Spacelike Biharmonic New Type B-Slant Helices According to Bishop Frame in the Lorentzian Heisenberg Group $H^3$

Talat Körpinar and Essin Turhan

Fırat University, Department of Mathematics 23119, Elazığ, Turkey
E-mails: talatkorpinar@gmail.com, essin.turhan@gmail.com

(Received: 20-5-12 / Accepted: 12-6-12)

Abstract

In this paper, we study biharmonic spacelike new type B-slant helices according to Bishop frame in the Lorentzian Heisenberg group $H^3$. We give necessary and sufficient conditions for new type B-slant helices to be biharmonic. We characterize these curves in the Lorentzian Heisenberg group $H^3$. Additionally, we illustrate our results.

Keywords: Bienergy, Bishop frame, Lorentzian Heisenberg group.

1 Introduction

Jiang derived the first and the second variation formula for the bienergy in [7,8], showing that the Euler--Lagrange equation associated to $E_2$ is

$$\tau_2(f) = -\mathcal{J}^f(\tau(f)) = -\Delta \tau(f) - \text{trace}R^N(df, \tau(f)) df = 0,$$

where $\mathcal{J}^f$ is the Jacobi operator of $f$. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic.
Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps. This study is organised as follows: Firstly, we give necessary and sufficient conditions for new type B-slant helices to be biharmonic. We characterize this curves in the Lorentzian Heisenberg group \( H^3 \). Secondly, we study biharmonic spacelike new type B-slant helices according to Bishop frame in the Lorentzian Heisenberg group \( H^3 \). Finally, we illustrate our results.

2 The Lorentzian Heisenberg Group \( H^3 \)

The Heisenberg group \( \text{Heis}^3 \) is a Lie group which is diffeomorphic to \( \mathbb{R}^3 \) and the group operation is defined as
\[
(x, y, z) \ast (\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x}, y + \bar{y}, z + \bar{z} - xy + x\bar{y}).
\]
The identity of the group is \((0,0,0)\) and the inverse of \((x, y, z)\) is given by \((-x,-y,-z)\). The left-invariant Lorentz metric on \( H^3 \) is
\[
g = -dx^2 + dy^2 + (xdy + dz)^2.
\]
The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:
\[
\begin{align*}
\mathbf{e}_1 &= \frac{\partial}{\partial z}, \\
\mathbf{e}_2 &= \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \\
\mathbf{e}_3 &= \frac{\partial}{\partial x}.
\end{align*}
\]
The characterising properties of this algebra are the following commutation relations, [13]:
\[
g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \ g(\mathbf{e}_3, \mathbf{e}_1) = -1.
\]

**Proposition 2.1.** For the covariant derivatives of the Levi-Civita connection of the left-invariant metric \( g \), defined above the following is true:
\[
\nabla = \frac{1}{2} \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix},
\]
where the \((i, j)\) -element in the table above equals \( \nabla_{\mathbf{e}_i} \mathbf{e}_j \) for our basis \( \{\mathbf{e}_k, k = 1,2,3\} \).
3 Spacelike Biharmonic New Type $B$-Slant Helices with Bishop Frame In The Lorentzian Heisenberg Group $H^3$

Let $\gamma: I \rightarrow \mathbb{H}^3$ be a non geodesic spacelike curve on the Lorentzian Heisenberg group $\mathbb{H}^3$ parametrized by arc length. Let $\{t, n, b\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}^3$ along $\gamma$ defined as follows:

- $t$ is the unit vector field $\gamma'$ tangent to $\gamma$,
- $n$ is the unit vector field in the direction of $\nabla_t t$ (normal to $\gamma$), and
- $b$ is chosen so that $\{t, n, b\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\nabla_t t = \kappa n,$$
$$\nabla_t n = \tau t + \kappa b,$$
$$\nabla_t b = m,$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion and

$$g(t, t) = 1, g(n, n) = -1, g(b, b) = 1,$$
$$g(t, n) = g(t, b) = g(n, b) = 0.$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\nabla_t t = k_1 m_1 - k_2 m_2,$$
$$\nabla_t m_1 = k_1 t,$$
$$\nabla_t m_2 = k_2 t,$$

where

$$g(t, t) = 1, g(m_1, m_1) = -1, g(m_2, m_2) = 1,$$
$$g(T, M_1) = g(t, m_2) = g(m_1, m_2) = 0.$$

Here, we shall call the set $\{t, m_1, m_2\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures.

Also, $\tau(s) = \psi'(s)$ and $\kappa(s) = \sqrt{k_2^2 - k_1^2}$. Thus, Bishop curvatures are defined by

$$k_1 = \kappa(s) \sinh \psi(s),$$
$$k_2 = \kappa(s) \cosh \psi(s).$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$t = r^1 e_1 + r^2 e_2 + r^3 e_3,$$
\( \mathbf{m}_1 = m_1^1 \mathbf{e}_1 + m_1^2 \mathbf{e}_2 + m_1^3 \mathbf{e}_3, \quad (3) \)
\( \mathbf{m}_2 = m_2^1 \mathbf{e}_1 + m_2^2 \mathbf{e}_2 + m_2^3 \mathbf{e}_3. \)

**Theorem 3.1.** \( \gamma : I \to \mathbb{H}^3 \) is a spacelike biharmonic curve with Bishop frame if and only if
\[
k_1^2 - k_2^2 = \text{constant} = C \neq 0, \]
\[
k_1 - \left[ k_1^2 - k_2^2 \right] k_1 = -k_1 \left[ 1 + \left( m_1^1 \right)^2 \right] + k_2 m_1^1 m_2^1, \quad (4) \]
\[
k_2 - \left[ k_1^2 - k_2^2 \right] k_2 = -k_1 m_1^1 m_2^1 - k_2 \left[ 1 + \left( m_1^1 \right)^2 \right] \]

To separate a spacelike new type slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as spacelike new type B-slant helix.

**Theorem 3.2.** Let \( \gamma : I \to \mathbb{H}^3 \) be a unit speed biharmonic spacelike new type B-slant helix with non-zero curvatures. Then the equation of biharmonic spacelike new type B-slant helix are
\[
x(s) = \frac{1}{C_0} \cos Q \cosh [C_0 s + C_1] + C_2, \]
\[
y(s) = \frac{1}{C_0} \cos Q \sinh [C_0 s + C_1] + C_3, \quad (5) \]
\[
z(s) = \sin Q s - \frac{C_2}{C_0} \cos Q \sinh [C_0 s + C_1] \]
\[
- \frac{1}{4C_0} \cos^2 Q (2[C_0 s + C_1] + \sinh 2[C_0 s + C_1]) + C_4, \]

where \( C_0, C_1, C_2, C_3 \) are constants of integration and
\[
C_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos Q} - \sin Q. \]

**Proof.** The vector \( \mathbf{m}_2 \) is a unit spacelike vector, we reach
\[
\mathbf{m}_2 = \cos Q \mathbf{e}_1 + \sin Q \cosh A(s) \mathbf{e}_2 + \sin Q \sinh A(s) \mathbf{e}_3, \quad (8) \]

On the other hand, using Bishop formulas Eq.(4) and Eq.(1), we have
\[
\mathbf{m}_1 = \sinh A(s) \mathbf{e}_2 + \cosh A(s) \mathbf{e}_3, \quad (9) \]
It is apparent that

$$t = \sin Q e_1 + \cos Q \cosh A(s) e_2 + \cos Q \sinh A(s) e_3.$$  \hspace{1cm} (10)

A straightforward computation shows that

$$\nabla_t t = (t_1') e_1 + (t_2' + t_1 t_3) e_2 + (t_3' + t_1 t_2) e_3.$$  \hspace{1cm} (11)

Therefore, we use Bishop formulas Eq.(4) and above equation we get

$$A(s) = \frac{\sqrt{k_2^2 - k_1^2}}{\cos Q} - \sin Q s + C_1,$$  \hspace{1cm} (12)

where $C_1$ is a constant of integration.

From Eq.(10), we get

$$t = (\cos Q \sinh[C_0 s + C_1], \cos Q \cosh[C_0 s + C_1], \sin Q - x \cos Q \cosh[C_0 s + C_1]),$$  \hspace{1cm} (13)

where, $$C_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos Q} - \sin Q.$$

Therefore, by Eq(13) and taking into account Eq.(12), we obtain the system Eq.(12). This completes the proof.

**Corollary 3.3.** Let $\gamma : I \to H^3$ be a unit speed biharmonic spacelike new type B–slant helix with non-zero Bishop curvatures. Then the equation of $\gamma$ is

$$\gamma(s) = [\sin Q s - C_2 \cos Q \sinh[C_0 s + C_1]$$

$$- \frac{1}{4 C_0} \cos^2 Q (2[C_0 s + C_1] + \sinh 2[C_0 s + C_1]) + C_4]$$

$$+ \left[ \frac{1}{C_0} \cos Q \sinh[C_0 s + C_1] + C_2 \right] \left[ \frac{1}{C_0} \cos Q \sinh[C_0 s + C_1] + C_1 \right] e_1$$

$$+ \left[ \frac{1}{C_0} \cos Q \sinh[C_0 s + C_1] + C_3 \right] e_2 + \left[ \frac{1}{C_0} \cos Q \cosh[C_0 s + C_1] + C_2 \right] e_3,$$

where $C_0, C_1, C_2, C_3$ are constants of integration and

$$C_0 = \frac{\sqrt{k_2^2 - k_1^2}}{\cos Q} - \sin Q.$$

If we use Mathematica in above system, we get:
Fig. 1.

References


