Lacunary Weak $I$-Statistical Convergence

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Abstract

In this study, we provide a new approach to $I$-statistical convergence. We introduce a new concept with $I$-statistical convergence and weak convergence together and we call it weak $I$-statistical convergence or $WS(I)-$convergence. Then we introduce this concept for lacunary sequences and we obtain lacunary weak $I$-statistical convergence i.e. $WS_{\theta}(I)-$convergence. $WN_{\theta}(I)-$convergence is any other definition in our study. After giving this description, we investigate their relationship and we have some results.

Keywords: $I$-statistical convergence, weak statistical convergence, lacunary sequence.

1 Introduction

In this area, statistical convergence is an important concept and Zygmund [15] gave it in the first edition of his monograph published in Warsaw in 1935. It was formally introduced by Fast and Steinhaus [5, 14] and later was reintroduced by Schoenberg, [13] This concept has a wide application area for example number theory [4], measure theory [10], trigonometric series [15], summability theory [6],
Fridy and Orhan studied statistical convergence with lacunary sequences. [8]

Let \( K \) be a subset of the set of all natural numbers \( \mathbb{N} \) and \( K_n = \{k \leq n : k \in K\} \) where the vertical bars indicate the number of elements in the enclosed set. The natural density of \( K \) is defined by \( \delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n| \). If a property \( P(k) \) holds for all \( k \in A \) with \( \delta(A) = 1 \) we say that \( P \) holds for almost all \( k \) that is a.a.k.

**Definition 1.1:** [14] A number sequence \( x = (x_k) \) is statistically convergent to \( x \) provided that for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0.
\]

In this case we write \( st - \lim x_k = x \).

Statistical convergence was extended to \( I - \) convergence in a metric space in Kostyrko, Salát and Wileżyński's study. [9]

**Definition 1.2:** A family of sets \( I \subseteq 2^\mathbb{N} \) is called an ideal if and only if

(i) \( \emptyset \in I \)

(ii) For each \( A, B \in I \) we have \( A \cup B \in I \)

(iii) For each \( A \in I \) and each \( B \subseteq A \) we have \( B \in I \)

An ideal is called non-trivial if \( \mathbb{N} \notin I \) and a non-trivial ideal is called admissible if \( \{n\} \in I \) for each \( n \in \mathbb{N} \).

**Definition 1.3:** A family of sets \( F \subseteq 2^\mathbb{N} \) is called a filter in \( \mathbb{N} \) if and only if

(i) \( \emptyset \notin F \)

(ii) For each \( A, B \in F \) we have \( A \cap B \in F \)

(iii) For each \( A \in F \) and each \( B \supseteq A \) we have \( B \in F \)

**Proposition 1.1** \( I \) is a non-trivial ideal in \( \mathbb{N} \) if and only if

\[ F = F(I) = \{M = N \setminus A : A \in I\} \]

is a filter in \( \mathbb{N} \).

Throughout the paper, \( I \) will be an admissible ideal.
**Definition 1.4:** A real sequence \( x = (x_k) \) is said to be \( I \)– convergent to \( L \in \mathbb{R} \) if and only if for each \( \varepsilon > 0 \) the set
\[
A_{\varepsilon} = \{ k \in \mathbb{N} : |x_k - L| \geq \varepsilon \}
\]
belongs to \( I \). The number \( L \) is called the \( I \)– limit of the sequence \( x \).

**Example 1.1:** Take for \( I \) class the \( I_f \) of all finite subsets of \( \mathbb{N} \). Then \( I_f \) is an admissible ideal and \( I_f \) – convergence coincides with the usual convergence.

In 2011, Das, Savas and Ghosal [3] have introduced the concept of \( I \) – statistical convergence and \( I \) – lacunary statistical convergence.

**Definition 1.5:** [3] A sequence \( x = (x_k) \) is said to be \( I \) – statistically convergent to \( L \) for each \( \varepsilon > 0 \) and \( \delta > 0 \),
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \geq \delta \right\} \in I.
\]

**Example 1.2:** Let us take the sequence \( (y_n) \) where \( y_n = \begin{cases} 1, & n = 1 \text{ to } 10 \\ n - 10, & n \geq 10 \end{cases} \) and the ideal \( I_d \) which is the ideal of density zero sets of \( \mathbb{N} \). Let \( A = \{1^2, 2^2, 3^2, \ldots\} \).
Define \( x = (x_k) \) in a normed linear space \( X \) by,
\[
x_k = \begin{cases} ku, & \text{for } n - \left\lfloor \sqrt{y_n} \right\rfloor + 1 \leq k \leq n, n \notin A \\ ku, & \text{for } n - y_n + 1 \leq k \leq n, n \in A \\ \theta, & \text{otherwise} \end{cases}
\]
where \( u \in X \) is a fixed element with \( \|u\| = 1 \) and \( \theta \) is the null element of \( X \). Then the sequence \( x = (x_k) \) is \( I \) – statistically convergent but it is not statistically convergent.

Now, we will give the definition of \( I \) – lacunary statistically convergent sequences from the paper of Das, Savas and Ghosal. But first, we need to remind lacunary sequence.

**Definition 1.6:** A lacunary sequence is an increasing integer sequence \( \theta = (k_r) \) such that \( k_0 = 0 \) and \( h_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \). The intervals determined by \( \theta \) will be denoted by \( J_r = (k_{r-1}, k_r] \) and the ratio \( \frac{k_r}{k_{r-1}} \) will be denoted by \( q_r \).
Definition 1.7: [3] Let \( \theta \) be a lacunary sequence. A sequence \( x = (x_k) \) is said to be \( I \) – lacunary statistically convergent to \( L \) for each \( \varepsilon > 0 \) and \( \delta > 0 \),

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left\{ \left| k \in J_r : \left| x_k - L \right| \geq \varepsilon \right\} \right\} \geq \delta \in I.
\]

Let’s continue to remind important concepts that we need for our study.

Definition 1.8: Let \( B \) be a Banach space, \( x = (x_k) \) be a \( B \)-valued sequence and \( x \in B \). The sequence \( x = (x_k) \) is weakly convergent to \( x \) provided that for any \( f \) in the continuous dual \( B^* \) of \( B \),

\[
\lim_{k \to \infty} f(x_k - x) = 0
\]

and in this case we write \( w- \lim x_k = x \).

Definition 1.9: Let \( B \) be a Banach space, \( x = (x_k) \) be a \( B \)-valued sequence and \( x \in B \). The sequence \( x = (x_k) \) is weakly \( C_1 \)-convergent to \( x \) provided that for any \( f \) in the continuous dual \( B^* \) of \( B \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k - x) = 0
\]

In 2000, Connor et al. [2], have introduced a new concept of weak statistical convergence and have characterized Banach spaces with separable duals via statistical convergence. Pehlivan and Karaev [12] have also used the idea of weak statistical convergence in strengthening a result of Gokhberg and Klein on compact operators. Bhardwaj and Bala have investigated some relations between weak convergent sequences and weakly statistically convergent sequences [1].

Following Connor et al. we define weak statistical convergence as follows:

Definition 1.10: [2] Let \( B \) be a Banach space, \( x = (x_k) \) be a \( B \)-valued sequence and \( x \in B \). The sequence \( x = (x_k) \) is weakly statistically convergent to \( x \) provided that for any \( f \) in the continuous dual \( B^* \) of \( B \) the sequence \( (f(x_k - x)) \) is statistically convergent to \( x \) i.e.

\[
\lim_{n \to \infty} \frac{1}{n} \left\{ k \leq n : \left| f(x_k - x) \right| \geq \varepsilon \right\} = 0
\]

and in this case we write \( W-st- \lim x_k = x \).

It is easy to see that the weak statistical limit of a weakly statistically convergent sequence is unique.

**Definition 1.11:** Let $B$ be a Banach space, $x = (x_k)$ be a $B$-valued sequence, $\theta$ be a lacunary sequence and $x \in B$. $x = (x_k)$ is weakly lacunary statistically convergent to $x$ or $WS_\theta$ convergent to $x$ provided that for any $f$ in the continuous dual $B'$ of $B$,

$$\lim_{r \to \infty} \frac{1}{h_r} \left\{ k \in J_r : |f(x_k - x)| \geq \varepsilon \right\} = 0.$$  

## 2 Lacunary Weak $I$- Statistical Convergence

**Definition 2.1:** Let $B$ be a Banach space, $x = (x_k)$ be a $B$-valued sequence and $x \in B$. The sequence $x = (x_k)$ is weakly $I$- convergent to $x$ provided that for any $f$ in the continuous dual $B'$ of $B$,

$$\{ k \in \mathbb{N} : |f(x_k - x)| \geq \varepsilon \} \in I.$$  

The set of all weakly $I$- convergent sequences is denoted by $WI$ and if we take $I = I_f$ the ideal of all finite subsets of $\mathbb{N}$, we have the usual weak convergence.

**Example 2.1:** $I_d$ is an admissible ideal and $WI_d$ – convergence coincides with the weak statistical convergence.

**Example 2.2:** Denote by $I_\phi$ the class of all $K \subset \mathbb{N}$ with

$$\lim_{r \to \infty} \frac{1}{h_r} \left\{ k \in J_r : k \in K \right\} = 0.$$  

Then $I_\phi$ is an admissible ideal and $WI_\phi$ – convergence coincides with the lacunary weak statistical convergence.

We now introduce our main definitions.

**Definition 2.2:** Let $B$ be a Banach space, $x = (x_k)$ be a $B$-valued sequence and $x \in B$. The sequence $x = (x_k)$ is weakly $I$ – statistically convergent to $x$ provided that for any $f$ in the continuous dual $B'$ of $B$ and every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left\{ k \leq n : |f(x_k - x)| \geq \varepsilon \right\} \geq \delta \right\} \in I.$$
The set of all weakly $I$–statistically convergent sequences is denoted by $WS(I)$.

**Definition 2.3:** Let $B$ be a Banach space, $x = (x_k)$ be a $B$-valued sequence, $x \in B$ and $\theta = (k_r)$ be a lacunary sequence. The sequence $x = (x_k)$ is lacunary weak $I$–statistically convergent to $x$ provided that for any $f$ in the continuous dual $B^*$ of $B$ and every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |f(x_k - x)| \geq \varepsilon \right\} \subseteq I,$$

The set of all lacunary weak $I$–statistically convergent sequences is denoted by $WS_\theta(I)$.

**Definition 2.4:** Let $B$ be a Banach space, $x = (x_k)$ be a $B$-valued sequence, $x \in B$ and $\theta = (k_r)$ be a lacunary sequence. The sequence $x = (x_k)$ is $WN_\theta(I)$–convergent to $x$ provided that for any $f$ in the continuous dual $B^*$ of $B$ and every $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |f(x_k - x)| \geq \varepsilon \right\} \subseteq I.$$  

**Theorem 2.1:** Let $\theta = (k_r)$ be a lacunary sequence. Then $(x_k)$ is $WN_\theta(I)$–convergent to $x$ if and only if $(x_k)$ is $WS_\theta(I)$–convergent to $x$.

**Proof:** Assume that $(x_k)$ is $WN_\theta(I)$–convergent to $x$ and $\varepsilon > 0$. We can write,

$$\frac{1}{h_r} \sum_{k \in J_r} |f(x_k - x)| \geq \frac{1}{h_r} \sum_{k \in J_r, \text{and} |f(x_k - x)| \geq \varepsilon} |f(x_k - x)|,$$

$$\geq \frac{\varepsilon}{h_r} \sum_{k \in J_r} |f(x_k - x)| \geq \varepsilon \left\{ k \in J_r : |f(x_k - x)| \geq \varepsilon \right\}$$

Then,

$$\frac{1}{h_r} \sum_{k \in J_r} |f(x_k - x)| \geq \frac{1}{h_r} \left\{ k \in J_r : |f(x_k - x)| \geq \varepsilon \right\}$$

and for any $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |f(x_k - x)| \geq \varepsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |f(x_k - x)| \geq \varepsilon \delta \right\}.$$  

We know that the right side is in ideal. So, the left side is also in ideal.
Now suppose that \((x_k)\) is \(WS_{\theta}(I)\)–convergent to \(x\). Since \(f \in B^*\), \(f\) is bounded. Then there exists a \(K \geq 0\) for all \(k \in \mathbb{N}\) such that \(|f(x_k - x)| \leq K\). Given \(\varepsilon > 0\), we get,

\[
\frac{1}{h_r} \sum_{k \in J, r \leq k \leq r + 1} |f(x_k - x)| = \frac{1}{h_r} \sum_{k \in J, r \leq k \leq r + 1} |f(x_k - x)| + \frac{1}{h_r} \sum_{k \in J, r \leq k \leq r + 1} |f(x_k - x)| \\
\leq K \frac{1}{h_r} \left( \sum_{k \in J, r \leq k \leq r + 1} |f(x_k - x)| \right) + \frac{\varepsilon}{2}.
\]

Consequently we have,

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J, r \leq k \leq r + 1} |f(x_k - x)| \geq \varepsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left( \sum_{k \in J, r \leq k \leq r + 1} |f(x_k - x)| \right) \geq \frac{\varepsilon}{2K} \right\} \in I.
\]

**Theorem 2.2:** Let \(\theta = (k_r)\) be a lacunary sequence with \(\liminf q_r > 1\). Then \(WS(I)\)–convergence implies \(WS_{\theta}(I)\)–convergence.

**Proof:** Assume that \(\liminf q_r > 1\). Then there exists an \(\alpha > 0\) such that \(q_r \geq 1 + \alpha\) for all sufficiently large \(r\). This implies \(\frac{h_r}{k_r} \geq \frac{\alpha}{1 + \alpha}\). Since \((x_k)\) is \(WS(I)\)–convergent to \(x\), for every \(\varepsilon > 0\) and sufficiently large \(r\) we have,

\[
\frac{1}{k_r} \left[ \{ k \leq k_r : |f(x_k - x)| \geq \varepsilon \} \right] \geq \frac{1}{k_r} \left[ \{ k \in J, r \leq k \leq r + 1 : |f(x_k - x)| \geq \varepsilon \} \right] \geq \frac{\alpha}{1 + \alpha} \frac{1}{h_r} \left[ \{ k \in J, r \leq k \leq r + 1 : |f(x_k - x)| \geq \varepsilon \} \right].
\]

Then for any \(\delta > 0\) we get

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left[ \{ k \in J, r \leq k : |f(x_k - x)| \geq \varepsilon \} \right] \geq \delta \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{k_r} \left[ \{ k \leq k_r : |f(x_k - x)| \geq \varepsilon \} \right] \geq \frac{\delta \alpha}{1 + \alpha} \right\} \in I.
\]

This proves the theorem.

**Theorem 2.3:** Let \(\theta = (k_r)\) be a lacunary sequence with \(\limsup q_r < \infty\). Then \(WS_{\theta}(I)\)–convergence implies \(WS(I)\)–convergence.

**Proof:** If \(\limsup q_r < \infty\) then there is a \(K > 0\) such that \(q_r < K\) for all \(r\). Suppose that \((x_k)\) is \(WS_{\theta}(I)\)–convergent to \(x\) and \(\varepsilon, \delta, \eta > 0\). Define the sets,
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\[ M = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left[ k \in J_r : |f(x_k - x)| \geq \varepsilon \right] < \delta \right\} \]

\[ R = \left\{ n \in \mathbb{N} : \frac{1}{n} \left[ k \leq n : |f(x_k - x)| \geq \varepsilon \right] < \eta \right\}. \]

Let \( F(I) \) be the filter associated with the ideal \( I \). It is obvious that \( M \in F(I) \). If we can show that \( R \in F(I) \) then we will have the proof. For all \( j \in M \) let,

\[ A_j = \frac{1}{j} \left[ k \in J_j : |f(x_k - x)| \geq \varepsilon \right] < \delta. \]

Choose \( n \in \mathbb{N} \) such that \( k_{r-1} < n < k_r \) for some \( r \in M \). Now,

\[
\frac{1}{n} \left[ k \leq n : |f(x_k - x)| \geq \varepsilon \right] \leq \frac{1}{k_{r-1}} \left[ k \leq k_r : |f(x_k - x)| \geq \varepsilon \right] \\
= \frac{k_{r-1}}{k_r} \left[ k \in J_1 : |f(x_k - x)| \geq \varepsilon \right] + \frac{k_r - k_{r-1}}{k_r} \left[ k \in J_2 : |f(x_k - x)| \geq \varepsilon \right] + \ldots + \frac{k_r - k_{r-1}}{k_r} A_r \\
\leq \sup_{j=M} \frac{k_j}{k_{r-1}} A_j < K \delta
\]

Choosing \( \eta = \frac{\delta}{K} \) and in view of the fact that \( \bigcup \left\{ n : k_{r-1} < n < k_r, r \in M \right\} \subset R \) then we have \( R \in F(I) \).

References


