On a Subclass of Meromorphic Function with Fixed Second Coefficient Involving Fox-Wright's Generalized Hypergeometric Function

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(Received: 13-4-15/ Accepted: 29-5-15)

Abstract

In this paper, we have introduced and studied new subclass of meromorphic function with fixed second coefficient involving Fox-Wright's generalized hypergeometric function. We have obtained coefficient estimates, extreme points, growth and distortion theorems, radii of meromorphically starlikeness and convexity for this new subclass and other interesting properties.

Keywords: Meromorphic functions, Hadamard product, Fixed second coefficient, coefficient inequalities, radii of meromorphically starlikeness and convexity.
1 Introduction

Let $\Sigma$ denote the class of normalized meromorphic functions of the form

\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \]  \hspace{1cm} (1)

defined on the punctured unit disk $\Delta^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$. A function $f \in \Sigma$ is meromorphic starlike of order $\alpha$, $0 \leq \alpha < 1$ if $-\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha (z \in \Delta^* = \Delta \setminus \{0\})$. The class of all such functions is denoted by $\Sigma^*(\alpha)$. A function $f \in \Sigma$ is meromorphic convex of order $\alpha$, $0 \leq \alpha < 1$ if $-\Re \left( 1 + \frac{zf'(z)}{f(z)} \right) > \alpha (z \in \Delta^* = \Delta \setminus \{0\})$. Let $\Sigma_p$ be the class of functions $f \in \Sigma$ with $a_n \geq 0$. The subclass of $\Sigma_p$ consisting of starlike functions of order $\alpha$ is denoted by $\Sigma_p^*(\alpha)$ and convex functions of order $\alpha$ by $\Sigma_p^c(\alpha)$. Various subclasses of $\Sigma$ have been defined and studied by various authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

For functions $f(z)$ given by (1) and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ we define the Hadamard product or convolution of $f$ and $g$ by

\[ (f \ast g) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n. \]

For positive real parameters $\alpha_1, A_1, \ldots, \alpha_l, A_l, \beta_1, B_1, \ldots, \beta_m, B_m (l, m \in \mathbb{N} = \{1, 2, 3, \ldots \})$ such that

\[ 1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k \geq 0, z \in \{ z \in \mathbb{C} : 0 < |z| < 1 \} \]

the Wright's generalized hypergeometric function

\[ \psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z] = _l \psi_m[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}; z] \]

is defined by

\[ _l \psi_m[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}; z] = \sum_{k=0}^{\infty} \prod_{t=0}^{l-1} \Gamma(\alpha_t + kA_t) \prod_{t=0}^{m-1} \Gamma(\beta_t + kB_t)^{-1} \frac{z^k}{k!}. \]

If $A_t = 1$ ($t = 1, 2, \ldots, l$) and $B_t = 1$ ($t = 1, 2, \ldots, m$) we have the relationship

\[ \Omega_4 \psi_m[(\alpha_t, A_t)_{1,l}, (\beta_t, B_t)_{1,m}; z] = _l F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \]

\[ = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k z^k}{(\beta_1)_k \cdots (\beta_m)_k k!}. \]
This is the generalized hypergeometric function (see [6]). Here $(\alpha_n)$ is the Pochhammer symbol and
\[
\Omega = \left( \prod_{t=0}^{l} \Gamma(\alpha_t) \right)^{-1} \left( \prod_{t=0}^{m} \Gamma(\beta_t) \right).
\]

Using the generalized hypergeometric function, we define a linear operator
\[
\mathcal{V}[\{\alpha_t, A_t\}_{1,l}, \{\beta_t, B_t\}_{1,m}] : \Sigma_p \to \Sigma_p.
\]
By
\[
\mathcal{V}[\{\alpha_t, A_t\}_{1,l}, \{\beta_t, B_t\}_{1,m}] f(z) = z^{-1} \Omega \sum_{\nu=0}^{\infty} \varpi_{\nu} \sum_{\mu=0}^{\infty} \nu_{\mu} (\alpha_t) \alpha_{\nu} z^\nu \tag{2}
\]
For convenience, we denote $\mathcal{V}[\{\alpha_t, A_t\}_{1,l}, \{\beta_t, B_t\}_{1,m}]$ by $\mathcal{V}[\alpha_1]$. If $f$ has the form (1) then,
\[
\mathcal{V}[\alpha_1] f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \sigma_n(\alpha_1) a_n z^n \tag{3}
\]
Where
\[
\sigma_n(\alpha_1) = \frac{\Omega(\alpha_1 + A_1(n+1)) \ldots \Gamma(\alpha_1 + A_1(n+1))}{(k+1)! \Gamma(\beta_1 + B_1(n+1)) \ldots \Gamma(\beta_1 + B_1(n+1))} \tag{4}
\]

Now, we define a new subclass of $\Sigma_p$ by using the linear operator $\mathcal{V}[\alpha_1]$ as follows.

For $0 \leq \eta < 1$ and $0 \leq \lambda < 1$ we let $\mathcal{N}(\lambda, \eta)$ denote a subclass of $\Sigma_p$ consisting functions of the form (1) satisfying the condition that
\[
\Re \left( \frac{z(\mathcal{V}[\alpha_1] f(z))'}{(\lambda - 1)(\mathcal{V}[\alpha_1] f(z))' + \lambda z(\mathcal{V}[\alpha_1] f(z))'} \right) > \eta \tag{5}
\]
Where $A_t = 1 \ (t = 1, 2, \ldots, l) \ and \ B_t = 1 \ (t = 1, 2, \ldots, m)$. 

Now we prove the coefficient inequality for $f \in \mathcal{N}(\lambda, \eta)$.

2 Coefficients Inequalities

Our first theorem gives a necessary and sufficient condition for a function $f$ to be in the class $\mathcal{N}(\lambda, \eta)$.

**Theorem 1:** Let $f \in \Sigma_p$ be given by (1). Then $f \in \mathcal{N}(\lambda, \eta)$ if and only if
\[ \sum_{n=1}^{\infty} \{ n + \eta - \eta \lambda (1 + n) \} \sigma_{n} (\alpha_{1}) a_{n} \leq (1 - \eta) \]  \hspace{1cm} (6)

**Proof:** At first suppose that \( f \in \Sigma_{p} \) given by (1) is in the class \( \mathcal{N}(\lambda, \eta) \). Then by (5) we have

\[
\Re \left( \frac{z (\mathcal{V}[\alpha_{1}] f(z))'}{(\lambda - 1)(\mathcal{V}[\alpha_{1}] f(z)) + \lambda z (\mathcal{V}[\alpha_{1}] f(z))'} > \eta \right)
\]

\[
\Re \left( \frac{-1 + \sum_{n=1}^{\infty} n \sigma_{n}(\alpha_{1}) a_{n} z^{n+1}}{-1 + \sum_{n=1}^{\infty} (\lambda - 1 + \lambda n) \sigma_{n}(\alpha_{1}) a_{n} z^{n+1}} > \eta \right).
\]

If \( z \to 1^{-} \), we have

\[
\Re \left( \frac{-1 + \sum_{n=1}^{\infty} n \sigma_{n}(\alpha_{1}) a_{n}}{-1 + \sum_{n=1}^{\infty} (\lambda - 1 + \lambda n) \sigma_{n}(\alpha_{1}) a_{n}} > \eta. \right.
\]

This means that (6) holds, conversely suppose that the inequality (6) holds. Let

\[ \omega = \frac{z (\mathcal{V}[\alpha_{1}] f(z))'}{(\lambda - 1)(\mathcal{V}[\alpha_{1}] f(z)) + \lambda z (\mathcal{V}[\alpha_{1}] f(z))'} \]

We have to prove that \( \Re \omega > \eta \). It is enough to prove that

\[ |\omega - 1| < |\omega + 1 - 2\eta| \]

\[
= \left| \frac{z (\mathcal{V}[\alpha_{1}] f(z))' - (\lambda - 1)(\mathcal{V}[\alpha_{1}] f(z)) + \lambda z (\mathcal{V}[\alpha_{1}] f(z))'}{z (\mathcal{V}[\alpha_{1}] f(z))' + (1 - 2\eta)(\lambda - 1)(\mathcal{V}[\alpha_{1}] f(z)) + \lambda z (\mathcal{V}[\alpha_{1}] f(z))'} \right|
\]

\[ \leq \left| \frac{\sum_{n=1}^{\infty} (1 - \lambda)(n + 1) \sigma_{n}(\alpha_{1}) a_{n} z^{n+1}}{2(1 - \eta) + \sum_{n=1}^{\infty} [n(1 + (1 - 2\eta) \lambda) + (1 - 2\eta)(\lambda - 1)] \sigma_{n}(\alpha_{1}) a_{n} z^{n+1}} \right| \leq 1 \]

Thus we have \( f \in \mathcal{N}(\lambda, \eta) \). \( \square \)

From (6) we have

\[ \sigma_{1} a_{1} \leq \frac{(1 - \eta)}{1 + \eta - 2\eta \lambda} \]  \hspace{1cm} (7)
\[\sigma_1 a_1 \leq \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda}, \quad 0 < c < 1. \quad (8)\]

**Definition 1:** The subclass \(\mathcal{N}(\lambda, \eta, c)\) of \(\mathcal{N}(\lambda, \eta)\) consists of all functions of the form
\[f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta \lambda} + \sum_{n=2}^{\infty} \sigma_n(\alpha_1)a_n z^n, \quad 0 < c < 1\] (9)

We now obtain the coefficient estimates, growth and distortion bounds, extreme points, radii of mero-morphically starlikeness and convexity for the class \(\mathcal{N}(\lambda, \eta)\) by fixing the second coefficient.

We now prove the coefficient inequality.

**Theorem 2:** Let \(f\) be defined by (9). Then \(f \in \mathcal{N}(\lambda, \eta, c)\) if and only if
\[\sum_{n=2}^{\infty}\{n + \eta - \eta \lambda(1 + n)\} \sigma_n(\alpha_1)a_n \leq (1 - \eta)(1 - c).\] (10)

The result is sharp.

**Proof:** \(f \in \mathcal{N}(\lambda, \eta, c)\) implies \(f \in \mathcal{N}(\lambda, \eta)\). Therefore by (6)
\[\sigma_1 a_1 \leq \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} \leq 1 - \eta + \frac{(1 - \eta)(1 - c)}{\sum_{n=2}^{\infty}\{n + \eta - \eta \lambda(1 + n)\} \sigma_n(\alpha_1)a_n} \leq 1 - \eta.\]

Using (8)
\[(1 - \eta)c + \sum_{n=2}^{\infty}\{n + \eta - \eta \lambda(1 + n)\} \sigma_n(\alpha_1)a_n \leq (1 - \eta).\]

From which we get (10). The result is sharp for the function
\[f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta \lambda} z + \frac{(1-\eta)(1-c)}{[n+\eta-\eta \lambda(1+n)] \sigma_n(\alpha_1)} z^n, \quad n \geq 2. \quad \blacksquare\] (11)

**Corollary 3:** If \(f\) defined by (9) is in the class \(\mathcal{N}(\lambda, \eta, c)\), then
\[a_n \leq \frac{(1-\eta)(1-c)}{[n+\eta-\eta \lambda(1+n)] \sigma_n(\alpha_1)}, \quad n \geq 2\] (12)

The result is sharp for the function given by (11).

### 3 Growth and Distortion Theorems

A growth and distortion property for the function \(f \in \mathcal{N}(\lambda, \eta, c)\) is given as follows:
**Theorem 4:** If \( f \) given by (9) is in the class \( N(\lambda, \eta, c) \), then for \( 0 < |z| = r < 1 \)

\[
|f(z)| \geq \frac{1}{r} - \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} r - \frac{(1 - \eta)(1 - c)}{2 + \eta - 3\eta \lambda} r^2
\]  

(13)

and

\[
|f(z)| \leq \frac{1}{r} + \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} r + \frac{(1 - \eta)(1 - c)}{2 + \eta - 3\eta \lambda} r^2.
\]  

(14)

The result is sharp for \( f(z) = \frac{1}{z} + \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} z + \frac{(1 - \eta)(1 - c)}{2 + \eta - 3\eta \lambda} z^2. \)

**Proof:** Since \( f \in N(\lambda, \eta, c) \) by Theorem 2

\[
\sigma_n(\alpha_1) a_n = \frac{(1 - \eta)(1 - c)}{n + \eta - \eta \lambda (1 + n)}
\]  

(15)

For \( 0 < |z| = r < 1 \),

\[
|f(z)| \leq \frac{1}{|z|} + \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} |z| + \sum_{n=2}^{\infty} \sigma_n(\alpha_1) a_n |z|^n
\]

\[
\leq \frac{1}{r} + \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} r + r^2 \sum_{n=2}^{\infty} \sigma_n(\alpha_1) a_n
\]

\[
\leq \frac{1}{r} + \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} r + \frac{(1 - \eta)(1 - c)}{2 + \eta - 3\eta \lambda} r^2.
\]

Similarly,

\[
|f(z)| \geq \frac{1}{|z|} - \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} |z| - \sum_{n=2}^{\infty} \sigma_n(\alpha_1) a_n |z|^n
\]

\[
\geq \frac{1}{r} - \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} r - r^2 \sum_{n=2}^{\infty} \sigma_n(\alpha_1) a_n
\]

\[
\geq \frac{1}{r} - \frac{(1 - \eta)c}{1 + \eta - 2\eta \lambda} r - \frac{(1 - \eta)(1 - c)}{2 + \eta - 3\eta \lambda} r^2.
\]

A distortion theorem for the function \( f \) to be in the class \( N(\lambda, \eta, c) \) is given as follow:
Theorem 5: If \( f \) given by (9) is in the class \( \mathcal{N}(\lambda, \eta, c) \) then for \( 0 < |z| = r < 1 \)
\[
|f'(z)| \geq \frac{1}{r^2} - \frac{(1-\eta)c}{1+\eta-2\eta\lambda} - \frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda} r
\]  
(16)

and

\[
|f'(z)| \leq \frac{1}{r^2} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} + \frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda} r.
\]  
(17)

The result is sharp for \( f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda} z^2 \).

4 Extreme Points

In this section, we determine the extreme points for functions in the class \( \mathcal{N}(\lambda, \eta, c) \).

Theorem 6: Let \( f_1(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z \), and
\[
f_n(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \sum_{n=2}^{\infty} \frac{(1-\eta)(1-c)}{(n+\eta-\eta\lambda(1+n))\sigma_n(\alpha_1)} z^n \text{ for } n \geq 2.
\]

Then \( f \in \mathcal{N}(\lambda, \eta, c) \) if and only if it can be expressed as
\[
f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \mu_n \geq 0, \sum_{n=1}^{\infty} \mu_n = 1.
\]

Proof: Suppose \( f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \mu_n \geq 0, \sum_{n=1}^{\infty} \mu_n = 1 \). Then
\[
f_n(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \sum_{n=2}^{\infty} \frac{(1-\eta)(1-c)}{(n+\eta-\eta\lambda(1+n))\sigma_n(\alpha_1)} \mu_n z^n.
\]

Now
\[
\sum_{n=2}^{\infty} \frac{(1-\eta)(1-c)\mu_n}{(n+\eta-\eta\lambda(1+n))\sigma_n(\alpha_1)} \frac{(n+\eta-\eta\lambda(1+n))\sigma_n(\alpha_1)}{(1-\eta)(1-c)} = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1.
\]

This implies \( f \in \mathcal{N}(\lambda, \eta, c) \). Conversely, let \( f \in \mathcal{N}(\lambda, \eta, c) \). Then
\[
a_n \leq \frac{(1-\eta)(1-c)}{(n+\eta-\eta\lambda(1+n))\sigma_n(\alpha_1)a_n}, \quad n \geq 2.
\]
Set \( \mu_n = \frac{(n+\eta-\eta\lambda(1+n))\sigma_n(\alpha_1)}{(1-\eta)(1-c)} a_n \), \( n \geq 2 \) and \( \mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n \). Then

\[
f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z). \]

**Theorem 7:** The class \( \mathcal{N}(\lambda, \eta, c) \) is closed under convex combination.

**Proof:** Let \( f, g \in \mathcal{N}(\lambda, \eta, c) \) such that

\[
f(z) = \frac{1}{z} + \frac{(1 - \eta)c}{1 + \eta - 2\eta\lambda} + \sum_{n=2}^{\infty} a_n z^n
\]

and

\[
g(z) = \frac{1}{z} + \frac{(1 - \eta)c}{1 + \eta - 2\eta\lambda} + \sum_{n=2}^{\infty} b_n z^n.
\]

For \( 0 \leq \mu \leq 1 \), let

\[
h(z) = \mu f(z) + (1 - \mu) g(z).
\]

Then

\[
h(z) = \frac{1}{z} + \frac{(1 - \eta)c}{1 + \eta - 2\eta\lambda} + \sum_{n=2}^{\infty} [a_n \mu + (1 - \mu) b_n] z^n.
\]

Therefore

\[
\sum_{n=2}^{\infty} \{n + \eta - \eta\lambda(1 + n)}\sigma_n(\alpha_1)[a_n \mu + (1 - \mu) b_n] \leq (1 - \eta)(1 - c).
\]

This implies \( h(z) = \mu f(z) + (1 - \mu) g(z) \in \mathcal{N}(\lambda, \eta, c) \). Hence \( \mathcal{N}(\lambda, \eta, c) \) is closed under convex combination. \( \blacksquare \)

### 5 Radii of Meromorphically Starlikeness and Convexity

The radii of starlikeness and convexity for the class \( \mathcal{N}(\lambda, \eta, c) \) is given by the following theorem:

**Theorem 8:** Let \( f \in \mathcal{N}(\lambda, \eta, c) \). Then \( f \) is meromorphically starlike of order \( \delta (0 \leq \delta < 1) \) in the disk \( |z| < r_1(\lambda, \eta, c, \delta) \), where \( r_1(\lambda, \eta, c, \delta) \) is the largest value for which

\[
\left( \frac{(3 - \delta)(1 - \eta)c}{1 + \eta - 2\eta\lambda} \right)^2 + \left( \frac{(n + 2 - \delta)(1 - \eta)(1 - c)}{(n + \eta - \eta\lambda(1 + n))} \right)^{n+1} \leq 1 - \delta, n \geq 2.
\]

(18)
Proof: It is enough to show that

\[
\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq (1 - \delta)
\]  

Then we write (19) as

\[
\frac{2(1 - \eta)cz^2}{1 + \eta - 2\eta \lambda} + \sum_{n=2}^{\infty} (n + 1)\sigma_n(\alpha_1)\alpha_n z^{n+1} \leq (1 - \delta) \left[ 1 + \frac{(1 - \eta)cz}{1 + \eta - 2\eta \lambda} + \sum_{n=2}^{\infty} \sigma_n(\alpha_1)\alpha_n z^{n+1} \right].
\]

That is

\[
\frac{(3 - \delta)(1 - \eta)c}{1 + \eta - 2\eta \lambda} r^2 + \sum_{n=2}^{\infty} (n + 2 - \delta)\alpha_n r^{n+1} \leq 1 - \delta.
\]

From Theorem 1, we may take

\[
a_n = \frac{(1 - \eta)(1 - c)}{[n + \eta - \eta \lambda(1 + n)]\sigma_n(\alpha_1)\alpha_n} \mu_n, \quad n \geq 2, \mu_n \geq 0, \quad \sum_{n=2}^{\infty} \mu_n = 1.
\]

For each fixed \(r\), we choose the positive integer \(n_0 = n_0(r)\) for which

\[
\frac{(n_0 + 2 - \delta)\sigma_n(\alpha_1)}{(n_0 + \eta - \eta \lambda(1 + n))} r^{n_0 + 1}
\]

is maximal. This implies

\[
\sum_{n=2}^{\infty} (n + 2 - \delta)\alpha_n r^{n+1} \leq \frac{(n_0 + 2 - \delta)(1 - \eta)(1 - c)}{(n_0 + \eta - \eta \lambda(1 + n))} r^{n_0 + 1}.
\]

Then \(f\) is starlike of order \(\delta\) in \(0 < |z| < r_1(\lambda, \eta, c, \delta)\) if

\[
\frac{(3 - \delta)(1 - \eta)c}{1 + \eta - 2\eta \lambda} r^2 + \frac{(n_0 + 2 - \delta)(1 - \eta)(1 - c)}{(n_0 + \eta - \eta \lambda(1 + n))} r^{n_0 + 1} \leq 1 - \delta.
\]

We have to find the value of \(r_0 = r_0(\lambda, \eta, c, \delta)\) and the corresponding integer \(n_0(r_0)\) so that

\[
\frac{(3 - \delta)(1 - \eta)c}{1 + \eta - 2\eta \lambda} r^2 + \frac{(n_0 + 2 - \delta)(1 - \eta)(1 - c)}{(n_0 + \eta - \eta \lambda(1 + n))} r^{n_0 + 1} = 1 - \delta.
\]

It is the value for which \(f(z)\) is starlike of order \(\delta\) in \(0 < |z| < r_0\).
We now state a result for radius of meromorphic convexity for the class $\mathcal{N}(\lambda, \eta, c)$ for which the proof is similar to above.

**Theorem 9:** Let $f \in \mathcal{N}(\lambda, \eta, c)$. Then $f$ is meromorphically convex of order $0 \leq \delta < 1$ in the disk $|z| < r_2(\lambda, \eta, c, \delta)$ where $r_2(\lambda, \eta, c, \delta)$ is the largest value for $n \geq 2$

$$
\left(\frac{(3-\delta)(1-\eta)c}{1+\eta-2\eta\lambda}\right)r^2 + \left(\frac{n(n+2-\delta)(1-\eta)(1-c)}{n+\eta-\eta\lambda(1+n)}\right)r^{n+1} \leq 1 - \delta. \quad (21)
$$

## 6 Integral Operators

In this section, we consider integral operators of functions in the class $\mathcal{N}(\lambda, \eta, c)$.

**Theorem 10:** Let $f \in \mathcal{N}(\lambda, \eta, c)$. Then the integral operator

$$
h(z) = x \int_0^1 u^x f(uz) \, du \quad (0 < u \leq 1, 0 < x < \infty)
$$

is in $\mathcal{N}(\lambda, \eta, c)$, where

$$
\varphi \leq \frac{(x + n + 1)(n + \eta - \eta\lambda(1 + n)) - xn(1 - \eta)(1 - c)}{x(1 - \eta)(1 - c)[1 - \lambda(1 + n)] + (x + n + 1)(n + \eta - \eta\lambda(1 + n))}.
$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \frac{(1-\eta)(1-c)}{2+\eta-3\eta\lambda} \, z^2$.

**Proof:** Let $f \in \mathcal{N}(\lambda, \eta, c)$. Then

$$
h(z) = x \int_0^1 u^x f(uz) \, du = \frac{1}{z} + \frac{(1-\eta)c}{1+\eta-2\eta\lambda} z + \sum_{n=2}^{\infty} \frac{x}{x + n + 1} \sigma_n(\alpha_1) a_n z^n.
$$

It is sufficient to show that

$$
\sum_{n=2}^{\infty} \frac{x[n + \varphi - \varphi\lambda(1 + n)]\sigma_n(\alpha_1)}{(x + n + 1)(1 - \varphi)(1 - c)} a_n \leq 1. \quad (22)
$$

Since $f \in \mathcal{N}(\lambda, \eta, c)$, we have

$$
\sum_{n=2}^{\infty} \frac{(n + \eta - \eta\lambda(1 + n))\sigma_n(\alpha_1)}{(1 - \eta)(1 - c)} a_n \leq 1.
$$

Therefore (22) is true if
\[
\frac{x[n + \varphi - \varphi \lambda (1 + n)]\sigma_n(\alpha_1)}{(x + n + 1)(1 - \varphi)(1 - c)} \leq \frac{(n + \eta - \eta \lambda (1 + n))\sigma_n(\alpha_1)}{(1 - \eta)(1 - c)}.
\]

Solving for \( \varphi \), we have
\[
\varphi \leq \frac{(x + n + 1)(n + \eta - \eta \lambda (1 + n)) - xn(1 - \eta)(1 - c)}{x(1 - \eta)(1 - c)[1 - \lambda (1 + n)] + (x + n + 1)(n + \eta - \eta \lambda (1 + n))} = \Psi(n).
\]

A simple computation will show that \( \Psi(n) \) is increasing and \( \Psi(n) \geq \Psi(1) \). ■

References