Suzuki Type n-Tupled Fixed Point Theorems in Ordered Metric Spaces

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Abstract

In this paper we prove a Suzuki type unique n-tupled common fixed point theorem in a partially ordered metric space.

Keywords: Partial order, Metric space, n-tupled fixed point, $W$-compatible maps.

1 Introduction and Preliminaries

Bhaskar and Lakshmikantham [13] introduced the notion of a coupled fixed point and proved some coupled fixed point theorems in partially ordered complete metric spaces under certain conditions. Later Lakshmikantham and Ciric [17] extended these results by defining the mixed $g$-monotone property to generalize the corresponding fixed point theorems contained in [13]. After that, Berinde and Borcut [16] introduced the concept of tripled fixed point and
proved some related theorems. In this continuation, Karapinar [4] introduced
the quadruple fixed point and proved some results on the existence and unique-
ess of quadruple fixed points.

Recently Imdad et al.[8] introduced the concept of n-tupled coincidence and
n-tupled common fixed point theorems for nonlinear \( \phi \)-contraction mappings.
For more details see [9, 10].

In 2008, Suzuki [14, 15] introduced generalized versions of both Banach’s
and Edelstain’s basic results. Many other works in this direction have been
considered, for example refer [1, 2, 3, 5, 6, 12] and the references threin.

Combining the concepts of n-tupled fixed point theorems and Suzuki type
theorems, in this paper, we prove n-tupled coincidence and n-tupled
common fixed point theorems of Suzuki-type in a partially ordered metric space.

Now we give some known definitions.

Let \((X, \preceq)\) be a partially ordered set and we denote \(X \times X \times X \times \cdots \times X\) (n times)
by \(X^n\). \(X^n\) is equipped with the following partial ordering: for \(x, y \in X^n\) where
\(x = (x^1, x^2, \ldots, x^n)\) and \(y = (y^1, y^2, \ldots, y^n)\), \(x \preceq y \Leftrightarrow x^i \preceq y^i\) if \(i\) is odd and
\(x^i \succeq y^i\) if \(i\) is even.

**Definition 1.1** ([8]) Let \((X, \preceq)\) be a partially ordered set. Let \(F : X^n \to X\) and \(g : X \to X\) be two mappings. Then the mapping \(F\) is said to have
the mixed \(g\)-monotone property if \(F\) is \(g\)-non decreasing in its odd position
arguments and \(g\)-non increasing in its even position arguments, that is, for all
\(x^i_1, x^i_2 \in X\),

\[
gx^i_1 \preceq gx^i_2 \Rightarrow \begin{cases} 
F(x^1, x^2, \ldots, x^i_1, \ldots, x^n) \preceq F(x^1, x^2, \ldots, x^i_2, \ldots, x^n) & \text{if } i \text{ is odd,} \\
F(x^1, x^2, \ldots, x^i_1, \ldots, x^n) \succeq F(x^1, x^2, \ldots, x^i_2, \ldots, x^n) & \text{if } i \text{ is even.}
\end{cases}
\]

**Definition 1.2** ([8]) An element \((x^1, x^2, \cdots, x^n) \in X\) is called a n-tupled
coincidence point of \(F : X^n \to X\) and \(g : X \to X\) if

\[
F(x^1, x^2, \cdots, x^n) = gx^1, \\
F(x^2, x^3, \cdots, x^n) = gx^2, \\
\vdots \\
F(x^n, x^1, x^2, \cdots, x^{n-1}) = gx^n.
\]
\textbf{Definition 1.3} ([8]) An element \((x^1, x^2, \ldots, x^n) \in X\) is called a \(n\)-tupled common fixed point of \(F : X^n \rightarrow X\) and \(g : X \rightarrow X\) if
\[
F(x^1, x^2, \ldots, x^n) = gx^1 = x^1,
F(x^2, x^3, \ldots, x^n) = gx^2 = x^2,
\vdots
F(x^n, x^1, x^2, \ldots, x^{n-1}) = gx^n = x^n.
\]

\textbf{Definition 1.4} ([7]) The mappings \(F : X \times X \rightarrow X\) and \(f : X \rightarrow X\) are called \(W\)-compatible if \(f(F(x, y)) = F(fx, fy)\) and \(f(F(y, x)) = F(fy, fx)\) whenever \(fx = F(x, y)\) and \(fy = F(y, x)\).

\textbf{Lemma 1.5} ([11]) Let \(X\) be a non-empty set and \(g : X \rightarrow X\) be a mapping. Then there exists a subset \(E\) of \(X\) such that \(g(E) = g(X)\) and the mapping \(g : E \rightarrow X\) is one-one.

Now we prove our main results.

\section{Main Results}

\textbf{Theorem 2.1} . Let \((X, \preceq, d)\) be a partially ordered metric space and \(F : X^n \rightarrow X\) and \(f : X \rightarrow X\) be two mappings such that \(F\) has the mixed \(g\)-monotone property on \(X\) and satisfying the following :

\begin{align*}
(2.1.1) & \quad F(X^n) \subseteq g(X) \text{ and } g(X) \text{ is complete,} \\
(2.1.2) & \quad \text{If there exists a constant } \theta \in [0, 1) \text{ such that }
\end{align*}
\[
\eta(\theta) \min \left\{ \begin{array}{ll}
d(gx^1, F(x^1, x^2, \ldots, x^n)), \\
d(gx^2, F(x^2, x^3, \ldots, x^n, x^1)), \\
\vdots \\
d(gx^n, F(x^n, x^1, \ldots, x^{n-1}))
\end{array} \right\} \leq \max \left\{ \begin{array}{ll}
d(gx^1, gy^1), \\
d(gx^2, gy^2), \\
\vdots \\
d(gx^n, gy^n)
\end{array} \right\}
\]

implies
\[
d(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n)) \\
\leq \theta \max \left\{ \begin{array}{ll}
d(gx^1, gy^1), d(gx^2, gy^2), \ldots, d(gx^n, gy^n), \\
d(gx^1, F(x^1, x^2, \ldots, x^n)), \ldots, d(gx^n, F(x^n, x^1, \ldots, x^{n-1})), \\
d(gy^1, F(x^1, x^2, \ldots, x^n)), \ldots, d(gy^n, F(x^n, x^1, \ldots, x^{n-1}))
\end{array} \right\}
\]

for all \(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n \in X\) for which \(gx^i\) and \(gy^i\) \((i = 1, 2, \ldots, n)\) are comparable, where \(\eta : [0, 1) \rightarrow \left(\frac{1}{2}, 1\right]\) defined by \(\eta(\theta) = \frac{1}{1+\theta}\) is a strictly
In view of (2.1.1), we construct sequences $\{x_0^1, x_0^2, \ldots, x_0^n\}$ in $X$ such that

$$
gx_0^i \preceq F(x_0^i, x_0^{i+1}, \ldots, x_0^n, x_0^1, x_0^2, \ldots, x_0^{i-1}) \quad \text{if } i \text{ is odd}
$$

$$
gx_0^i \succeq F(x_0^i, x_0^{i+1}, \ldots, x_0^n, x_0^1, x_0^2, \ldots, x_0^{i-1}) \quad \text{if } i \text{ is even.}
$$

(2.1.4) (a) Suppose $F$ and $g$ are continuous

or

(b) $X$ has the following properties:

(i) If a non-decreasing sequence $\{x_m\} \rightarrow x$, then $x_m \preceq x$, for all $m$,

(ii) If a non-increasing sequence $\{y_m\} \rightarrow y$, then $y \preceq y_m$, for all $m$.

Then $F$ and $g$ have a n-tupled coincidence point in $X$.

**Proof.** Let $x_0^1, x_0^2, \ldots, x_0^n \in X$ be satisfying (2.1.3).

In view of (2.1.1), we construct sequences $\{x_m^1\}, \{x_m^2\}, \ldots, \{x_m^n\}$ in $X$ as follows:

$$
gx_m^1 = F(x_m^1, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1}),
$$

$$
gx_m^2 = F(x_m^2, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1}),
$$

$$
\vdots
$$

$$
gx_m^n = F(x_m^n, x_m^{i+1}, \ldots, x_m^1, x_m^1, \ldots, x_m^{i-1}),
$$

for all $m \geq 1.$

We claim for all $m \geq 0$, that

$$
gx_m^i \preceq gx_{m+1}^i \quad \text{if } i \text{ is odd and } gx_m^i \succeq gx_{m+1}^i \quad \text{if } i \text{ is even} \tag{2}
$$

Relations (2.1.3) and (1) implies that (2) holds for $m = 0$.

Suppose (2) holds for $m = k > 0$.

For odd $i$, consider $x_{k+1}^i$ and using mixed $g$-monotone property of $F$, we get

$$
gx_{k+2}^i = F(x_{k+1}^i, x_{k+2}^{i+1}, \ldots, x_{k+1}^n, x_{k+1}^1, \ldots, x_{k+1}^{i-1}) \preceq F(x_{k+1}^i, x_{k+1}^{i+1}, \ldots, x_{k+1}^n, x_{k+1}^1, \ldots, x_{k+1}^{i-1})
$$

$$
\vdots
$$

$$
gx_{k+1}^i = x_{k+1}^i.
$$

For even $i$, consider
Hence by mathematical induction, (2) holds for all \( m \geq 0 \).

Suppose \( gx^1_{m+1} = gx^1_m, gx^2_{m+1} = gx^2_m, \ldots, gx^n_{m+1} = gx^n_m \) for some \( m \).
Then \( (x^1_m, x^2_m, \ldots, x^n_m) \) is a \( n \)-tupled coincidence point of \( F \) and \( g \).
Assume that \( gx^1_{m+1} \neq gx^1_m \) or \( gx^2_{m+1} \neq gx^2_m \), or \( \cdots \) or \( gx^n_{m+1} \neq gx^n_m \) for all \( m \).
Since

\[
\eta(\theta) \min \left\{ \begin{array}{l}
d(gx^1_0, F(x^1_0, x^2_0, \ldots, x^n_0), \\
n \vdots \\
d(gx^n_0, F(x^n_0, x^1_0, \ldots, x^{n-1}_0), \\
\end{array} \right\} \leq \min \left\{ \begin{array}{l}
d(gx^1_0, gx^1_1), \\
n \vdots \\
d(gx^n_0, gx^n_1) \\
\end{array} \right\}
\]

\[
\leq \max \left\{ \begin{array}{l}
d(gx^1_0, gx^1_1), \\
n \vdots \\
d(gx^n_0, gx^n_1) \\
\end{array} \right\},
\]

by (2.1.2) we have

\[
d(gx^1_1, gx^2_2) = d(F(x^1_0, x^2_0, \ldots, x^n_0), F(x^1_1, x^2_1, \ldots, x^n_1)) \\
\leq \theta \max \left\{ \begin{array}{l}
d(gx^1_0, gx^1_1), \\
n \vdots \\
d(gx^n_0, gx^n_1) \\
\end{array} \right\}
\]

\[
= \theta \max \left\{ \begin{array}{l}
d(gx^1_0, gx^1_1), \\
n \vdots \\
d(gx^n_0, gx^n_1) \\
\end{array} \right\}.
\]

Thus

\[
\max \left\{ \begin{array}{l}
d(gx^1_1, gx^2_2), \\
n \vdots \\
d(gx^n_1, gx^2_2) \\
\end{array} \right\} \leq \theta \max \left\{ \begin{array}{l}
d(gx^1_0, gx^1_1), \\
n \vdots \\
d(gx^n_0, gx^n_1) \\
\end{array} \right\}.
\]

Continuing in this way, we obtain

\[
\max \left\{ \begin{array}{l}
d(gx^1_m, gx^1_{m+1}), \\
n \vdots \\
d(gx^n_m, gx^n_{m+1}) \\
\end{array} \right\} \leq \theta \max \left\{ \begin{array}{l}
d(gx^1_{m-1}, gx^1_m), \\
n \vdots \\
d(gx^n_{m-1}, gx^n_m) \\
\end{array} \right\}
\]

\[
\leq \theta^2 \max \left\{ \begin{array}{l}
d(gx^1_{m-2}, gx^1_{m-1}), \\
n \vdots \\
d(gx^n_{m-2}, gx^n_{m-1}) \\
\end{array} \right\}
\]

\[
\vdots
\]

\[
\leq \theta^m \max \left\{ \begin{array}{l}
d(gx^1_0, gx^1_1), \\
n \vdots \\
d(gx^n_0, gx^n_1) \\
\end{array} \right\}.
\]

\[
(3)
\]
For $m > l$, consider
\[
\begin{align*}
d(gx^1_m, gx^1_l) &\leq d(gx^1_l, gx^1_{l+1}) + d(gx^1_{l+1}, gx^1_{l+2}) + \cdots + d(gx^1_{m-1}, gx^1_m) \\
&\leq (\theta^l + \theta^{l+1} + \cdots + \theta^{m-1}) \max\left\{ \frac{d(gx^1_0, gx^1_1)}{d(gx^1_0, gx^1_1)}, \ldots \right\} \quad \text{from (3)} \\
&\leq \frac{\theta^l}{1 - \theta} \max\left\{ \frac{d(gx^1_0, gx^1_1)}{d(gx^1_0, gx^1_1)}, \ldots \right\} \\
&\rightarrow 0 \quad \text{as} \quad l \rightarrow \infty.
\end{align*}
\]
Hence $\{gx^1_m\}$ is a Cauchy sequence in $g(X)$. Similarly we can show that $\{gx^2_m\}, \ldots, \{gx^m_m\}$ are Cauchy sequences in $g(X)$.

Since $g(X)$ is complete, there exist $p^1, p^2, \ldots, p^n, z^1, z^2, \ldots, z^n \in X$ such that
\[
gx^1_m \rightarrow p^1 = gz^1, \quad gx^2_m \rightarrow p^2 = gz^2, \quad \ldots, \quad gx^m_m \rightarrow p^n = gz^n.
\]
(4)

Suppose (2.1.4)(a) holds, i.e $F$ and $g$ are continuous.

From Lemma 1.5, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and the mapping $g : E \rightarrow X$ is one - one. Let us define $G : [g(E)]^n \rightarrow X$ by $G(gx^1, gx^2, \ldots, gx^n) = F(x^1, x^2, \ldots, x^n)$ for all $gx^1, gx^2, \ldots, gx^n \in g(E)$.

Since $F$ and $g$ are continuous, it follows that $G$ is continuous.

Now, we have
\[
F(z^1, z^2, \ldots, z^n) = G(gz^1, gz^2, \ldots, gz^n) = \lim_{n \rightarrow \infty} G(gx^1_m, gx^2_m, \ldots, gx^m_m) = \lim_{n \rightarrow \infty} F(x^1_m, x^2_m, \ldots, x^n_m) = \lim_{n \rightarrow \infty} gx^1_{m+1} = gz^1.
\]

Similarly we have
\[
gz^2 = F(z^2, \ldots, z^n, z^1), \ldots, gz^n = F(z^n, z^1, \ldots, z^{n-1}).
\]

Thus $(z^1, z^2, \ldots, z^n)$ is a $n$-tupled coincidence point of $F$ and $g$.

Suppose (2.1.4)(b) holds.

Since $gx^1_m \neq gx^1_{m+1}$ or $gx^2_m \neq gx^2_{m+1}$ or $\cdots$ or $gx^n_m \neq gx^n_{m+1}$ for all $m$ and $gx^1_m \rightarrow gz^1, \quad gx^2_m \rightarrow gz^2, \quad \ldots, \quad gx^n_m \rightarrow gz^n$ it follows that
\[
\max\{d(gx^1_m, gz^1), d(gx^2_m, gz^2), \ldots, d(gx^n_m, gz^n)\} > 0 \quad \text{for infinitely many } m.
\]

Claim: $\max\left\{ \frac{d(gz^1, F(x^1, x^2, \ldots, x^n))}{d(gz^1, F(x^1, x^2, \ldots, x^n))}, \ldots, \frac{d(gz^n, F(x^1, x^2, \ldots, x^n))}{d(gz^n, F(x^1, x^2, \ldots, x^n))} \right\} \leq \theta \max\left\{ \frac{d(gz^1, gx^1), \ldots, d(gz^n, gx^n)}{d(gz^1, gx^n)} \right\}$

for all $x^1, x^2, \ldots, x^n \in X$ with $gz^i \not\geq gz^i$ for $i$ is odd and $gz^i \geq gz^i$ for $i$ is even and $\max\{d(gz^1, gx^1), \ldots, d(gz^n, gx^n)\} > 0$.

Let $x^1, x^2, \ldots, x^n \in X$ with $gz^i \not\geq gz^i$ for $i$ is odd and $gz^i \geq gz^i$ for $i$ is even and $\max\{d(gz^1, gx^1), \ldots, d(gz^n, gx^n)\} > 0$. 
Since \( gx^i_m \to gz^i \), for \( i = 1, 2, \ldots, n \) there exists a positive integer \( m_0 \) such that for \( m \geq m_0 \) we have

\[
\max \left\{ \frac{d(gx^1_m, gz^1), \ldots, d(gx^n_m, gz^n)}{d(gx^1_m, gzx^1), \ldots, d(gx^n_m, gzx^n)} \right\} \leq \frac{1}{6} \max \left\{ \frac{d(gz^1, gx^1), \ldots, d(gz^n, gx^n)}{d(gz^1, gzx^1), \ldots, d(gz^n, gzx^n)} \right\} \tag{5}
\]

Now for \( m \geq m_0 \), consider

\[
\eta(\theta) \min \left\{ \frac{d(gx^1_m, F(x^1_m, x^2_m, \ldots, x^n_m)), \ldots, d(gx^n_m, F(x^1_m, x^2_m, \ldots, x^n_m))}{d(gx^1_m, gzx^1), \ldots, d(gx^n_m, gzx^n)} \right\} \leq \max \left\{ \frac{d(gx^1_m, gx^1), \ldots, d(gx^n_m, gx^n)}{d(gx^1_m, gzx^1), \ldots, d(gx^n_m, gzx^n)} \right\}
\]

\[
\leq \max \left\{ \frac{d(gx^1_m, gz^1) + \ldots + d(gx^n_m, gz^n)}{d(gx^1_m, gz^1) + \ldots + d(gx^n_m, gz^n)} \right\} + \max \left\{ \frac{d(gz^1, gx^1) + \ldots + d(gz^n, gx^n)}{d(gz^1, gx^1) + \ldots + d(gz^n, gx^n)} \right\}
\]

\[
\leq \frac{5}{6} \max \left\{ \frac{d(gz^1, gx^1), \ldots, d(gz^n, gx^n)}{d(gz^1, gx^1), \ldots, d(gz^n, gx^n)} \right\} \quad \text{from (5)}
\]

\[
= \frac{5}{6} \left[ \max \left\{ d(gz^1, gx^1), \ldots, d(gz^n, gx^n) \right\} - \frac{1}{6} \max \left\{ d(gz^1, gx^1), \ldots, d(gz^n, gx^n) \right\} \right]
\]

\[
\leq \frac{5}{6} \left[ \max \left\{ d(gx^1_m, gx^1), \ldots, d(gx^n_m, gx^n) \right\} - \max \left\{ d(gx^1_m, gx^1), \ldots, d(gx^n_m, gx^n) \right\} \right] \quad \text{from (5)}
\]

\[
\leq \max \left\{ \frac{d(gz^1, gx^1) - d(gx^1, gz^1), \ldots, d(gz^n, gx^n) - d(gx^1, gz^n)}{d(gx^1, gz^1), \ldots, d(gx^n, gz^n)} \right\}
\]

\[
\leq \max \left\{ d(gx^1_m, gx^1), \ldots, d(gx^n_m, gx^n) \right\}.
\]

From (2), (4) and (2.1.4)(b), we have \( gx^i_m \leq gz^i \) if \( i \) is odd and \( gz^i \leq gx^i_m \) if \( i \) is even for all \( m \). Hence for all \( m \), we have

\[
gz^i_m \leq gz^i \leq gx^i \quad \text{for i is odd and} \quad gx^i \leq gz^i \leq gx^i_m \quad \text{for i is even.} \tag{6}
\]

Hence by (2.1.2), we get

\[
d(F(x^1_m, x^2_m, \ldots, x^n_m), F(x^1, x^2, \ldots, x^n))
\]

\[
\leq \theta \max \left\{ \frac{d(gx^1_m, gx^1), \ldots, d(gx^n_m, gx^n)}{d(gx^1_m, gzx^1), \ldots, d(gx^n_m, gzx^n)}, \frac{d(gx^1_m, gx^1), \ldots, d(gx^n_m, gx^n)}{d(gx^1_m, gzx^1), \ldots, d(gx^n_m, gzx^n)} \right\}.
\]

Letting \( m \to \infty \), we get

\[
d(gz^1, F(x^1, x^2, \ldots, x^n)) \leq \theta \max \left\{ d(gz^1, gx^1), \ldots, d(gz^n, gx^n) \right\}.
\]
Analogously we can prove that
\[
\begin{align*}
&d(gz^2, F(x^2, x^3, \cdots, x^n, x^1)) \leq \theta \max \left\{ d(gz^1, gx^1), \cdots, d(gz^n, gx^n) \right\} \\
&\vdots \\
&d(gz^n, F(x^n, x^1, \cdots, x^{n-1})) \leq \theta \max \left\{ d(gz^1, gx^1), \cdots, d(gz^n, gx^n) \right\}.
\end{align*}
\]
Thus
\[
\max \left\{ \begin{array}{c}
d(gz^1, F(x^1, x^2, \cdots, x^n)), \\
\quad \cdots \\
\quad d(gz^n, F(x^n, x^1, \cdots, x^{n-1}) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{c}
d(gz^1, gx^1), \\
\quad \cdots \\
\quad d(gz^n, gx^n) \end{array} \right\}
\] (7)

Hence the claim.

Now consider
\[
d(gx^1, F(x^1, x^2, \cdots, x^n)) \leq d(gx^1, gz^1) + d(gz^1, F(x^1, x^2, \cdots, x^n))
\]
\[
\leq d(gx^1, gz^1) + \theta \max \left\{ \begin{array}{c}
d(gz^1, gx^1), \\
\quad \cdots \\
\quad d(gz^n, gx^n) \end{array} \right\}
\]
\[
\leq (1 + \theta) \max \left\{ \begin{array}{c}
d(gx^1, gz^1), \\
\quad \cdots \\
\quad d(gz^n, gx^n) \end{array} \right\}
\]

Thus
\[
\eta(\theta)d(gx^1, F(x^1, x^2, \cdots, x^n)) \leq \max \left\{ \begin{array}{c}
d(gx^1, gz^1), \\
\quad \cdots \\
\quad d(gz^n, gx^n) \end{array} \right\}.
\]

Hence
\[
\eta(\theta) \min \left\{ \begin{array}{c}
d(gx^1, F(x^1, x^2, \cdots, x^n)), \\
\quad d(gx^1, F(x^n, x^1, \cdots, x^{n-1})) \end{array} \right\} \leq \max \left\{ \begin{array}{c}
d(gz^1, gz^1), \\
\quad \cdots \\
\quad d(gz^n, gx^n) \end{array} \right\}.
\]

Now from (2.1.2), we have
\[
d(F(x^1, x^2, \cdots, x^n), F(z^1, z^2, \cdots, z^n))
\]
\[
\leq \theta \max \left\{ \begin{array}{c}
d(gx^1, gz^1), \cdots, d(gz^n, gz^n), \\
d(gx^1, F(x^1, x^2, \cdots, x^n)), \cdots, d(gx^1, F(x^n, x^1, \cdots, x^{n-1})), \\
d(gz^1, F(x^1, x^2, \cdots, x^n)), \cdots, d(gz^n, F(x^n, x^1, \cdots, x^{n-1})) \end{array} \right\}
\] (8)

Now from (8), we obtain
\[
d(F(x^1_m, x^2_m, \cdots, x^n_m), F(z^1, z^2, \cdots, z^n))
\]
\[
\leq \theta \max \left\{ \begin{array}{c}
d(gx^1_m, gz^1), \cdots, d(gz^n_m, gz^n), \\
d(gx^1_m, gx^1_{m+1}), \cdots, d(gz^n_m, gx^1_{m+1}), \\
d(gz^1, gx^1_{m+1}), \cdots, d(gz^n, gx^1_{m+1}) \end{array} \right\}.
\]
Letting $m \to \infty$, we get
\[ d(gz^1, F(z^1, z^2, \ldots, z^n)) \leq 0 \] so that \( gz^1 = F(z^1, z^2, \ldots, z^n) \).

Analogously, we can show that \( gz^2 = F(z^2, z^3, \ldots, z^n, z^1) \), \ldots, \( gz^n = F(z^n, z^1, \ldots, z^{n-1}) \).

Thus \((z^1, z^2, \ldots, z^n)\) is a $n$-tupled coincidence point of $F$ and $g$.

**Theorem 2.2** In addition to the hypotheses of Theorem 2.1, suppose that for any \((x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n\), there exists \((u^1, u^2, \ldots, u^n) \in X^n\) such that \( (F(u^1, u^2, \ldots, u^n), F(u^2, u^3, \ldots, u^n, u^1), \ldots, F(u^n, u^1, \ldots, u^{n-1})) \)

is comparable with \( (F(x^1, x^2, \ldots, x^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, \ldots, x^{n-1})) \) and

\( (F(y^1, y^2, \ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(y^n, y^1, \ldots, y^{n-1})) \). Further more assume that $F$ and $g$ are $W$-compatible, then $F$ and $g$ have a unique $n$-tupled common fixed point.

**Proof.** From Theorem 2.1, the set of $n$-tupled coincidence points of $F$ and $g$ is non-empty.

Let \((x^1, x^2, \ldots, x^n)\) and \((y^1, y^2, \ldots, y^n)\) be two $n$-tupled coincidence points of $F$ and $g$. That is
\[
F(x^1, x^2, \ldots, x^n) = gx^1, F(y^1, y^2, \ldots, y^n) = gy^1,
F(x^2, x^3, \ldots, x^n, x^1) = gx^2, F(y^2, y^3, \ldots, y^n, y^1) = gy^2,
\vdots
F(x^n, x^1, \ldots, x^{n-1})) = gx^n, F(y^n, y^1, \ldots, y^{n-1})) = gy^n.
\]

Now we shall show that
\[
gx^1 = gy^1, gx^2 = gy^2, \ldots, gx^n = gy^n. \tag{9}
\]

By the assumption, there exists \((u^1, u^2, \ldots, u^n) \in X \times X\) such that
\( (F(u^1, u^2, \ldots, u^n), F(u^2, u^3, \ldots, u^n, u^1), \ldots, F(u^n, u^1, \ldots, u^{n-1})) \)

is comparable with \( (F(x^1, x^2, \ldots, x^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, \ldots, x^{n-1})) \) and

\( (F(y^1, y^2, \ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(y^n, y^1, \ldots, y^{n-1})) \).

Put \( u_0^1 = u^1, u_0^2 = u^2, \ldots, u_0^n = u^n \) and choose \( u_1^1, u_2^1, \ldots, u_n^1 \in X \) such that
\[
gu_1^1 = F(u_0^1, u_0^2, \ldots, u_0^n)
gu_2^1 = F(u_0^2, u_0^3, \ldots, u_0^n, u_0^1)
\vdots
gu_1^n = F(u_0^n, u_0^1, \ldots, u_0^{n-1})
\]

As in in the proof of Theorem 2.1, we can define the sequences \( \{u_m^1\}, \{u_m^2\}, \ldots, \{u_m^n\} \) such that
\[
\begin{align*}
gu_m^1 &= F(u_{m-1}^1, u_{m-1}^2, \ldots, u_{m-1}^n) \\
gu_m^2 &= F(u_{m-1}^2, u_{m-1}^3, \ldots, u_{m-1}^n, u_{m-1}^1) \\
&\vdots \\
gu_m^n &= F(u_{m-1}^n, u_{m-1}^1, \ldots, u_{m-1}^{n-1}) \text{ for } m \geq 1.
\end{align*}
\]
Further, set \( x_0^1 = x^1, x_0^2 = x^2, \ldots, x_0^n = x^n \) and \( y_0^1 = y^1, y_0^2 = y^2, \ldots, y_0^n = y^n \) in the same way, we define the sequences \( \{gx_m^1\}, \{gx_m^2\}, \ldots, \{gx_m^n\} \) and \( \{gy_m^1\}, \{gy_m^2\}, \ldots, \{gy_m^n\} \) by

\[
gx_m^1 = F(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n), \quad gy_m^1 = F(y_{m-1}^1, y_{m-1}^2, \ldots, y_{m-1}^n),
\]

\[
gx_m^2 = F(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^n), \quad gy_m^2 = F(y_{m-1}^2, y_{m-1}^3, \ldots, y_{m-1}^n),
\]

\[
\vdots
\]

\[
gx_m^n = F(x_{m-1}^n, x_{m-1}^{n-1}, \ldots, x_{m-1}^1), \quad gy_m^n = F(y_{m-1}^n, y_{m-1}^{n-1}, \ldots, y_{m-1}^1).
\]

Without loss of generality assume that

\[
(F(x^1, x^2, \ldots, x^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, \ldots, x^{n-1})) \leq (F(u^1, u^2, \ldots, u^n), F(u^2, u^3, \ldots, u^n, u^1), \ldots, F(u^n, u^1, \ldots, u^{n-1}))
\]

and

\[
(F(y^1, y^2, \ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(y^n, y^1, \ldots, y^{n-1})) \leq (F(u^1, u^2, \ldots, u^n), F(u^2, u^3, \ldots, u^n, u^1), \ldots, F(u^n, u^1, \ldots, u^{n-1})).
\]

Then we have \( gx^i \preceq gu^i_m \) for \( i \) is odd and \( gx^i \succeq gu^i_m \) for \( i \) is even.

As in Theorem 2.1, we have \( gu^i_m \preceq gu^i_{m+1} \) for \( i \) is odd and \( gu^i_m \succeq gu^i_{m+1} \) for \( i \) is even for all \( m \).

Hence \( gx^i \preceq gu^i_m \) for \( i \) is odd and \( gx^i \succeq gu^i_m \) for \( i \) is even for all \( m \).

Since

\[
\eta(\theta) \min \left\{ \begin{array}{c}
d(gx^1, F(x^1, x^2, \ldots, x^n)), \\
\vdots \\
d(gx^n, F(x^n, x^1, \ldots, x^{n-1})
\end{array} \right\} = 0 \leq \max \left\{ \begin{array}{c}
d(gx^1, gu^1_m), \\
\vdots \\
d(gx^n, gu^n_m)
\end{array} \right\}.
\]

We have by (2.1.2) that

\[
d(F(x^1, x^2, \ldots, x^n), F(u^1_m, u^2_m, \ldots, u^n_m)) 
\leq \theta \max \left\{ \begin{array}{c}
d(gx^1, gu^1_m), \ldots, d(gx^n, gu^n_m), \\
d(gx^1, F(x^1, x^2, \ldots, x^n)), \ldots, d(gx^n, F(x^n, x^1, \ldots, x^{n-1})) \\
d(gu^1_m, F(x^1, x^2, \ldots, x^n)), \ldots, d(gu^n_m, F(x^n, x^1, \ldots, x^{n-1}))
\end{array} \right\}
\]

which implies that

\[
d(gx^1, gu^1_{m+1}) \leq \theta \max \left\{ \begin{array}{c}
d(gx^1, gu^1_m), \ldots, d(gx^n, gu^n_m), \\
0, \ldots, 0 \\
d(gu^1_m, gx^1), \ldots, d(gu^n_m, gx^n)
\end{array} \right\}
\]

\[
= \theta \max \{d(gx^1, gu^1_m), \ldots, d(gx^n, gu^n_m)\}.
\]

Similarly, for \( i = 2, 3, \ldots, n \) we can we show that

\[
d(gx^i, gu^i_{m+1}) \leq \theta \max \{d(gx^1, gu^1_m), \ldots, d(gx^n, gu^n_m)\}.
\]
Thus
\[
\max \left\{ \frac{d(gx^1, gu^1_m)}{d(gx^n, gu^n_m)}, \ldots, \frac{d(gx^n, gu^n_m)}{d(gx^n, gu^n_m)} \right\} \leq \theta \max \left\{ \frac{d(gx^1, gu^1_m)}{d(gx^n, gu^n_m)} \right\}. \tag{11}
\]

Let \( r_m = \max \{d(gx^1, gu^1_m), \ldots, d(gx^n, gu^n_m)\} \).

Then from (11), we have \( r_{m+1} \leq \theta r_m \).

Hence \( r_{m+1} \leq \theta r_m \leq \theta^2 r_{m-1} \leq \ldots \leq \theta^m r_0 \to 0 \) as \( m \to \infty \).

Hence
\[
\lim_{m \to \infty} d(gx^i, gu^i_m) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{12}
\]

Similarly, we can show that
\[
\lim_{m \to \infty} d(gy^i, gu^i_m) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{13}
\]

Hence \( gx^i = gy^i \) for \( i = 1, 2, \ldots, n \).

Thus (9) is proved.

Since \( gx^1 = F(x^1, x^2, \ldots, x^n), gx^2 = F(x^2, x^3, \ldots, x^n, x^1), \ldots, gx^n = F(x^n, x^1, \ldots, x^{n-1}) \), by \( W \)-compatibility of \( F \) and \( g \), we have
\[
g(gx^1) = g(F(x^1, x^2, \ldots, x^n)) = F(gx^1, gx^2, \ldots, gx^n),
g(gx^2) = g(F(x^2, x^3, \ldots, x^n, x^1)) = F(gx^2, gx^3, \ldots, gx^n, gx^1),
\vdots
\]
\[
g(gx^n) = g(F(x^n, x^1, \ldots, x^{n-1})) = F(gx^n, gx^1, \ldots, gx^{n-1}),
\]

Denote \( gx^1 = z^1, gx^2 = z^2, \ldots, gx^n = z^n \) Then
\[
gz^1 = F(z^1, z^2, \ldots, z^n),
gz^2 = F(z^2, z^3, \ldots, z^n, z^1),
\vdots
\]
\[
gz^n = F(z^n, z^1, \ldots, z^{n-1}),
\]

Thus \( (z^1, z^2, \ldots, z^n) \) is a \( n \)-tupled coincidence point of \( F \) and \( g \). Then from (9), we have \( gx^1 = gz^1, gx^2 = gz^2, \ldots, gx^n = gz^n \) so that
\[
z^1 = gz^1, z^2 = gz^2, \ldots, z^n = gz^n. \tag{15}
\]

Now by (14) and (15), we conclude that \( (z^1, z^2, \ldots, z^n) \) is a \( n \)-tupled common fixed point of \( F \) and \( g \).

To prove the uniqueness of \( n \)-tupled common fixed point of \( F \) and \( g \), assume that \( (s^1, s^2, \ldots, s^n) \) is another \( n \)-tupled common fixed point of \( F \) and \( g \).

Then from (9), we have \( gz^1 = gs^1, gz^2 = gs^2, \ldots, gz^n = gs^n \) which yields that \( z^1 = s^1, z^2 = s^2, \ldots, z^n = s^n \).

Hence \( (z^1, z^2, \ldots, z^n) \) is the unique \( n \)-tupled common fixed point of \( F \) and \( g \).

Now we illustrate Theorem 2.2 with an example when \( n = 4 \).
Example 2.3 Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Let us define $\preceq$ by ordering $\leq$.

Define $F : X^4 \to X$ and $g : X \to X$ by

$$F(x^1, x^2, x^3, x^4) = \frac{x^1 - 2x^2 + 3x^3 - 4x^4}{64}, \quad gx = \frac{x}{4}.$$ 

Then for $(x^1, x^2, x^3, x^4), (y^1, y^2, y^3, y^4)$ in $X^4$, we have

$$d(F(x^1, x^2, x^3, x^4), F(y^1, y^2, y^3, y^4)) = \frac{|x^1 - 2x^2 + 3x^3 - 4x^4 - y^1 - 2y^2 + 3y^3 - 4y^4|}{64} \leq \frac{1}{16} \left[ \frac{|x^1 - y^1|}{4} + 2 \frac{|x^2 - y^2|}{4} + 3 \frac{|x^3 - y^3|}{4} + 4 \frac{|x^4 - y^4|}{4} \right]$$

$$= \frac{1}{16} \left[ d(gx^1, gy^1) + 2d(gx^2, gy^2) + 3d(gx^3, gy^3) + 4d(gx^4, gy^4) \right]$$

$$\leq \frac{5}{8} \max \left\{ d(gx^1, gy^1), d(gx^2, gy^2), d(gx^3, gy^3), d(gx^4, gy^4) \right\}.$$ 

Thus (2.1.2) is satisfied with $\theta = \frac{5}{8}$ and $\eta(\theta) = \frac{8}{13}$. Clearly $F$ and $g$ are $W$-compatible. One can easily verify the remaining conditions of Theorem 2.2. Clearly $(0, 0, 0, 0)$ is a $n$-tupled unique common fixed point of $F$ and $g$.

References


[16] V. Berinde and M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, *Nonlinear Analysis, Theory, Methods and Applications*, 70(12) (2009), 4341-4349.