \( R \) be a prime ring of characteristic not 2 and \( I \) a nonzero ideal of \( R \). If \( R \) admits a nonzero semiderivation \( f \) with associated function \( g \) such that \( f([x, y]) = [x, y] \) for all \( x, y \in I \), then one of the following holds:

(1) \( R \) is commutative;

(2) \( f(x) = x - g(x) \) for all \( x \in R \), with \( g([R, R]) = 0 \);

(3) \( f(x) = x \), for all \( x \in I \) and \( g(I) = 0 \).

Our aim in this work is to investigate the commutativity of semiprime rings admitting semiderivations. In the first section we extend the above mentioned result of Chang [7, Theorem 2] for prime rings to semiprime rings, extend two results of Bell and Martindale ([5, Lemma 4], [5, Lemma 5]) for prime rings to semiprime rings, and give a counter example to [5, Lemma 2] in the semiprime ring case. In the second section we study commutativity for a semiprime ring \( R \) admitting a semiderivation \( f \) associated with an epimorphism \( g \) of \( R \) which satisfies \([f(x), f(y)] = [x, y]\) for all \( x, y \) belonging to an ideal of \( R \), or satisfies \([f([x, y]) = \pm[x, y]\) for all \( x, y \in R \), or admits an additive map \( f \) and a derivation \( d \) which satisfy \([f(x), d(y)] = [x, y]\) for all \( x, y \) belonging to an ideal of \( R \).

In order to prove our aims we need the following results:

**Theorem 1.1.** [2, Theorem 2.3.2]. Let \( R \) be a semiprime ring, \( Q = Q_{mr}(R) \), the maximal right ring of quotients of \( R \), \( _R U_R \subseteq_R Q_R \) a submodule of \( Q \) and \( f : _R U_R \rightarrow_R Q_R \) a homomorphism of bimodules. Then there exists an element \( \lambda \in C \) (the extended centroid of \( R \)) such that \( f(u) = \lambda u \) for all \( u \in U \).

**Lemma 1.2.** [8, Lemma 1]. Let \( R \) be a semiprime ring and \( I \) a nonzero ideal of \( R \). If \( x \) in \( R \) centralizes the set \([I, I]\), then \( x \) centralizes \( I \).
Lemma 1.3. [3, Lemma 1.]. If $R$ is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of $R$; in particular, any commutative one-sided ideal is contained in the center of $R$.

Remark 1.4. [2, Remark 2.1.4]. If $U$ is an essential two-sided ideal of a semiprime ring $R$, then $l(U) = r(U) = (0)$.

2 Semiderivations on Semiprime Rings

In this section we begin with a theorem that extends Chang’s theorem ([7, Theorem 2]) from prime rings to semiprime rings, and also generalizes Daif’s theorem ([9, Theorem 2.1]) for derivations to semiderivations. To achieve this goal we modify Theorem 3 of [4] from the case of derivations to the case of semiderivations. Also we extend two results of Bell and Martindale ([5, Lemma 4], [5, Lemma 5]) on derivations to semiderivations, and give a counter example to [5, Lemma 2] in the semiprime ring case.

Lemma 2.1. Let $R$ be a semiprime ring. If $R$ admits a nonzero semiderivation $f$ with associated surjective map $g$ of $R$ which is commuting on $R$, then $R$ contains a nonzero central ideal.

Proof. We have for all $x \in R$ that $[x, f(x)] = 0$. Replacing $x$ by $u + v$, we get

$$[u, f(v)] + [v, f(u)] = 0 \text{ for all } u, v \in R. \quad (2.1)$$

Replacing $u$ by $x$ and $v$ by $yx$, and using our hypothesis and (2.1), we get

$$[x, g(y)]f(x) = 0 \text{ for all } x, y \in R. \quad (2.2)$$

Since $g$ is onto we have

$$[x, y]f(x) = 0 \text{ for all } x, y \in R. \quad (2.3)$$

Replacing $y$ by $wy$ and using (2.3), we get $[x, w]yf(x) = 0$, which implies that

$$[x, w]Rx = \{0\} \text{ for all } x, w \in R. \quad (2.4)$$

Since $R$ is semiprime, consider the set $\{P_\alpha\}$ of prime ideals of $R$ such that $\bigcap P_\alpha = \{0\}$. Then for each $P_\alpha$ either

(a) $[x, w] \in P_\alpha$ for all $x, w \in R$, \hspace{1cm} (2.5)

or

(b) $f(x) \in P_\alpha$ for all $x \in R$. \hspace{1cm} (2.6)
Call $P_\alpha$ a type-one prime if it satisfies (a), and call $P_\alpha$ a type-two prime if it satisfies (b). Let $P_1$ and $P_2$ be, respectively, the intersections of all type-one and type-two primes. Note that $P_1 \cap P_2 = \{0\}$.

We now investigate a typical type-two prime $P = P_\alpha$. From (b), we have

$$Rf(R) \subseteq P \quad (2.7)$$

Now consider the left ideal $V = Rf(R)$; we shall show that $V$ is commutative, hence a two-sided central ideal. A typical element of $V$ is a sum of elements of the form $rf(s)$, where $r, s \in R$. Thus we need only show that commutators of the form $[r_1f(s_1), r_2f(s_2)]$ are all trivial, clearly this commutator is in $P_1$ by (a) and in $P_2$ by (2.7), hence belongs to $P_1 \cap P_2 = \{0\}$.

Assume that $V = \{0\}$ in which case $Rf(R) = \{0\}$, hence $f(R)Rf(R) = \{0\}$, since $R$ is semiprime we have $f(R) = \{0\}$ which is a contradiction. Hence $V \neq \{0\}$. By Lemma 1.3, $R$ contains a nonzero central ideal. \hfill \Box

Now, we are ready to prove the first theorem of this section.

**Theorem 2.2.** If $R$ is a two torsion free semiprime ring and $f$ is a nonzero semiderivation of $R$ associated with an epimorphism $g$ of $R$ such that $[f(R), f(R)] = \{0\}$, then $R$ contains a nonzero central ideal.

**Proof.** We have $[f(x), f(y)] = 0$ for all $x, y \in R$, replacing $y$ by $yf(z)$, then yields

$$[f(x), f(y)]f(z) + f(y)[f(x), f(z)] + g(y)[f(x), f^2(z)] + [f(x), g(y)]f^2(z)$$

$$= 0 \text{ for all } x, y, z \in R. \quad (2.8)$$

Using our hypothesis, then $[f(x), g(y)]f^2(z) = 0$ for all $x, y, z \in R$. Since $g$ is onto, we have

$$[f(x), y]f^2(z) = 0 \text{ for all } x, y, z \in R. \quad (2.9)$$

Replacing $y$ by $yw$ and using (2.9), we get

$$[f(x), y]Rf^2(z) = \{0\} \text{ for all } x, y, z \in R. \quad (2.10)$$

Consider the set of prime ideals $P_\alpha$ of $R$ such that $\cap P_\alpha = \{0\}$. For each $P_\alpha$, from (2.10) we either have

(a) $[f(x), y] \in P_\alpha$ for all $x, y \in R$,

or

(b) $f^2(R) \subseteq P_\alpha$.

Call $P_\alpha$ an (a)-prime ideal or a (b)-prime according to which of these conditions is satisfied.
Now consider a (b)-prime ideal \( P_\alpha \). Since \( f^2(xy) = f^2(x)g^2(y) + f(x)Ef(g(y)) + f(x)f(g(y)) + xf^2(y) \), then \( 2f(x)f(g(y)) \in P_\alpha \), and since \( g \) is onto we get

\[
2f(x)f(y) \in P_\alpha, \text{ for all } x, y \in R. \tag{2.11}
\]

Now replacing \( y \) by \( zy \), we get \( 2f(x)f(z)g(y) + 2f(x)zf(y) \in P_\alpha \), which implies

\[
2f(x)zf(y) \in P_\alpha, \text{ for all } x, y, z \in R. \tag{2.12}
\]

Since \( P_\alpha \) is prime, we either have \( 2f(x) \in P_\alpha \) for all \( x \in R \) or \( f(y) \in P_\alpha \) for all \( y \in R \). In either case, we have \( 2[f(x), y] \in P_\alpha \) for all (b)-prime \( P_\alpha \). Also from (a), \( 2[f(x), y] \in P_\alpha \) for all (a)-prime \( P_\alpha \). So \( 2[f(x), y] \in \bigcap P_\alpha = \{0\} \). Since \( R \) is two torsion free, then \( [f(x), y] = 0 \) for all \( x, y \in R \), in particular \( [f(x), x] = 0 \) for all \( x \in R \). By Lemma 2.1, \( R \) contains a nonzero central ideal.

**Lemma 2.3.** [see 5, Lemma 1] Let \( R \) be a semiprime ring. If \( f \neq 0 \) is a semiderivation on \( R \) associated with a function \( g \) of \( R \), and \( U \) is an essential ideal of \( R \), then \( f \neq 0 \) on \( U \).

**Proof.** Suppose \( f(U) = 0 \). Then for \( u \in U, x \in R \) we have \( 0 = f(ux) = f(u)g(x) + uf(x) = uf(x) \), which implies \( 0 = Uf(x) \). From Remark 1.4, we have \( f(x) = 0 \), which is a contradiction. \( \square \)

**Theorem 2.4.** [see 5, Lemma 4] Let \( R \) be a semiprime ring, and \( f \) be a semiderivation on \( R \) associated with an endomorphism \( g \) of \( R \). If there exists a nonzero essential ideal \( U \) of \( R \) for which \( U \cap g(R) = 0 \), then there exists \( \lambda \in C \) (the extended centroid of \( R \)) such that \( f(x) = \lambda(x - g(x)) \) for all \( x \in R \).

**Proof.** We let \( W \) be the ideal \( \sum U(x - g(x))U \) and note that \( W \neq 0 \) (otherwise \( g \) would be the identity mapping, contradicting that \( U \cap g(R) = 0 \)). We define a mapping \( \phi : W \rightarrow R \) according to the rule \( \sum u_i(x_i - g(x_i))v_i \rightarrow u_i f(x_i)v_i \) where \( u_i, v_i \in U \) and \( x_i \in R \). Of course our main problem is to prove that \( \phi \) is well-defined, consequently \( \phi \) is an \((R, R)-\)bimodule map of \( W \) into \( R \). Suppose that

\[
\sum u_i(x_i - g(x_i))v_i = 0. \tag{2.13}
\]

We attempt to show that \( \phi(\sum u_i(x_i - g(x_i))v_i) = 0 \), i.e., \( u_i f(x_i)v_i = 0 \). Applying \( f \) to 2.13, we see that \( 0 = f(\sum u_i(x_i - g(x_i))v_i) \)

\[
= \sum [u_i f(x_i) v_i] + f(u_i)g(x_i)v_i - f(u_i g(x_i))g(v_i) - u_i g(x_i)f(v_i)
\]

\[
= \sum [u_i f(x_i) v_i] + u_i g(x_i)f(v_i) + f(u_i)g(x_i)g(v_i)
\]

\[
- f(u_i)g(x_i)v_i - g(u_i)f(g(x_i))g(v_i) - u_i g(x_i)f(v_i)
\]

\[
= \sum [u_i f(x_i) v_i] - g(u_i)f(g(x_i))g(v_i)
\]

\[
= \sum u_i f(x_i) v_i - g(\sum u_i f(x_i)v_i). \] Therefore \( \sum u_i f(x_i) v_i = g(\sum u_i f(x_i)v_i) \in U \cap g(R) = 0 \), which implies \( \sum u_i f(x_i)v_i = 0 \), then \( \phi \) is well-defined. Since \( \phi \) is an \((R, R)-\)bimodule map of \( W \) into \( R \), from Theorem 1.1, there exists
\( \lambda \in C \) (the extended centroid of \( R \)) such that \( \lambda w = \phi(w) \) for all \( w \in W \). Now, regarding \( R \) as a subring of the central closure \( RC \), we have for all \( u,v \in U \) and \( x \in R \) that \( u\lambda(x - g(x))v = \lambda(u(x - g(x)))v = \phi(u(x - g(x)))v = uf(x)v \),

which implies \( u[\lambda(x - g(x)) - f(x)]v = 0 \) for all \( u,v \in U, x \in R \), i.e., \( U[\lambda(x - g(x)) - f(x)]v = 0 \) for all \( v \in U, x \in R \). From Remark 1.4, we have \( [\lambda(x - g(x)) - f(x)]v = 0 \) for all \( v \in U, x \in R \); i.e., \( [\lambda(x - g(x)) - f(x)]U = 0 \) for all \( x \in R \). From Remark 1.4 we have \( \lambda(x - g(x)) - f(x) = 0 \), which implies \( f(x) = \lambda(x - g(x)), \lambda \in C \). □

**Theorem 2.5.** [see 5, Lemma 5] Let \( R \) be a semiprime ring, and \( f \neq 0 \) be a semiderivation of \( R \) associated with an endomorphism \( g \) of \( R \). If \( g \) is not one-one and \( V \) is an essential ideal of \( R \) contained in ker \( g \), then

(a) \( f(V) \) is a nonzero ideal of \( R \), and

(b) there exists \( \lambda \in C \) such that \( f(x) = \lambda(x - g(x)) \) for all \( x \in R \).

**Proof.** (a) For \( v \in V \) and \( r \in R \), we see immediately from \( f(vr) = f(v)r + g(v)f(r) = f(v)r \) and \( f(rv) = rf(v) + f(r)g(v) = rf(v) \) that \( f(V) \) is an ideal of \( R \). Furthermore \( f(V) \neq 0 \) in view of Lemma 2.3, and so (a) is proved.

(b) The argument establishing (a) also shows that \( f \) is an \( (R,R) \)-bimodule map of \( V \) into \( R \). From Theorem 1.1, there exists \( \lambda \in C \) such that \( \lambda v = f(v) \) for all \( v \in V \). For \( v \in V \) and \( r \in R \) we then see that \( \lambda vr = f(vr) = vrf(r) + f(v)g(r) = vrf(r) + \lambda vg(r) \). In other words, \( v(f(r) + \lambda g(r) - \lambda r) = 0 \), which implies \( V(f(r) + \lambda g(r) - \lambda r) = 0 \), and from Remark 1.4, we get \( f(r) + \lambda g(r) - \lambda r = 0 \), which yields \( f(r) = \lambda(r - g(r)) \) for all \( r \in R \). □

In the next remark we give a counter example to [5, Lemma 2] when \( R \) is semiprime.

**Remark 2.6.** We notice that [5, Lemma 2] is not true in the case when \( R \) is semiprime. Let \( R = R_1 \oplus R_2 \) where \( R_1 \) and \( R_2 \) are prime rings, \( R \) is a semiprime ring. Let \( \alpha : R_1 \to R_2 \) be an additive map and \( \beta : R_2 \to R_2 \) be a nonzero left and right \( R_2 \)-module map which is not a derivation. Define \( f : R \to R \) such that \( f((r_1, r_2)) = (0, \beta(r_2)) \) and \( g : R \to R \) such that \( g((r_1, r_2)) = (\alpha(r_1), 0), r_1 \in R_1, r_2 \in R_2 \). Then \( f \) is a semiderivation on \( R \). Consider the subset \( U = \{(0, r_2), r_2 \in R_2 \} \), then \( U \) is an ideal of \( R \). Let \( a = (a_1, 0) \neq 0 \) be an element of \( R \), we see that \( af(U) = 0 \) but neither \( a \) nor \( f(U) \) is zero.

### 3 Commutativity Results for Semiprime Rings with Derivations and Semiderivations

In this section, we study commutativity for a semiprime ring \( R \) admitting a semiderivation \( f \) associated with an epimorphism \( g \) of \( R \) which satisfies
[f(x), f(y)] = [x, y] for all x, y belonging to an ideal of R, or satisfies f([[x, y]]) = ±[x, y] for all x, y ∈ R, or admits an additive map f and a derivation d which satisfy [f(x), d(y)] = [x, y] for all x, y belonging to an ideal of R. We generalize [3, Theorem 1] of Bell and Daif and [8, Theorem 2] of Daif and Bell from the case of derivations to the case of semiderivations.

**Theorem 3.1.** Let R be a semiprime ring admitting a semiderivation f associated with an epimorphism g of R. Suppose that U is a nonzero ideal of R such that f is scp on U and g(U) = U. Then U ⊆ Z.

**Note that:** The condition g(U) = U may be seen as U is a g–ideal.

**Proof.** For x, y ∈ U, we have [x, xy] = [f(x), f(xy)], which yields

\[ f(x)[f(x), g(y)] + [f(x), x]f(y) = 0 \text{ for all } x, y ∈ U. \]  

(3.1)

Replacing y by yr, r ∈ R, gives

\[ f(x)[f(x), g(y)]g(r) + f(x)g(y)[f(x), g(r)] + [f(x), x]f(y)g(r) + [f(x), x]yf(r) = 0 \text{ for all } x, y ∈ U, r ∈ R. \]  

(3.2)

Comparing with (3.1) yields

\[ f(x)g(y)[f(x), g(r)] + [f(x), x]yf(r) = 0 \text{ for all } x, y ∈ U, r ∈ R. \]  

(3.3)

Since g(U) = U, letting x = g(x), we see that f(g(x))g(y)[f(g(x)), g(r)] + [f(g(x)), g(x)]yf(r) = 0 for all x, y ∈ U, r ∈ R. Letting r = f(x), we see that

\[ [f(g(x)), g(x)]yf^2(x) = 0 \text{ for all } x, y ∈ U. \]  

(3.4)

Therefore (3.4) implies that

\[ [f(g(x)), g(x)]URf^2(x) = \{0\} \text{ for all } x ∈ U. \]  

(3.5)

Since R is semiprime, it must contain a family \( \{P_\alpha|\alpha ∈ \Lambda\} \) of prime ideals such that \( \cap P_\alpha = \{0\} \). If P is a typical member of these and \( x ∈ U \), (3.5) shows that \( f^2(x) ∈ P \) or \( [f(g(x)), g(x)]U ⊆ P \). For a fixed P, the sets of \( x ∈ U \) for which these two conditions hold are additive subgroups of U whose union is U; therefore

\[ f^2(U) ⊆ P \text{ or } [f(g(x)), g(x)]U ⊆ P \text{ for all } x ∈ U. \]  

(3.6)

Suppose that \( f^2(U) ⊆ P \), then for each \( y ∈ U \) we get \( [x, yf(x)] = [f(x), f(yf(x))] \); expanding this equation to \( y[x, f(x)] = [f(x), g(y)]f^2(x) + g(y)[f(x), \)

}\]
$f^2(x)$ implies $y[x, f(x)] \in P$, then so $UR[x, f(x)] \subseteq P$. By the primeness of $P$ we reach to $U \subseteq P$ or $[x, f(x)] \in P$ for all $x \in U$. Either of these cases implies

$$[x, f(x)]U \subseteq P \text{ for all } x \in U. \quad (3.7)$$

From (3.6) now suppose that $[f(g(x)), g(x)]U \subseteq P$ for all $x \in U$, since $g(U) = U$ we get

$$[f(x), x]U \subseteq P \text{ for all } x \in U. \quad (3.8)$$

From (3.7) and (3.8) we have $[x, f(x)]U = \{0\}$ and from (3.3) we have $f(x)g(y)[f(x), g(r)] = 0$ for all $x, y \in U, r \in R$. Since $g$ is onto, $f(x)g(y)[f(x), r] = 0$. Moreover, since $g(U) = U$ we have $f(x)y[f(x), r] = 0$, which implies

$$f(x)UR[f(x), r] = \{0\} \text{ for all } x \in U, r \in R. \quad (3.9)$$

Since $R$ is semiprime, it must contain a family $\{P_\alpha|\alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If $P$ is a typical member of these and $x \in U$, (3.9) shows that $f(x)U \subseteq P$ for all $x \in U$ or $[f(x), r] \in P$ for all $x \in U, r \in R$. For a fixed $P$, the sets of $x \in U$ for which these two conditions hold are additive subgroups of $U$ whose union is $U$; therefore

$$f(U)U \subseteq P \text{ or } [f(U), R] \subseteq P. \quad (3.10)$$

Suppose that $f(U)U \subseteq P$, then $f(U)RU \subseteq P$, that is, $f(U) \subseteq P$ or $U \subseteq P$. In either event $[f(U), f(U)] \subseteq P$. Now (3.10) yields $[f(U), f(U)] = \{0\}$, then $[U, U] = \{0\}$, $U$ is commutative, by Lemma 1.3, $U \subseteq Z$. □

The following two corollaries are immediate from the previous theorem.

**Corollary 3.2.** Let $R$ be a semiprime ring. If $R$ admits a semiderivation $f$ which is scp on $R$ associated with an epimorphism $g$ of $R$, then $R$ is commutative.

**Corollary 3.3.** Let $R$ be a prime ring, $U$ a nonzero ideal, and $R$ admit a semiderivation $f$ which is scp on $U$ associated with an epimorphism $g$ of $R$. If $g(U) = U$, then $R$ is commutative.

**Theorem 3.4.** Let $R$ be a semiprime ring and $U$ a nonzero ideal of $R$. If $R$ admits an additive map $f$ and a derivation $d$ such that $[f(x), d(y)] = [x, y]$ for all $x, y \in U$, then $U \subseteq Z$.

**Proof.** For $x, y \in U$, we have $[x, xy] = [f(x), d(xy)]$, which yields

$$d(x)[f(x), y] + [f(x), x]d(y) = 0 \text{ for all } x, y \in U. \quad (3.11)$$

Replacing $y$ by $yr$ gives

$$d(x)[f(x), yr] + [f(x), x]d(yr) = 0 \text{ for all } x, y \in U, r \in R. \quad (3.12)$$
Comparing with (3.11) yields
\[ d(x)y[f(x), r] + [f(x), x]yd(r) = 0 \text{ for all } x, y \in U, r \in R. \quad (3.13) \]

Letting \( r = f(x) \), we see that \([f(x), x]yd(f(x)) = 0\) for all \(x, y \in U\), which implies
\[ [f(x), x]Ud(f(x)) = 0 = [f(x), x]URd(f(x)) \text{ for all } x \in U. \quad (3.14) \]

Since \(R\) is semiprime, it must contain a family \(\{P_\alpha | \alpha \in \wedge\}\) of prime ideals such that \(\cap P_\alpha = \{0\}\). If \(P\) is a typical member of these and \(x \in U\), (3.14) shows that \(d(f(x)) \in P\) or \([f(x), x]U \subseteq P\). For a fixed \(P\), the sets of \(x \in U\) for which these two conditions hold are additive subgroups of \(U\) whose union is \(U\). Therefore,
\[ d(f(U)) \subseteq P \text{ or } [f(x), x]U \subseteq P \text{ for all } x \in U. \quad (3.15) \]

Suppose that \(d(f(U)) \subseteq P\), for \(x, y \in U\), we get \([x, yf(x)] = [f(x), d(yf(x))]\), which implies \(U[x, f(x)] \subseteq P\) and \(UR[x, f(x)] \subseteq P\), by the primeness of \(P\) we reach to \(U \subseteq P\) or \([x, f(x)] \in P\) for all \(x \in U\). In either case
\[ [x, f(x)]U \subseteq P \text{ for all } x \in U. \quad (3.16) \]

From (3.15) we have \([x, f(x)]U = \{0\}\) and from (3.13) we have \(d(x)y[f(x), r] = 0\) and
\[ d(x)UR[f(x), r] = \{0\} \text{ for all } x \in U, r \in R. \quad (3.17) \]

Since \(R\) is semiprime, it must contain a family \(\{P_\alpha | \alpha \in \wedge\}\) of prime ideals such that \(\cap P_\alpha = \{0\}\). If \(P\) is a typical member of these and \(x \in U\), (3.17) shows that \(d(x)U \subseteq P\) or \([f(x), R] \subseteq P\). For a fixed \(P\), the sets of \(x \in U\) for which these two conditions hold are additive subgroups of \(U\) whose union is \(U\). Therefore,
\[ d(U)U \subseteq P \text{ or } [f(U), R] \subseteq P. \quad (3.18) \]

Suppose that \(d(U)U \subseteq P\), then \(d(U)RU \subseteq P\). By the primeness of \(P\) we reach to \(d(U) \subseteq P\) or \(U \subseteq P\), in either case \(Ud(U) \subseteq P\), then \(y[f(x), d(z)] \in P\) for all \(x, y, z \in U\). By our hypothesis, then \(y[x, z] \in P\) which implies that \(UR[U, U] \subseteq P\), by the primeness of \(P\) we reach to \(U \subseteq P\) or \([U, U] \subseteq P\). In either case \([U, U] \subseteq P\). By our hypothesis \([f(U), d(U)] \subseteq P\). From (3.18) we have \([f(U), d(U)] = \{0\}\), then \([U, U] = \{0\}\), \(U\) is commutative, by Lemma 1.3, \(U \subseteq Z\).

The following three corollaries are immediate from the previous theorem.

**Corollary 3.5.** Let \(R\) be a semiprime ring and \(U\) a nonzero ideal of \(R\). If \(R\) admits a semiderivation \(f\) and a derivation \(d\) such that \([f(x), d(y)] = [x, y]\) for all \(x, y \in U\), then \(U \subseteq Z\).
Corollary 3.6. Let $R$ be a semiprime ring. If $R$ admits a semiderivation $f$ and a derivation $d$ such that $[f(x), d(y)] = [x, y]$ for all $x, y \in R$, then $R$ is commutative.

Corollary 3.7. Let $R$ be a prime ring and $U$ a nonzero ideal of $R$. If $R$ admits a semiderivation $f$ and a derivation $d$ such that $[f(x), d(y)] = [x, y]$ for all $x, y \in U$, then $R$ is commutative.

In the next theorem, we prove Daif and Bell result ([8, Theorem 2]) in the setting of semiderivations.

Theorem 3.8. Let $R$ be a semiprime ring admitting a semiderivation $f$ associated with an epimorphism $g$ of $R$ for which either $xy + f(xy) = yx + f(yx)$ for all $x, y \in R$, or $xy - f(xy) = yx - f(yx)$ for all $x, y \in R$. Then $R$ is commutative.

Proof. Suppose first

$$xy + f(xy) = yx + f(yx) \text{ for all } x, y \in R. \quad (3.19)$$

This can be written as

$$[x, y] = -f([x, y]) \text{ for all } x, y \in R. \quad (3.20)$$

From (3.19) replace $x$ by $[x, y]$ and $y$ by $z$ and using (3.20) and our hypothesis we get, $[g(x), g(y)]f(z) = f(z)[g(x), g(y)]$. Since $g$ is onto we have $[x, y]f(z) = f(z)[x, y]$, which shows that $f(z)$ centralizes $[R, R]$. From Lemma 1.2, $f(z)$ centralizes $R$. By using (3.19), we get

$$[x, y] \in Z(R) \text{ for all } x, y \in R. \quad (3.21)$$

From Lemma 1.2, $R$ centralizes $R$, which implies that $R$ is commutative. \qed

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References


