Weakly $\delta$-$b$-Continuous Functions

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Abstract

The aim of this paper is to introduce and investigate some properties of weakly $\delta$-$b$-continuous functions. We also define the notion of $\delta$-$b$-continuous functions by using $\delta$-$b$-open sets. We obtain that the notion of weak $\delta$-$b$-continuity is weaker than $\delta$-$b$-continuity, but stronger than both the weak $b$-continuity and weak $e$-continuity. In order to show coincidencies in functions whose range space is regular, we introduce and investigate some properties of the notions of faint $\delta$-$b$-continuity and strong $\theta$-$\delta$-$b$-continuity. Finally, we obtain some properties of weakly $\delta$-$b$-continuous functions related to some separation axioms and graphic functions for $\delta$-$b$-open sets.

Keywords: $\delta$-$b$-open sets, $\delta$-$b$-continuity, faint $\delta$-$b$-continuity, strong $\theta$-$\delta$-$b$-continuity, weak $\delta$-$b$-continuity.

1 Introduction

Of course, continuity is one of important topic for study in topological spaces. This notion is based on open sets. So, generalizations of continuity are given by using weaker types of open sets such as $\alpha$-open sets [12], semi-open sets [9], preopen sets [11], $b$-open sets [1]. On the other hand, the notion of weak continuity is defined by Levine [8] and their modifications are studied by several authors such as [4], [11], [14], [13], [20].

In this paper, first we introduce and give some characterizations of $\delta$-$b$-continuous functions. We also define and investigate some properties of weakly $\delta$-$b$-continuous functions weaker than this notation. Then, we consider two types of continuous functions are called faintly $\delta$-$b$-continuous and
strongly $\theta$-$\delta$-continuous, respectively. In order to show coincidences among weak $\delta$-$b$-continuity, $\delta$-$b$-continuity, faint $\delta$-$b$-continuity and strong $\theta$-$\delta$-continuity. Finally, we investigate relations between weak $\delta$-$b$-continuity and covering properties (resp. connetededness).

2 Preliminaries

Through in this paper, $(X, \tau)$ and $(Y, \varphi)$ denote nonempty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of $(X, \tau)$, the closure of $A$ and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$, respectively. $\tau(x)$ represents all open neighbourhoods of the point $x \in X$.

A set $A$ is called $\theta$-open [10] (resp. $\delta$-open [21]) if every point of $A$ has an open neighbourhood whose closure (resp. interior of closure) is contained in $A$. The $\theta$-interior $[10]$ (resp. $\delta$-interior $[21]$) of $A$ in $(X, \tau)$ is the union of all $\theta$-open (resp. $\delta$-open) subsets of $A$ and is denoted by $Int_\theta(A)$ (resp. $Int_\delta(A)$). Of course, the complement of a $\theta$-open (resp. $\delta$-open) set is called $\theta$-closed $[10]$ (resp. $\delta$-closed $[21]$). That is, $Cl_\theta(A) = \{x \in X \mid \forall U \in \tau(x), Cl(U) \cap A \neq \emptyset\}$ (resp. $Cl_\delta(A) = \{x \in X \mid \forall U \in \tau(x), Int(Cl(U)) \cap A \neq \emptyset\}$).

A subset $A$ of $(X, \tau)$ is called $\delta$-semi-open $[16]$ (resp. preopen $[11]$, $b$-open $[1]$ or $\gamma$-open $[6]$, $e$-open $[5]$ and $\delta$-$b$-open $[7]$) if $A \subseteq Cl(Int_\delta(A))$ (resp. $A \subseteq Int(Cl(A)), A \subseteq Int(Cl(A)) \cup Cl(Int(A)), A \subseteq Int(Cl_\delta(A)) \cup Cl(Int_\delta(A)), A \subseteq Int(Cl(A)) \cup Cl(Int_\delta(A)))$.


The family of all $\delta$-$b$-open and $\delta$-$b$-closed sets of $(X, \tau)$ are denoted by $\delta SO(X, \tau)$ and $\delta SC(X, \tau)$, respectively. The family of all $\delta$-$b$-open sets of $(X, \tau)$ containing a point $x \in X$ is denoted by $\delta BO(X, x)$.

If $A$ is a subset of a space $(X, \tau)$, then the $\delta$-$b$-closure of $A$, denoted by $bCl_\delta(A)$, is the smallest $\delta$-$b$-closed set containing $A$. The $\delta$-$b$-interior of $A$, denoted by $bInt_\delta(A)$, is the largest $\delta$-$b$-open set contained in $A$.

We have the following statements related to two operators $\delta$-$b$-closure, $\delta$-$b$-interior and $\delta$-$b$-closed sets according to $[7]$.

**Lemma 2.1** For a subset $A$ of a space $(X, \tau)$, the following properties are hold:

1. $bCl_\delta(A) = A \cup (Int(Cl_\delta(A)) \cap Cl(Int(A)))$;
2. $bInt_\delta(A) = A \cap (Int(Cl(A)) \cup Cl(Int_\delta(A)))$;
3. $bCl_\delta(X - A) = X - bInt_\delta(A)$;
4. $x \in bCl_\delta(A)$ if and only if $A \cap U \neq \emptyset$ for every $U \in \delta BO(X, x)$;
5. $A \in \delta BC(X)$ if and only if $A = bCl_\delta(A)$. 

Definition 2.2 A function \( f : (X, \tau) \to (Y, \varphi) \) is said to be
(1) \( \gamma \)-continuous [6] (resp. \( e \)-continuous [5]) if \( f^{-1}(V) \) is \( b \)-open (resp. \( e \)-open) in \((X, \tau)\) for every open set \( V \) of \((Y, \varphi)\);
(2) weakly \( b \)-continuous [20] (resp. weakly \( e \)-continuous [15]) if for each \( x \in X \) and each open set \( V \) of \((Y, \varphi)\) containing \( f(x) \), there exists a \( b \)-open (resp. an \( e \)-open) set \( U \) of \((X, \tau)\) containing \( x \) such that \( f(U) \subseteq Cl(V) \).

First of all, we define a new type of continuity whose name is \( \delta \)-\( b \)-continuity and give some characterizations of it.

Definition 2.3 A function \( f : (X, \tau) \to (Y, \varphi) \) is said to be \( \delta \)-\( b \)-continuous if \( f^{-1}(V) \) is \( \delta \)-\( b \)-open in \((X, \tau)\) for every open set \( V \) of \((Y, \varphi)\).

Theorem 2.4 For a function \( f : (X, \tau) \to (Y, \varphi) \), the following properties are equivalent:
(1) \( f \) is \( \delta \)-\( b \)-continuous;
(2) For each \( x \in X \) and each \( V \in \varphi(f(x)) \), there exists \( U \subseteq \delta BO(X, x) \) such that \( f(U) \subseteq V \);
(3) The inverse image of each closed set in \((Y, \varphi)\) is \( \delta \)-\( b \)-closed in \((X, \tau)\);
(4) \( \text{Int} (Cl_{\delta} (f^{-1}(B))) \cap Cl (\text{Int} (f^{-1}(B))) \subseteq f^{-1} (Cl(B)) \) for each \( B \subseteq Y \);
(5) \( f(\text{Int} (Cl_{\delta} (A)) \cap Cl (\text{Int} (A))) \subseteq Cl(f(A)) \) for each \( A \subseteq X \).

Proof: (1) \( \implies \) (2) : Let \( x \in X \) and \( V \in \varphi(f(x)) \). Then \( f^{-1}(V) \in \delta BO(X, x) \). If we consider \( U = f^{-1}(V) \), we obtain \( f(U) \subseteq V \).
(2) \( \implies \) (1) : Let \( V \subseteq Y \) be open and \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and thus there exists \( U_x \in \delta BO(X, x) \) such that \( f(U_x) \subseteq V \). Then \( x \in U_x \subseteq f^{-1}(V) \), and so \( f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x \). According to [7], since the union of any family of \( \delta \)-\( b \)-open sets is a \( \delta \)-\( b \)-open set, we have \( \cup_{x \in f^{-1}(V)} U_x \in \delta BO(X) \) and then \( f^{-1}(V) \in \delta BO(X) \). This shows that \( f \) is \( \delta \)-\( b \)-continuous.
(1) \( \implies \) (3) : Obvious.
(3) \( \implies \) (1) : Obvious.
(3) \( \implies \) (4) : Let \( B \subseteq Y \). Then, \( f^{-1} (Cl(B)) \) is \( \delta \)-\( b \)-closed in \((X, \tau)\). Really, we have
\[
\text{Int} (Cl_{\delta} (f^{-1}(B))) \cap Cl (\text{Int} (f^{-1}(B))) \subseteq \text{Int} (Cl_{\delta} (f^{-1}(Cl(B))))
\]
\[
\cap Cl (\text{Int} (f^{-1}(Cl(B)))) \subseteq f^{-1} (Cl(B)) \text{ is obtained.}
\]
(4) \( \implies \) (5) : Let \( A \subseteq X \). If we consider \( B = f(A) \) in (4), then we have \( \text{Int} (Cl_{\delta} (f^{-1}(f(A)))) \cap Cl (\text{Int} (f^{-1}(f(A)))) \subseteq f^{-1} (Cl(f(A))) \). Since for every subset \( A \) of \( X \), \( A \subseteq f^{-1}(f(A)) \) is true, we obtain \( \text{Int} (Cl_{\delta} (A)) \cap Cl (\text{Int} (A)) \subseteq f^{-1}(Cl(f(A))) \) and hence \( f(\text{Int} (Cl_{\delta} (A)) \cap Cl (\text{Int} (A))) \subseteq Cl(f(A)) \).
(5) \( \implies \) (1) : Let \( V \in \varphi \). If we consider \( W = Y - V \) and \( A = f^{-1}(W) \), we have \( f (\text{Int} (Cl_{\delta} (f^{-1}(Y - V)))) \cap Cl (\text{Int} (f^{-1}(Y - V)))) \subseteq Cl(f (f^{-1}(Y - V))) \subseteq Cl (Y - V) = Y - V \) by using ” for every \( B \subseteq Y \), \( f (f^{-1}(B)) \subseteq B” \) and \( V \in \varphi \). Therefore, \( f^{-1}(W) = f^{-1}(Y - V) \) is \( \delta \)-\( b \)-closed in \((X, \tau)\). This shows that \( f \) is \( \delta \)-\( b \)-continuous.
3 Weakly $\delta$-$b$-Continuous Functions

**Definition 3.1** A function $f : (X, \tau) \rightarrow (Y, \varphi)$ is said to be weakly $\delta$-$b$-continuous (briefly $w.\delta.b.c.$) at $x \in X$ if for each open set $V$ of $(Y, \varphi)$ containing $f(x)$, there exists a $\delta$-$b$-open set $U$ of $(X, \tau)$ containing $x$ such that $f(U) \subseteq \text{Cl}(V)$. The function $f$ is $w.\delta.b.c.$ iff $f$ is $w.\delta.b.c.$ for all $x \in X$.

We have the following Diagram from Definitions 1, 2 and 3.

$$
\text{e-continuity} \quad \leftarrow \quad \delta\text{-b-continuity} \quad \rightarrow \quad b\text{-continuity} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{weak e-continuity} \quad \leftarrow \quad \text{weak } \delta\text{-b-continuity} \quad \rightarrow \quad \text{weak } b\text{-continuity}
$$

Diagram

We state that the converses of these implications are not true in generally, as shown in the [5] and the following examples.

**Example 3.2** Let $(X, \tau)$ and $(Y, \varphi)$ are two topological space such that $X = \{a, b, c, d\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, c, d\}\}$, $Y = \{a, b\}$ and $\varphi = \{\varnothing, Y, \{a\}\}$. A function $f : (X, \tau) \rightarrow (Y, \varphi)$ defined as follows: $f(a) = f(b) = a$ and $f(c) = f(d) = b$. Then, $f$ is $b$-continuous but not $\delta$-$b$-continuous.

**Example 3.3** Let $(X, \tau)$ and $(Y, \varphi)$ are two topological spaces such that $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, $Y = \{a, b\}$ and $\varphi = \{\varnothing, Y, \{a\}\}$. A function $f : (X, \tau) \rightarrow (Y, \varphi)$ defined as follows: $f(c) = a$ and $f(a) = f(b) = b$. Then, $f$ is $\delta$-$b$-continuous but not $\delta$-$b$-continuous.

**Example 3.4** Let $(X, \tau)$ and $(X, \varphi)$ are two topological spaces such that $X = \{a, b, c, d\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, c, d\}\}$ and $\varphi = \{\varnothing, X, \{a, b\}, \{c, d\}\}$. Then, the identity function $f : (X, \tau) \rightarrow (X, \varphi)$ is weakly $b$-continuous but not weakly $\delta$-$b$-continuous.

**Example 3.5** Let $(X, \tau)$ and $(X, \varphi)$ are two topological spaces such that $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and $\varphi = \{\varnothing, X, \{a\}, \{b, c\}\}$. A function $f : (X, \tau) \rightarrow (X, \varphi)$ defined as follows: $f(c) = a$ and $f(a) = f(b) = b$. Then, $f$ is weakly $\delta$-$b$-continuous but not weakly $\delta$-$b$-continuous.

**Example 3.6** Let $X = \{a, b, c, d, e\}$, $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$ and $\varphi = \{\varnothing, X, \{b, c, d\}\}$. Then, the identity function $f : (X, \tau) \rightarrow (X, \varphi)$ is weakly $\delta$-$b$-continuous but not $\delta$-$b$-continuous.

Now, we give some characterizations of weak $\delta$-$b$-continuity at one point.
**Proposition 3.7** For a function \( f : (X, \tau) \longrightarrow (Y, \varphi) \), the following properties are equivalent:

1. \( f \) is w.δ.b.c. at \( x \in X \);
2. \( x \in Cl(\text{Int}_\delta(f^{-1}(Cl(V)))) \cup \text{Int}(Cl(f^{-1}(Cl(V)))) \) for each open neighbourhood \( V \) of \( f(x) \);
3. \( f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(Cl(V))) \) for each \( V \in \varphi \).

**Proof:** (1) \( \implies \) (2). Let \( V \) be any open subset of \( (Y, \varphi) \) such that containing \( f(x) \). Since \( f \) is w.δ.b.c. at \( x \), there exists \( U \in \delta \text{BO}(X, x) \) such that \( f(U) \subseteq Cl(V) \) and hence \( U \subseteq f^{-1}(Cl(V)) \). Since \( U \) is δ-open, \( x \in U \subseteq Cl(\text{Int}_\delta(U)) \cup \text{Int}(Cl(U)) \subseteq Cl(\text{Int}_\delta(f^{-1}(Cl(V)))) \cup \text{Int}(Cl(f^{-1}(Cl(V)))) \) and hence \( x \in b\text{Int}_\delta(f^{-1}(Cl(V))) \). Consequently, we obtain \( f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(Cl(V))) \).

(2) \( \implies \) (3). Let \( x \in f^{-1}(V) \). Then, we have \( f(x) \in V \). Since \( V \subseteq Cl(V) \) for every subset of \( (Y, \varphi) \), we have \( x \in f^{-1}(Cl(V)) \). By hypothesis since \( x \in Cl(\text{Int}_\delta(f^{-1}(Cl(V)))) \cup \text{Int}(Cl(f^{-1}(Cl(V)))) \), we have \( x \in (f^{-1}(Cl(V)) \cap \text{Int}(Cl(f^{-1}(Cl(V)))) \cup \text{Int}(Cl(f^{-1}(Cl(V))))\) and hence \( x \in b\text{Int}_\delta(f^{-1}(Cl(V))) \). Consequently, we obtain \( f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(Cl(V))) \).

(3) \( \implies \) (1). Let \( V \) be any open neighbourhood of \( f(x) \). Then, \( x \in f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(Cl(V))) \). If we consider \( U = b\text{Int}_\delta(f^{-1}(Cl(V))) \), we have \( U \subseteq \delta \text{BO}(X, x) \) and \( f(U) \subseteq Cl(V) \). Consequently, this shows that \( f \) is w.δ.b.c. at \( x \in X \).

The following three theorems are related to some characterizations of weak δ-b-continuity.

**Theorem 3.8** For a function \( f : (X, \tau) \longrightarrow (Y, \varphi) \), the following properties are equivalent:

1. \( f \) is w.δ.b.c.;
2. \( bCl_\delta(f^{-1}(\text{Int}(Cl(B)))) \subseteq f^{-1}(Cl(B)) \) for every subset \( B \) of \( (Y, \varphi) \);
3. \( bCl_\delta(f^{-1}(\text{Int}(F))) \subseteq f^{-1}(F) \) for every regular closed set \( F \) of \( (Y, \varphi) \);
4. \( bCl_\delta(f^{-1}(V)) \subseteq f^{-1}(Cl(V)) \) for every open set \( V \) of \( (Y, \varphi) \);
5. \( f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(Cl(V))) \) for every open set \( V \) of \( (Y, \varphi) \);
6. \( f^{-1}(V) \subseteq Cl(\text{Int}_\delta(f^{-1}(Cl(V)))) \cup \text{Int}(Cl(f^{-1}(Cl(V)))) \) for every open set \( V \) of \( (Y, \varphi) \).

**Proof:** (1) \( \implies \) (2). Let \( B \) be any subset of \( (Y, \varphi) \). Suppose that \( x \in (X - f^{-1}(Cl(B))) \). Then, \( f(x) \in (Y - Cl(B)) \) and there exists an open set \( V \) containing \( f(x) \) such that \( V \cap B = \emptyset \); hence \( Cl(V) \cap \text{Int}(Cl(B)) = \emptyset \). Since \( f \) is w.δ.b.c., there exists \( U \in \delta \text{BO}(X, x) \) such that \( f(U) \subseteq Cl(V) \). Therefore, we obtain \( U \cap f^{-1}(\text{Int}(Cl(B))) = \emptyset \) and hence \( x \in (X - bCl_\delta(f^{-1}(\text{Int}(Cl(B)))) \). So, we have \( (X - bCl_\delta(f^{-1}(\text{Int}(Cl(B)))) \subseteq f^{-1}(Cl(B)) \).

(2) \( \implies \) (3). Let \( F \) be any regular closed set of \( (Y, \varphi) \). Then, we have \( bCl_\delta(f^{-1}(\text{Int}(F))) = bCl_\delta(f^{-1}(\text{Int}(Cl(f^{-1}(F)))) \subseteq f^{-1}(Cl(\text{Int}(F))) = f^{-1}(F) \).
(3) $\implies$ (4). For any open set $V$ of $(Y, \varphi)$. Therefore, we have $b\text{Cl}_\delta(f^{-1}(V)) \subseteq b\text{Cl}_\delta(f^{-1}(\text{Int}(\text{Cl}(V)))) \subseteq f^{-1}(\text{Cl}(V))$.

(4) $\implies$ (5). Let $V$ be any open set of $(Y, \varphi)$. Then, $(Y - \text{Cl}(V))$ is open in $(Y, \varphi)$ and by using Lemma 1, we have $(X - \text{Int}_\delta(f^{-1}(\text{Cl}(V)))) = b\text{Cl}_\delta(f^{-1}(Y - \text{Cl}(V)))) \subseteq f^{-1}(\text{Cl}(Y - \text{Cl}(V)))) \subseteq (X - f^{-1}(V))$. Hence, we obtain $f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(\text{Cl}(V))))$.

(5) $\implies$ (6). Let $V$ be any open set of $(Y, \varphi)$. By Lemma 1, we have $f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(\text{Cl}(V)))) \subseteq \text{Cl}(\text{Int}_\delta(f^{-1}(\text{Cl}(V)))) \cup \text{Int}(\text{Cl}(f^{-1}(\text{Cl}(V))))$.

(6) $\implies$ (1). Let $x$ be any point of $(X, \tau)$ and $V$ be any open set of $(Y, \varphi)$ such that containing $f(x)$. Then, $x \in f^{-1}(V) \subseteq \text{Cl}(\text{Int}_\delta(f^{-1}(\text{Cl}(V)))) \cup \text{Int}(\text{Cl}(f^{-1}(\text{Cl}(V))))$. So, we obtain $f$ is $w.\delta.b.c.$ from Proposition 3.

**Theorem 3.9** For a function $f : (X, \tau) \rightarrow (Y, \varphi)$, the following properties are equivalent:

1. $f$ is $w.\delta.b.c.$;
2. $b\text{Cl}_\delta(f^{-1}(\text{Int}(\text{Cl}(V)))) \subseteq f^{-1}(\text{Cl}(V))$ for every $\delta$-b-open set $V$ of $(Y, \varphi)$;
3. $b\text{Cl}_\delta(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}(V))$ for every preopen set $V$ of $(Y, \varphi)$;
4. $f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(\text{Cl}(V))))$ for every preopen set $V$ of $(Y, \varphi)$.

**Proof:** (1) $\implies$ (2). This is obvious from Theorem 4.2.

(2) $\implies$ (3). Since every preopen set is $\delta$-b-open set and $V \subseteq \text{Int}(\text{Cl}(V))$, this is obvious.

(3) $\implies$ (4). This proof is similar to the proof of the implication (4) $\implies$ (5) in Theorem 2.

(4) $\implies$ (1). Since every open set is preopen, it is obtained from Theorem 4.

**Theorem 3.10** For a function $f : (X, \tau) \rightarrow (Y, \varphi)$, the following properties are equivalent:

1. $f$ is $w.\delta.b.c.$;
2. $f(\text{Cl}_\delta(A)) \subseteq \text{Cl}_\delta(f(A))$ for each subset $A$ of $(X, \tau)$;
3. $b\text{Cl}_\delta(f^{-1}(B)) \subseteq f^{-1}(\text{Cl}_\delta(B))$ for each subset $B$ of $(Y, \varphi)$;
4. $b\text{Cl}_\delta(f^{-1}(\text{Int}(\text{Cl}_\delta(B)))) \subseteq f^{-1}(\text{Cl}_\delta(B))$ for each subset $B$ of $(Y, \varphi)$.

**Proof:** (1) $\implies$ (2). Let $x \in b\text{Cl}_\delta(A)$, $V$ be any open set of $(Y, \varphi)$ containing $f(x)$. Then, there exists $U \in \delta \text{BO}(X, x)$ such that $f(U) \subseteq \text{Cl}(V)$. Then, we have $U \cap A \neq \emptyset$ and $\emptyset \neq f(U) \cap f(A) \subseteq \text{Cl}(V) \cap f(A)$, so that $f(x) \in \text{Cl}_\delta(f(A))$. The proof is completed.

(2) $\implies$ (3). Let $B$ be any subset of $(Y, \varphi)$. Set $A = f^{-1}(B)$ in (2), then we have $f(\text{Cl}_\delta(f^{-1}(B))) \subseteq \text{Cl}_\delta(B)$ and $b\text{Cl}_\delta(f^{-1}(B)) \subseteq f^{-1}(f(b\text{Cl}_\delta(f^{-1}(B)))) \subseteq f^{-1}(\text{Cl}_\delta(B))$.

(3) $\implies$ (4). Let $B$ be any subset of $(Y, \varphi)$. Since $\text{Cl}_\delta(B)$ is closed in $(Y, \varphi)$, we have $b\text{Cl}_\delta(f^{-1}(\text{Int}(\text{Cl}_\delta(B)))) \subseteq f^{-1}(\text{Cl}_\delta(\text{Int}(\text{Cl}_\delta(B)))) \subseteq f^{-1}(\text{Cl}_\delta(B))$. 

(4) \implies (1). Let \( V \) be any open set of \((Y, \varphi)\). Then, we have \( V \subseteq \text{Int}(\text{Cl}(V)) = \text{Int}(\text{Cl}_\theta(V)) \) and so \( b\text{Cl}_\delta(f^{-1}(V)) \subseteq b\text{Cl}_\delta(f^{-1}(\text{Int}(\text{Cl}_\theta(V)))) \subseteq f^{-1}(\text{Cl}_\theta(V)) \subseteq f^{-1}(\text{Cl}(V)) \). Consequently, we obtain \( f \) is \( w.\delta.b.c. \) from Theorem 2.

We have the following two properties as results of Theorems 5 and 6.

**Corollary 3.11** If \( f : (X, \tau) \rightarrow (Y, \varphi) \) is \( w.\delta.b.c. \), then \( f^{-1}(V) \) is \( \delta-b\)-closed (resp. \( \delta-b\)-open) in \((X, \tau)\) for every \( \theta \)-closed (resp. \( \theta \)-open) set \( V \) of \((Y, \varphi)\).

**Proof:** (a) If \( V \) is \( \theta \)-closed, we obtain \( b\text{Cl}_\delta(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}_\theta(V)) = f^{-1}(V) \) and hence \( f^{-1}(V) \) is \( \delta-b\)-closed from Theorem 6.3.

(b) Although this is obvious considering complement of (a), we prove it alternatively as the following.

If \( V \) is \( \theta \)-open, then \((Y - V) \) is \( \theta \)-closed and so \( b\text{Cl}_\delta(f^{-1}(Y - V)) \subseteq f^{-1}(\text{Cl}_\theta(Y - V)) = f^{-1}(Y - V) \) from Theorem 5. Therefore, we have \( b\text{Cl}_\delta(X - f^{-1}(V)) \subseteq X - f^{-1}(V) \) and hence \( X - b\text{Int}_\delta(f^{-1}(V)) \subseteq X - f^{-1}(V) \). Consequently, we have \( f^{-1}(V) \subseteq b\text{Int}_\delta(f^{-1}(V)) \) and \( f^{-1}(V) \) is \( \delta-b\)-open.

**Corollary 3.12** Let \( f : (X, \tau) \rightarrow (Y, \varphi) \) be a function. If \( f^{-1}(\text{Cl}_\theta(B)) \) is \( \delta-b\)-closed in \((X, \tau)\) for every subset \( B \) of \((Y, \varphi)\), then \( f \) is \( w.\delta.b.c. \).

**Proof:** Since \( f^{-1}(\text{Cl}_\theta(B)) \) is \( \delta-b\)-closed in \((X, \tau)\), we have \( b\text{Cl}_\delta(f^{-1}(B)) \subseteq b\text{Cl}_\delta(f^{-1}(\text{Cl}_\theta(B))) = f^{-1}(\text{Cl}_\theta(B)) \). So, \( f \) is \( w.\delta.b.c. \) by using Theorem 6.

Now, we define a new type of faintly continuity by using \( \delta-b\)-open sets.

**Definition 3.13** A function \( f : (X, \tau) \rightarrow (Y, \varphi) \) is said to be faintly \( \delta-b \)-continuous if for each \( x \in X \) and each \( \theta \)-open set \( V \) of \((Y, \varphi)\) containing \( f(x) \), there exists a \( \delta-b \)-open set \( U \) of \((X, \tau)\) containing \( x \) such that \( f(U) \subseteq V \).

We give some characterizations of faintly \( \delta-b \)-continuity.

**Proposition 3.14** For a function \( f : (X, \tau) \rightarrow (Y, \varphi) \), the following properties are equivalent:

1. \( f \) is faintly \( \delta-b \)-continuous;
2. The inverse image of every \( \theta \)-open set in \((Y, \varphi)\) is \( \delta-b \)-open set in \((X, \tau)\);
3. The inverse image of every \( \theta \)-closed set in \((Y, \varphi)\) is \( \delta-b \)-closed set in \((X, \tau)\).

**Proof:** (1) \implies (2). Let \( V \) be any \( \theta \)-open set of \((Y, \varphi)\) and \( x \in f^{-1}(V) \). Then, we have \( f(x) \in V \) and so there exists \( U_x \in \delta BO(X, x) \) such that \( f(U_x) \subseteq V \). Then, we have \( x \in U_x \subseteq f^{-1}(V) \) and so \( f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x \). Since the union of any family of \( \delta-b \)-open sets is a \( \delta-b \)-open set from Theorem
2 in [7], we have \((\bigcup_{x \in f^{-1}(V)} U_x) \in \delta BO(X)\). Therefore, we obtain \(f^{-1}(V)\) is a δ-b-open set of \((X, \tau)\).

(2) \(\implies\) (1). Let \(x \in X\) and \(V\) be any \(\theta\)-open subset of \((Y, \varphi)\) containing \(f(x)\). Then, we have \(f^{-1}(V) \subseteq \delta BO(X, x)\). If we consider \(U = f^{-1}(V)\), then we obtain \(f(U) \subseteq V\). This shows that \(f\) is faintly δ-b-continuous.

(1) \(\implies\) (3). This proof is similar to the proof of (1) \(\implies\) (2).

(3) \(\implies\) (1). This proof is similar to the proof of (2) \(\implies\) (1).

(2) \(\iff\) (3). Since the complement of every \(\theta\)-closed set is \(\theta\)-open, proofs are obvious.

Now, we give a new type of strongly \(\theta\)-b-continuous by using δ-b-open set.

**Definition 3.15** A function \(f : (X, \tau) \rightarrow (Y, \varphi)\) is said to be strongly \(\theta\)-δ-b-continuous (briefly st.\(\theta\).δ.b.c.) if for each \(x \in X\) and each open set \(V\) of \((Y, \varphi)\) containing \(f(x)\), there exists a δ-b-open set \(U\) of \((X, \tau)\) containing \(x\) such that \(f(bCl_δ(U)) \subseteq V\).

Immediately, we give the following equivalence.

**Proposition 3.16** Let \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function and \((Y, \varphi)\) be a regular space. Then \(f\) is st.\(\theta\).δ.b.c. if and only if \(f\) is δ.b.continuous.

**Proof:** Because of necessity is obvious, we only prove sufficiency. Let \(x \in X\) and \(V\) be any open subset of \((Y, \varphi)\) containing \(f(x)\). Since \((Y, \varphi)\) is regular, there exists an open set \(G\) such that \(f(x) \in G \subseteq Cl(G) \subseteq V\). If \(f\) is δ.b.continuous, there exists \(U \in \delta BO(x)\) such that \(f(U) \subseteq G\). Now we shall show that \(f(bCl_δ(U)) \subseteq Cl(G)\). Assume that \(y \notin Cl(G)\). There exists an open set \(W\) containing \(y\) such that \(W \cap G = \emptyset\). Since \(f\) is δ.b.continuous, \(f^{-1}(W) \in \delta BO(X)\) and \(f^{-1}(W) \cap U = \emptyset\), and hence \(f^{-1}(W) \cap bCl_δ(U) = \emptyset\). Therefore, we have \(W \cap f(bCl_δ(U)) = \emptyset\) and \(y \notin f(bCl_δ(U))\). As a result, we have \(f(bCl_δ(U)) \subseteq Cl(G) \subseteq V\) and \(f\) is st.\(\theta\).δ.b.c.

The next theorem is important. If the range space \((Y, \varphi)\) of a function \(f : (X, \tau) \rightarrow (Y, \varphi)\) is regular, then it is stated that st.\(\theta\).δ.b.c., δ.b.c., \(w\).δ.b.c. and \(f.δ.b.c.\) are coincide each other.

**Theorem 3.17** Let \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function and \((Y, \varphi)\) be a regular space. Then, the following properties are equivalent:

1. \(f\) is st.\(\theta\).δ.b.c.;
2. \(f\) is δ.b.c.;
3. \(f^{-1}(Cl_δ(B))\) is δ-b-closed set in \((X, \tau)\) for every subset \(B\) of \((Y, \varphi)\);
4. \(f\) is \(w\).δ.b.c.;
5. \(f\) is \(f.δ.b.c.\).
Theorem 4.1 Let $g \circ f : (X, \tau) \rightarrow (Z, \psi)$ be the composition for two functions $f : (X, \tau) \rightarrow (Y, \varphi)$ and $g : (Y, \varphi) \rightarrow (Z, \psi)$. Then, the following properties are hold:

1. If $f$ is $w.\delta.b.c.$ and $g$ is continuous, then the composition $g \circ f$ is $w.\delta.b.c.$.
2. If $f$ is open continuous surjection and $g \circ f$ is $w.\delta.b.c.$, then $g$ is $w.\delta.b.c.$.

Proof: (1) Let $x \in X$ and $G$ be any open subset of $(Z, \psi)$ containing $g(f(x))$. Then, $g^{-1}(G)$ is an open subset of $(Y, \varphi)$ containing $f(x)$ and there exists $U \in \delta BO(X, x)$ such that $f(U) \subseteq Cl(g^{-1}(G))$ by using hypothesis. Since $g$ is continuous, we obtain $(g \circ f)(U) \subseteq g(Cl(g^{-1}(G))) \subseteq g(g^{-1}(Cl(G))) \subseteq Cl(G)$. This shows that $g \circ f$ is $w.\delta.b.c.$.

(2) Let $G$ be an open set of $(Z, \psi)$. By hypothesis because of $g \circ f : (X, \tau) \rightarrow (Z, \psi)$ is $w.\delta.b.c.$ and $f$ is continuous, it is obvious that $(g \circ f)^{-1}(G) \subseteq Cl(Int_{\delta}(g \circ f)^{-1}(Cl(G))) \cup Int(Cl((g \circ f)^{-1}(Cl(G)))) = Cl(Int_{\delta}(f^{-1}(g^{-1}(Cl(G)))) \cup Int(Cl(f^{-1}(g^{-1}(Cl(G))))).$ Since $f$ is open continuous surjection, there exist the following relations: $g^{-1}(G) = f(f^{-1}(g^{-1}(G)))$.

Example 3.18 Let $f : (X, \tau) \rightarrow (Y, \varphi)$ function is as same as in Example 5. Then, $f$ is faintly $\delta$-continuous but not strongly $\theta$-$\delta$-continuous.
continuous surjection, we have natural projections. Consequently, from Theorem 2, we obtain \( g \) is w.δ.b.c.

Let \( \{X_\alpha \mid \alpha \in \Delta \} \) and \( \{Y_\alpha \mid \alpha \in \Delta \} \) be any two families of spaces with the same index set \( \Delta \). Let \( f_\alpha : X_\alpha \rightarrow Y_\alpha \) be a function for each \( \alpha \in \Delta \). The product space \( \Pi \{X_\alpha \mid \alpha \in \Delta \} \) will be denoted by \( \Pi X_\alpha \) and \( f : \Pi X_\alpha \rightarrow \Pi Y_\alpha \) will be denote the product function defined by \( f (\{x_\alpha\}) = \{f_\alpha (x_\alpha)\} \) for every \( \{x_\alpha\} \in \Pi X_\alpha \). Moreover, let \( \rho_\beta : \Pi X_\alpha \rightarrow X_\beta \) and \( q_\beta : \Pi Y_\alpha \rightarrow Y_\beta \) be the natural projections.

As a result of Theorem 12, we give the following theorem.

**Theorem 4.2** If a function \( f : \Pi X_\alpha \rightarrow \Pi Y_\alpha \) is w.δ.b.c., then \( f_\alpha : X_\alpha \rightarrow Y_\alpha \) for each \( \alpha \in \Delta \).

**Proof:** Assume that \( f \) is w.δ.b.c.. Since \( q_\beta \) is continuous, we have \( q_\beta \circ f = f_\beta \circ \rho_\beta \) is w.δ.b.c. by using Theorem 12.1. Besides, since \( \rho_\beta \) is open continuous surjection, we have \( f_\beta \) is w.δ.b.c. from Theorem 12.2.

**Definition 4.3** A topological space \((X, \tau)\) is said to be

1. Urysohn [23] if for each pair of distinct points \( x \) and \( y \) in \((X, \tau)\), there exist open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( Cl(U) \cap Cl(V) = \emptyset \);

2. \( \delta\)-b-T\(_1\) [3] if for each pair of distinct points \( x \) and \( y \) in \((X, \tau)\), there exist \( \delta\)-b-open sets \( U \) and \( V \) of \((X, \tau)\) containing \( x \) and \( y \), respectively, such that \( y \notin U \) and \( x \notin V \);

3. \( \delta\)-b-T\(_2\) [3], [17] if for each pair of distinct points \( x \) and \( y \) in \((X, \tau)\), there exist \( \delta\)-b-open sets \( U \) and \( V \) of \((X, \tau)\) containing \( x \) and \( y \), respectively, such that \( U \cap V = \emptyset \).

**Theorem 4.4** Let \( f : (X, \tau) \rightarrow (Y, \varphi) \) be a w.δ.b.c. injective function. Then, the following properties hold:

1. If \((Y, \varphi)\) is Urysohn, then \((X, \tau)\) is \( \delta\)-b-T\(_2\);
2. If \((Y, \varphi)\) is Hausdorff, then \((X, \tau)\) is \( \delta\)-b-T\(_1\).

**Proof:** (1) Let \( x_1 \) and \( x_2 \) be any distinct points in \((X, \tau)\). Then \( f(x_1) \neq f(x_2) \) and there exist open sets \( V_1 \) and \( V_2 \) of \((Y, \varphi)\) containing \( f(x_1) \) and \( f(x_2) \), respectively, such that \( Cl(V_1) \cap Cl(V_2) = \emptyset \). Since \( f \) is w.δ.b.c., there exists \( U_i \in \delta BO(X, x_i) \) such that \( f(U_i) \subseteq Cl(V_i) \), for \( i = 1, 2 \). Since \( f^{-1}(Cl(V_1)) \) and \( f^{-1}(Cl(V_2)) \) are disjoint, we have \( U_1 \cap U_2 = \emptyset \). Therefore, \((X, \tau)\) is \( \delta\)-b-T\(_2\).

(2) Let \( x_1 \) and \( x_2 \) be any distinct points in \((X, \tau)\). Then \( f(x_1) \neq f(x_2) \) and there exist open sets \( V_1 \) and \( V_2 \) of \((Y, \varphi)\) such that \( f(x_1) \in V_1 \) and \( f(x_2) \in V_2 \). Then, we have \( f(x_1) \notin Cl(V_2) \) and \( f(x_2) \notin Cl(V_1) \). Since \( f \) is w.δ.b.c., there exists \( U_i \in \delta BO(X, x_i) \) such that \( f(U_i) \subseteq Cl(V_i) \), for \( i = 1, 2 \). Therefore, we obtain \( x_1 \notin U_2 \) and \( x_2 \notin U_1 \). Consequently, \((X, \tau)\) is \( \delta\)-b-T\(_1\).
Theorem 4.5 If \( f : (X, \tau) \rightarrow (Y, \varphi) \) is w.δ.b.c. and \( A \) is \( \theta \)-closed set of \( X \times Y \), then \( \rho_x (A \cap G_f) \) is \( \delta \)-b-closed in \( (X, \tau) \) where \( \rho_x \) represents the projection of \( X \times Y \) onto \( (X, \tau) \) and \( G_f \) denotes the graph of \( f \).

Proof: Let \( A \) be a \( \theta \)-closed set of \( X \times Y \) and \( x \in bCl_\delta (\rho_x (A \cap G_f)) \). Let \( U \) be any open set of \( (X, \tau) \) containing \( x \) and \( V \) any open set of \( (Y, \varphi) \) containing \( f(x) \). Since \( f \) is w.δ.b.c., by Theorem 4, we have \( x \in f^{-1} (V) \subseteq bInt_\delta (f^{-1} (Cl (V))) \). Since \( x \in bCl_\delta (\rho_x (A \cap G_f)) \) by Lemma 1, \([U \cap bInt_\delta (f^{-1} (Cl (V)))] \cap \rho_x (A \cap G_f)\) contains some points \( y \) of \((X, \tau)\). This shows that \((y, f(y)) \in A \) and \( f(U) \subseteq Cl (V) \). Hence we obtain \( \varnothing \neq (U \times Cl (V)) \cap A \subseteq Cl (U \times V) \cap A \) and hence \((x, f(x)) \in Cl_\theta (A) \). Since \( A \) is \( \theta \)-closed, \((x, f(x)) \in (A \cap G_f)\) and \( x \in \rho_x (A \cap G_f) \). Then by using Lemma 1, \( \rho_x (A \cap G_f) \) is \( \delta \)-b-closed.

Corollary 4.6 If \( f : (X, \tau) \rightarrow (Y, \varphi) \) has \( \theta \)-closed graph and \( g : (X, \tau) \rightarrow (Y, \varphi) \) is w.\( \delta \)-b.c., then the set \( \{x \in X \mid f(x) = g(x)\} \) is \( \delta \)-b-closed in \( (X, \tau) \).

Proof: Since \( G_f \) is \( \theta \)-closed and \( \rho_x (G_f \cap G_g) = \{x \in X \mid f(x) = g(x)\} \), we have that \( \{x \in X \mid f(x) = g(x)\} \) is \( \delta \)-b-closed by using Theorem 15.

Definition 4.7 A function \( f : (X, \tau) \rightarrow (Y, \varphi) \) is said to have a \( \delta \)-b-strongly closed graph if for each \((x, y) \in (X \times Y)-G_f\), there exist a \( \delta \)-b-open subset \( U \) of \((X, \tau)\) and an open subset \( V \) of \((Y, \varphi)\) such that \((x, y) \in (U \times V)\) and \((U \times Cl(V)) \cap G_f = \varnothing\).

Theorem 4.8 If \((Y, \varphi)\) is Urysohn space and \( f : (X, \tau) \rightarrow (Y, \varphi) \) is w.\( \delta \)-b.c., then \( G_f \) is \( \delta \)-b-strongly closed.

Proof: Let \((x, y) \in (X \times Y)-G_f\). Then \( y \neq f(x) \) and there exist open set \( V_1 \) and \( V_2 \) of \((Y, \varphi)\) containing \( f(x) \) and \( y \), respectively, such that \( Cl (V_1) \cap Cl (V_2) = \varnothing \). Since \( f \) is w.\( \delta \)-b.c., there exists a \( \delta \)-b-open subset \( U \) of \((X, \tau)\) containing \( x \) such that \( f(U) \subseteq Cl (V_1) \). Hence, we have \( f(U) \cap Cl (V_2) = \varnothing \) and hence \((U \times Cl(V_2)) \cap G_f = \varnothing\). This shows that \( G_f \) is \( \delta \)-b-strongly closed.

Theorem 4.9 Let \( f : (X, \tau) \rightarrow (Y, \varphi) \) be a w.\( \delta \)-b.c. function such that have a \( \delta \)-b-strongly closed graph \( G_f \). If \( f \) is injective, then \((X, \tau)\) is \( \delta \)-b-\( T_2 \).

Proof: Let \( x_1 \) and \( x_2 \) be any distinct points in \((X, \tau)\). Since \( f \) is injective, \( f(x_1) \neq f(x_2) \) and \((x_1, f(x_2)) \notin G_f \). Since \( G_f \) is \( \delta \)-b-strongly closed, there exist \( U \in \delta BO(X, x_1) \), and so \( f(U) \cap Cl (V) = \varnothing \). Since \( f \) is w.\( \delta \)-b.c., there exists a \( G \in \delta BO(X, x_2) \) such that \( f(G) \subseteq Cl (V) \). Hence, we have \( f(U) \cap f(G) = \varnothing \) and hence \( U \cap G = \varnothing \). This shows that \((X, \tau)\) is \( \delta \)-b-\( T_2 \).

From now on, we investigate covering properties which is another separation axiom. Recall that a Hausdorff space \((X, \tau)\) is called semicompact [24] at a
point $x$ if every neighbourhood $U_x$ contains a $V_x$ such that $B(V_x)$, the boundary of $V_x$, is compact. Of course, it is called semicompact, if it has this property at every point.

**Theorem 4.10** If $f : (X, \tau) \rightarrow (Y, \varphi)$ be a $w.\delta.b.c.$ and $(Y, \varphi)$ be a semi-compact Hausdorff space, then $f$ is $\delta.b.c.$.

**Proof:** Since every semicompact Hausdorff space is regular, we obtain $f$ is is $\delta.b.c.$ by using Theorem 11.

It is well known that Veličko [21] introduced the notion of $H$-set as the following. A subset $A$ of a space $(X, \tau)$ is said to be an $H$-set if for every cover $\{U_\alpha \mid \alpha \in \Delta\}$ of $A$ by open sets of $(X, \tau)$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $A \subseteq \bigcup\{\text{Cl}(U_\alpha) \mid \alpha \in \Delta_0\}$. This notion is renamed as quasi-$H$-closed relative to $X$ by Porter et al. [18].

**Definition 4.11** A topological space $(X, \tau)$ is said to be

(1) almost compact [13] or quasi-$H$-closed [18] (resp. almost lindelöf [13]) if every open cover of $X$ has a finite resp. countable) subcover whose closures cover $X$;

(2) $\delta b$-compact ( resp. $\delta b$-lindelöf ) if every $\delta b$-open cover of $X$ has a finite (resp. countable) subcover.

Now, we have the following theorem.

**Theorem 4.12** For a $w.\delta.b.c.$ surjection function $f : (X, \tau) \rightarrow (Y, \varphi)$, then the following properties hold:

(1) If $(X, \tau)$ is $\delta b$-compact, then $(Y, \varphi)$ is almost compact;

(2) If $(X, \tau)$ is $\delta b$-lindelöf, then $(Y, \varphi)$ is almost lindelöf.

**Proof:** (1) Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a cover of $Y$ by open subset of $(Y, \varphi)$. For each point $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $w.\delta.b.c.$, there exists a $\delta b$-open set $U_x$ of $X$ containing $x$ such that $f(U_x) \subseteq \text{Cl}(V_{\alpha(x)})$. The family $\{U_x \mid x \in X\}$ is a cover of $X$ by $\delta b$-open subset of $X$, and so there exists a finite subset $X_0$ of $X$ such that $X = \bigcup_{x \in X_0} U_x$. Hence, we have $Y = f(X) = \bigcup_{x \in X_0} \text{Cl}(V_{\alpha(x)})$. Consequently, this shows that $(Y, \varphi)$ is almost compact.

(2) This proof is similar to the proof of (1).

**Theorem 4.13** If a function $f : (X, \tau) \rightarrow (Y, \varphi)$ has a $\delta b$-strongly closed graph $G_f$, then $f(A)$ is $\theta$-closed in $(Y, \varphi)$ for each subset $A$ which is $\delta b$-compact relative to $X$.
Proof: Let $A$ be $\delta$-b-compact relative to $X$ and $y \notin f(A)$. Then, $y \in (Y - f(A))$ and for each $x \in A$ we have $(x, y) \notin G_f$. So, there exist $U_x \in \delta BO(X, x)$ and an open $V_x$ of $(Y, \varphi)$ containing $y$ such that $f(U_x) \subseteq Cl(V_x) = \emptyset$. The collection $\{U_x \mid x \in A\}$ is a cover of $A$ by $\delta$-b-open subsets of $X$. Since $A$ is $\delta$-b-compact relative to $X$, there exists a finite subset $A_0$ of $A$ such that $A \subseteq \cup \{U_x \mid x \in A_0\}$. If we consider $V = \cap_{x \in A_0} V_x$, then we obtain that $V$ is an open set in $(Y, \varphi)$, $y \in V$ and $f(A) \cap Cl(V) \subseteq [\cup_{x \in A_0} f(U_x)] \cap Cl(V) \subseteq [\cup_{x \in A_0} f(U_x) \cap Cl(V)] = \emptyset$. Hence $y \notin Cl_0(f(A))$ and hence $f(A)$ is $\theta$-closed in $(Y, \varphi)$.

Definition 4.14 A topological space $(X, \tau)$ is said to be $\delta$-b-connected (resp. $\gamma$-connected [6]) if it cannot be written as the union of two nonempty disjoint $\delta$-b-open (resp. $\gamma$-open) sets.

Lemma 4.15 [7] For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $\gamma$-connected,
2. $X$ cannot be expressed as the union of two nonempty disjoint $\delta$-b-open sets.

It is obvious that Lemma 22 states a topological space is $\gamma$-connected if and only if it is $\delta$-b-connected.

Theorem 4.16 If $f : (X, \tau) \to (Y, \varphi)$ is w.d.b.c. surjection and $(X, \tau)$ is $\delta$-b-connected ( $\gamma$-connected ), then $(Y, \varphi)$ is connected.

Proof: Assume that $(Y, \varphi)$ is not connected. There exist nonempty open sets $V_1$ and $V_2$ of $(Y, \varphi)$ such that $V_1 \cup V_2 = Y$ and $V_1 \cap V_2 = \emptyset$. Then $V_1$ and $V_2$ are clopen in $(Y, \varphi)$. In this state, we obtain $f^{-1}(V_1) \subseteq bInt_\delta(f^{-1}(Cl(V_1))) = bInt_\delta(f^{-1}(V_1))$ and hence $f^{-1}(V_1)$ is $\delta$-b-open in $(X, \tau)$ by using Proposition 3.3. Similarly, we have $f^{-1}(V_2)$ is $\delta$-b-open in $(X, \tau)$. Besides, we have $f^{-1}(V_1) \cup f^{-1}(V_2) = X$, $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1)$, $f^{-1}(V_2)$ are nonempty. So, $(X, \tau)$ is not $\delta$-b-connected.

We give the following result by using Theorem 23.

Corollary 4.17 If $f : (X, \tau) \to (Y, \varphi)$ is $\delta$-b.c. surjection and $(X, \tau)$ is $\delta$-b-connected ( $\gamma$-connected ), then $(Y, \varphi)$ is connected.

Lemma 4.18 Let $A$ and $B$ be subsets of a space $(X, \tau)$. If $A \in \delta BO(X)$ and $B \in \delta O(X)$, then $(A \cap B) \in \delta BO(B)$.

Proof: Since $A$ is $\delta$-b-open set and $B$ is $\delta$-open set, we have $A \subseteq Int(Cl(A)) \cup Cl(Int_\delta(A))$ and $B \subseteq Int_\delta(B)$. Then, we obtain $(A \cap B) \subseteq [(Int(Cl(A)) \cup Cl(Int_\delta(A))] \cap Int_\delta(B)$.
Weakly $\delta$-b-Continuous Functions

$=$ \[\text{Int} (\text{Cl}(A)) \cap \text{Int}_\delta(B) \] \[\subseteq [\text{Int}(\text{Cl}(A)) \cap \text{Int}(B)] \cup [\text{Cl}(\text{Int}_\delta(A)) \cap \text{Int}_\delta(B)]\]

\[= \text{Int}[\text{Cl}(A) \cap \text{Int}(B)] \cup \text{Cl}(\text{Int}_\delta(A) \cap \text{Int}_\delta(B))\]

\[\subseteq \text{Int}[\text{Cl}(A \cap B)] \cup \text{Cl}[\text{Int}_\delta(A \cap B)]\]

\[\subseteq \text{Int}[\text{Cl}_B(A \cap B)] \cup \text{Cl}[\text{Int}_\delta(A \cap B)]\]

\[\subseteq \text{Int}_B[\text{Cl}_B(A \cap B)] \cup \text{Cl}_B[\text{Int}_\delta(A \cap B)].\]

Of course, this shows that $(A \cap B)$ is $\delta$-b-open set in subspace $(B, \tau_B)$ of $(X, \tau)$.

**Theorem 4.19** Let \( \{U_\alpha | \alpha \in \Delta\} \) be any $\delta$-open cover of a space $(X, \tau)$. If a function $f : (X, \tau) \to (Y, \varphi)$ is w.$\delta$-b.c., then the restriction $f : (X, \tau) \to (Y, \varphi)$ is w.$\delta$-b.c.

**Theorem 4.20** Let \( \{U_\alpha | \alpha \in \Delta\} \) be any $\delta$-open cover of a space $(X, \tau)$. If a function $f : (X, \tau) \to (Y, \varphi)$ is w.$\delta$-b.c., then the restriction $f |_{U_\alpha}$: $(U_\alpha, \tau_{U_\alpha}) \to (Y, \varphi)$ is w.$\delta$-b.c. for each $\alpha \in \Delta$.

**Proof:** Let $\alpha$ be an arbitrary fixed index and $U_\alpha$ be $\delta$-open in $(X, \tau)$. Let $x$ be any point of $U_\alpha$ and $V$ be any open set of $(Y, \varphi)$ containing $(f |_{U_\alpha})(x) = f(x)$. Since $f$ is w.$\delta$-b.c., there exists $U \in \delta BO(X, x)$ such that $f(U) \subseteq \text{Cl}(V)$.

Since $U_\alpha$ is open ( $\delta$-open ) in $(X, \tau)$, by Lemma 22, $(U \cap U_\alpha) \in \delta BO(X, x)$ and $(f |_{U_\alpha})(U \cap U_\alpha) = f(U \cap U_\alpha) \subseteq f(U) \subseteq \text{Cl}(V)$. This shows that $f |_{U_\alpha}$ is w.$\delta$-b.c.

It is well-known that a topological space $(X, \tau)$ is said to be

(a) submaximal [19], [2] if every dense subset of $(X, \tau)$ is open,

(b) extremally disconnected [2] if the closure of each open set of $(X, \tau)$ is open.

**Theorem 4.21** Let $(X, \tau)$ be a submaximal, extremally disconnected space. If

$f : (X, \tau) \to (Y, \varphi)$ has $\delta$-b-strongly closed graph, then $f^{-1}(F)$ is closed in $(X, \tau)$ for each subset $F$ which is $H$-set in $(Y, \varphi)$.

**Proof:** Let $F$ be $H$-set of $(Y, \varphi)$ and $x \notin f^{-1}(F)$. For each $y \in F$, we have $(x, y) \in (X \times Y) \setminus G_f$ and there exist a $\delta$-b-open set $U_y$ of $(X, \tau)$ containing $x$ and an open set $V_y$ of $(Y, \varphi)$ containing $y$ such that $f(U_y) \cap \text{Cl}(V_y) = \emptyset$ and hence $U_y \cap f^{-1}(\text{Cl}(V_y)) = \emptyset$. The collection $\{V_y \mid y \in F\}$ is a cover of $F$ by open sets of $(Y, \varphi)$. Since $F$ is $H$-set in $(Y, \varphi)$, there exists a finite subset $F_0$ of $F$ such that $F \subseteq \cup \{\text{Cl}(V_y) \mid y \in F_0\}$. Since $(X, \tau)$ is submaximal and extremally disconnected space, for each $U_y$ is an open in $(X, \tau)$ we consider $U = \cap_{y \in F_0} U_y$. Then $U$ is an open set containing $x$ and $f(U) \cap F \subseteq \cup_{y \in F_0} [f(U) \cap \text{Cl}(V_y)] \subseteq \cup_{y \in F_0} [f(U_y) \cap \text{Cl}(V_y)] = \emptyset$. Therefore, we have $U \cap f^{-1}(F) = \emptyset$ and hence $f^{-1}(F)$ is closed in $(X, \tau)$. 
Recall that a topological space \((X, \tau)\) is said to be \(C\)-compact \([22]\) if for each closed subset \(A \subseteq X\) and each open cover \(\{U_\alpha \mid \alpha \in \Delta\}\) of \(A\), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(A \subseteq \bigcup \{\text{Cl}(U_\alpha) \mid \alpha \in \Delta_0\}\).

**Corollary 4.22** Let \(f : (X, \tau) \rightarrow (Y, \phi)\) be a function with a \(\delta b\)-strongly closed graph, from a submaximal, extremally disconnected space \((X, \tau)\) into a \(C\)-compact space \((Y, \phi)\). Then, \(f\) is continuous.

**Proof:** Let \(A\) be a closed subset in the \(C\)-compact space \((Y, \phi)\). Then, \(A\) is an \(H\)-set and \(f^{-1}(A)\) is closed in \((X, \tau)\) by Theorem 28. Hence, \(f\) is continuous.

**References**


Weakly $\delta$-$b$-Continuous Functions


