On Degree of Approximation
by Product Means \((E,q)(N,p_n)\) of Fourier Series

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Abstract

In this paper a theorem on degree of Approximation of a function \(f \in \text{Lip } \alpha\) by product summability \((E,q)(N,p_n)\) of Fourier series associated with \(f\).

Keywords: Degree of Approximation, \(f \in \text{Lip } \alpha\) class of function, \((E,q)\) mean, \((N,p_n)\) mean, \((E,q)(N,p_n)\) product mean, Fourier series, Lebesgue integral
1 Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

\[ P_n = \sum_{v=0}^{n} p_v \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty \quad (P_{-i} = p_{-i} = 0, \quad i \geq 0) \]

The sequence –to-sequence transformation

\[ t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \]

defines the sequence $\{t_n\}$ of the $(N, p_n)$ -mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

\[ t_n \rightarrow s \quad \text{as} \quad n \rightarrow \infty \]

then the series $\sum a_n$ is said to be $(N, p_n)$ summable to $s$.

The conditions for regularity of Nörlund summability $(N, p_n)$ are easily seen to be

(i) $\frac{P_n}{n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$

(ii) $\sum_{k=0}^{n} p_k = O(P_n) \quad \text{as} \quad n \rightarrow \infty$

The sequence –to-sequence transformation, [1]

\[ T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v \]

defines the sequence $\{T_n\}$ of the $(E, q)$ mean of the sequence $\{s_n\}$. If

\[ T_n \rightarrow s \quad \text{as} \quad n \rightarrow \infty \]

then the series $\sum a_n$ is said to be $(E, q)$ summable to $s$.

Clearly $(E, q)$ method is regular.

Further, the $(E, q)$ transform of the $(N, p_n)$ transform of $\{s_n\}$ is defined by
\[
\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} T_k
\]

(1.6)

\[
= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{k-\nu} s_{\nu} \right\}
\]

If

(1.7)

\[
\tau_n \to s \quad \text{as} \quad n \to \infty
\]

then \[\sum a_n\] is said to be \((E, q)(N, p_n)\)-summable to \(s\).

Let \(f(t)\) be a periodic function with period \(2\pi\), L-integrable over \((-\pi, \pi)\), The Fourier series associated with \(f\) at any point \(x\) is defined by

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \sum_{n=0}^{\infty} A_n(x)
\]

(1.8)

Let \(s_n(f; x)\) be the n-th partial sum of (1.8).

The \(L_\infty\)-norm of a function \(f : R \to R\) is defined by

\[
\|f\|_\infty = \sup \{f(x) : x \in R\}
\]

(1.9)

and the \(L_\nu\)-norm is defined by

\[
\|f\|_\nu = \left( \int_0^{2\pi} |f(x)|^\nu dx \right)^{\frac{1}{\nu}}, \quad \nu \geq 1
\]

(1.10)

The degree of approximation of a function \(f : R \to R\) by a trigonometric polynomial \(P_n(x)\) of degree \(n\) under norm \(\|\cdot\|_\infty\) is defined by [4].

\[
\|P_n - f\|_\infty = \sup \{|P_n(x) - f(x)| : x \in R\}
\]

(1.11)

and the degree of approximation \(E_n(f)\) of a function \(f \in L_\nu\) is given by

\[
E_n(f) = \min_{P_n} \|P_n - f\|_\nu
\]

(1.12)
This method of approximation is called Trigonometric Fourier approximation. A function \( f \in \text{Lip} \alpha \) if

\[
|f(x + t) - f(x)| = O(|t|^{\alpha}), \quad 0 < \alpha \leq 1
\]

We use the following notation throughout this paper:

\[
\phi(t) = f(x + t) + f(x - t) - 2f(x),
\]

and

\[
K_n(t) = \frac{1}{2\pi(1 + q)} \sum_{k=0}^{n} \left( \frac{n}{k} \right) q^{n-k} \left\{ \frac{1}{P_k} \sum_{k=0}^{k} p_{k-v} \right\} \sin \left( \frac{v + \frac{1}{2}}{2} \right) t.
\]

Further, the method \((E,q)(N,p_n)\) is assumed to be regular and this case is supposed throughout the paper.

## 2 Known Theorem

Dealing with the degree of approximation by the product \((E,q)(c,1)\)-mean of Fourier series, Nigam [2] proved the following theorem:

**Theorem 2.1** If a function \( f \), \(2\pi\)-periodic, belonging to class \( \text{Lip} \alpha \), then its degree of approximation by \((E,q)(c,1)\) summability mean on its Fourier series

\[
\sum_{n=0}^{\infty} A_n(t) \text{ is given by } \left\|E_n^q c_n^1 - f\right\|_\infty = o \left( \frac{1}{(n+1)^{\alpha}} \right), \quad 0 < \alpha < 1,
\]

where \( E_n^q c_n^1 \) represents the \((E,q)\) transform of \((c,1)\) transform of \( s_n(f;x) \).

## 3 Main Theorem

In this paper, we have proved a theorem on degree of approximation by the product mean \((E,q)(N,p_n)\) of Fourier series (1.8). We prove

**Theorem 3.1** If \( f \) is a \(2\pi\)-Periodic function of class \( \text{Lip} \alpha \), then degree of approximation by the product \((E,q)(N,p_n)\) summability means on its Fourier series (1.8) is given by
\[ \| \tau_n - f \|_\infty = O\left( \frac{1}{(n+1)^\alpha} \right), \quad 0 < \alpha < 1 \quad \text{where} \quad \tau_n \quad \text{on defined in (1.6)}. \]

4 Required Lemmas

We require the following Lemmas to prove the theorem.

**Lemma 4.1**

\[ |K_n(t)| = O(n) \quad , 0 \leq t \leq \frac{1}{n+1} \]

**Proof:**

For \( 0 \leq t \leq \frac{1}{n+1} \), we have \( \sin nt \leq \sin nt \) then

\[
|K_n(t)| = \frac{1}{2\pi (1+q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{k-\nu} \frac{\sin \left( \nu + \frac{1}{2} \right)t}{\sin \frac{t}{2}} \right\} \right| 
\]

\[
\leq \frac{1}{2\pi (1+q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{k-\nu} \frac{(2\nu + 1)\sin \frac{t}{2}}{\sin \frac{t}{2}} \right\} \right| 
\]

\[
\leq \frac{1}{2\pi (1+q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{n-k} (2k + 1) \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{k-\nu} \right\} \right| 
\]

\[
\leq \frac{(2n+1)}{2\pi (1+q)^n} \left| \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) q^{n-k} \right| 
\]

= O(n).

This proves the lemma.

**Lemma 4.2**

\[ |K_n(t)| = O\left( \frac{1}{t} \right), \quad \text{for} \quad \frac{1}{n+1} \leq t \leq \pi. \]
Proof:

For \( \frac{1}{n+1} \leq t \leq \pi \), we have by Jordan’s lemma, \( \sin \left( \frac{t}{2} \right) \geq \frac{t}{\pi} \cdot \sin nt \leq 1 \)

Then \( |K_n(t)| = \frac{1}{2\pi(1+q)^n} \left\{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} \frac{\pi p_{k-\nu}}{t} \right\} \right\} \)

\[ \leq \frac{1}{2\pi(1+q)^n} \left\{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} \frac{\pi p_{k-\nu}}{t} \right\} \right\} \]

\[ = \frac{1}{2(1+q)^n} \left\{ \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} p_{k-\nu} \right\} \right\} \]

\[ = O\left( \frac{1}{t} \right) \cdot \]

This proves the lemma.

5 Proof of Theorem 3.1

Using Riemann –Lebesgue theorem, we have for the n-th partial sum \( s_n(f; x) \) of the Fourier series (1.8) of \( f(x) \),

\[ s_n(f; x) - f(x) = \frac{1}{2\pi} \int_{0}^{\pi} \phi(t) \frac{\sin \left( \frac{n+\frac{1}{2}}{2} \right) t}{\sin \left( \frac{t}{2} \right)} dt, \]

following Titechmarch [3], the \((N, p_n)\) transform of \( s_n(f; x) \) using (1.2) is given by

\[ t_n - f(x) = \frac{1}{2\pi P_n} \int_{0}^{\pi} \phi(t) \sum_{k=0}^{n} p_{n-k} \frac{\sin \left( \frac{n+\frac{1}{2}}{2} \right) t}{\sin \left( \frac{t}{2} \right)} dt, \]
Directing the \((E,q)(N, p_n)\) transform of \(s_n(f; x)\) by \(\tau_n\), we have

\[
\|\tau_n - f\| = \frac{1}{2\pi(1+q)^n} \int_0^\pi \phi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{k-\nu} \frac{\sin \left( \phi(t) + \frac{1}{2} \right)}{\sin \frac{t}{2}} \right\} dt
\]

\[
= \int_0^\pi \phi(t) K_n(t) dt
\]

\[
\begin{aligned}
&= \left\{ \int_0^{1/n+1} + \int_{1/n+1}^\pi \right\} \phi(t) K_n(t) dt \\
\end{aligned}
\]

\[
(5.1)
= I_1 + I_2, \text{ say}
\]

Now

\[
|I_1| = \frac{1}{2\pi(1+q)^n} \left| \int_0^{1/n+1} \phi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{k-\nu} \frac{\sin \left( \phi(t) + \frac{1}{2} \right)}{\sin \frac{t}{2}} \right\} dt \right|
\]

\[
\leq O(n) \int_0^{1/n+1} |\phi(t)| dt, \text{ using Lemma 4.1}
\]

\[
= O(n) \int_0^{1/n+1} |t^\alpha| dt
\]

\[
= O(n) \left[ t^{\alpha+1} \right]_0^{1/n+1}
\]

\[
= O(n) \left[ \frac{1}{\alpha+1}(n+1)^{\alpha+1} \right].
\]
(5.2) \[ = O\left[ \frac{1}{(n+1)^{\alpha+1}} \right] \]

Next
\[
|I_2| \leq \int_1^{\frac{\pi}{n+1}} |\phi(t)| |K_n(t)| \, dt
\]
\[
= \int_1^{\frac{\pi}{n+1}} \phi(t)\left| O\left( \frac{1}{t} \right) \right| \, dt \quad \text{using Lemma 4.2}
\]
\[
= \int_1^{\frac{\pi}{n+1}} t^\alpha \left| O\left( \frac{1}{t} \right) \right| \, dt
\]
\[
= \int_1^{\frac{\pi}{n+1}} t^{\alpha-1} \, dt
\]
\[= O\left( \frac{1}{(n+1)^{\alpha}} \right) \]

(5.3)

Then from (5.2) and (5.3), we have
\[
|\tau_n - f(x)| = O\left( \frac{1}{(n+1)^{\alpha}} \right), \text{ for } 0 < \alpha < 1
\]
\[
\|\tau_n - f(x)\|_\infty = \sup_{-\pi < x < \pi} |\tau_n - f(x)| = O\left( \frac{1}{(n+1)^{\alpha}} \right), 0 < \alpha < 1.
\]

This completes the proof of the theorem.

6 Corollaries

The following corollaries can be derived from our main theorem.

**Corollary 6.1** If \( p_n = 1 \), \( \forall n \in N \), *theorem 2.1 follows from theorem 3.1.*
Corollary 6.2 If $p_n = 1, \forall n$ and $q = 1$ then the theorem 3.1 reduces to degree of approximation for $(E,1)(C,1)$ method of Fourier series.

References


