A-Quasi Normal Operators in Semi Hilbertian Spaces

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Abstract

In this paper we introduce the concept of A-quasinormal operators acting on semi Hilbertian spaces $H$ with inner product $\langle \cdot, \cdot \rangle_A$. The object of this paper is to study conditions on $T$ which imply A-quasi normality. If $S$ and $T$ are A-quasi normal operators, we shall obtain conditions under which their sum and product are A-quasi normal.

Keywords: A-adjoint, A-Normal, Semi inner product, and Moore-Penrose inverse and quasinormal.

1 Introduction

Throughout this paper $H$ denotes a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. $L(H)$ stands the Banach algebra of all bounded linear operators
on $H.I=I_H$ being the identity operator and if $V\subset H$ is a closed subspace, $P_V$ is the orthogonal projection onto $V$.

$L(H)^+$ is the cone of positive operators,

i.e. $L(H)^+ = \{ A \in L(H) : \langle Ax, x \rangle \geq 0, \forall x \in H \}$.

Any positive operator $A \in L(H)^+$ defines a positive semi-definite sesquilinear form

$$\langle \langle \cdot, \cdot \rangle \rangle_A : H \times H \to \mathbb{C}, \langle x, y \rangle_A = \langle Ax, y \rangle.$$ 

By $\| \cdot \|_A$, we denote the semi norm induced by $\langle \langle \cdot, \cdot \rangle \rangle_A$, i.e. $\| x \|_A = \frac{1}{2} \| x, x \|_A^2$. Note that $\| x \|_A = 0$ if and only if $x \in N(A)$. Then $\| \cdot \|_A$ is a norm on $H$ if and only if $A$ is an injective operator, and the semi-normed space $(L(H), \| \cdot \|_A)$ is complete if and only if $R(A)$ is closed. Moreover $\langle \langle \cdot, \cdot \rangle \rangle_A$ induces a semi norm on the subspace

$$\{ T \in L(H) : \exists c > 0, \|Tx\|_A \leq c\|x\|_A, \forall x \in H \}.$$ 

For this subspace of operators it holds

$$\| T \|_A = \sup_{x \in R(A)} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$ 

Moreover $\| T \|_A = \sup \left\{ \left| \langle Tx, y \rangle_A \right| : x, y \in H \text{ and } \|x\|_A \leq 1, \|y\|_A \leq 1 \right\}$.

For $x, y \in H$, we say that $x$ and $y$ are $A$-orthogonal if $\langle x, y \rangle_A = 0$.

The following theorem due to Douglas will be used (for its proof refer [5].)

**Theorem 1.1** Let $T, S \in L(H)$. The following conditions are equivalent.

(i) $R(S) \subset R(T)$.

(ii) There exists a positive number $\lambda$ such that $SS^* \leq \lambda TT^*$.

(iii) There exists $W \in L(H)$ such that $TW = S$.

From now on, $A$ denotes a positive operator on $H$ (i.e. $A \in L(H)^+$).

**Definition 1.2** Let $T \in L(H)$, an operator $W \in L(H)$ is called an $A$-adjoint of $T$ if $\langle Tu, v \rangle_A = \langle u, Wv \rangle_A$ for every $u, v \in H$, or equivalently $AW = T^*A$, $T$ is called $A$-selfadjoint if $AT = T^*A$ and $T$ is called $A$-positive if $AT$ is positive.

By Douglas Theorem, an operator $T \in L(H)$ admits an $A$-adjoint if and only if $R(T^*A) \subset R(A)$ and if $W$ is an $A$-adjoint of $T$ and $AZ = 0$ for some $Z \in L(H)$ then
$W + Z$ is also an $A$-adjoint of $T$. Hence neither the existence nor the uniqueness of an $A$-adjoint operator is guaranteed. In fact an operator $T \in L(H)$ may admit none, one or many $A$-adjoints.

From now on, $L_A(H)$ denotes the set of all $T \in L(H)$ which admit an $A$-adjoint, i.e.,

$$L_A(H) = \left\{ T \in L(H) : R(T^*A) \subset R(A) \right\}$$

$L_A(H)$ is a subalgebra of $L(H)$ which is neither closed nor dense in $L(H)$.

On the other hand the set of all $A$-bounded operators in $L(H)$ (i.e. with respect the semi norm $\| \cdot \|_A$ is

$$L_A^1(H) = \left\{ T \in L(H) : \frac{1}{2} R(A^\frac{1}{2}) \subset R(A^\frac{1}{2}) \right\} = \left\{ T \in L(H) : R(A^\frac{1}{2}T^*A^\frac{1}{2}) \subset R(A) \right\}$$

Note that $L_A(H) \subset L_A^1(H)$, which shows that if $T$ admits an $A$-adjoint then it is $A$-bounded.

If $T \in L(H)$ with $R(T^*A) \subset R(A)$, then $T$, admits an $A$-adjoint operator. Moreover there exists a distinguished $A$-adjoint operator of $T$, namely, the reduced solution of the equation $ATX = T^*$, i.e. $T^* = A^*T^*A$, where $A^*$ is the Moore-Penrose inverse of $T$. The $A$-adjoint operator $T^*$ verifies

$$AT^* = T^*A, R(T^*) \subset \overline{R(A)} \text{ and } N(T^*) = N(T^*A).$$

In the next we give some important properties of $T^*$ without proof (refer [3], [4] and [5]).

**Theorem 1.3** Let $T \in L_A(H)$. Then

1. If $AT = TA$ then $T^* = PT^*$.
2. $T^*T$ and $TT^*$ are $A$-self adjoint and $A$-positive.
3. $\|T\|_A^2 = \|T^*\|_A^2 = \|T^*T\| = \|TT^*\|$
4. $\|S\|_A = \|T^*\|_A$ for every $S \in L(H)$ which is an $A$-adjoint of $T$.
5. If $S \in L_A(H)$ then $ST \in L_A(H)$, $(ST)^* = T^*S^*$ and $\|ST\|_A = \|ST\|_A$.
6. $T^* \in L_A(H), (T^*)^* = PTP$ and $(T^*)^* = T^*$.

**Definition 1.4** An operator $T \in L_A(H)$ is called $A$-normal if $T^*T = TT^*$ (for more details refer [1]).
2  \textbf{A-Quasinormal Operators}

\textbf{Definition 2.1} An operator $T \in L_A(H)$ is called $A$-quasinormal if $T$ commutes with $T^*T$ i.e. $T(T^*T) = (T^*T)T$.

Let $T = U + V \in L_A(H)$ where $U = T + T^* \frac{2}{2}$ and $V = T - T^* \frac{2}{2}$. We shall write $B = TT^*$ and $C = T^*T$ where $B$ and $C$ are non-negative definite. We give necessary and sufficient conditions for an operator to be $A$-quasinormal \cite{2} and \cite{6}.

\textbf{Theorem 2.2} $T$ is $A$-quasinormal with $N(A)$ is invariant subspace for $T$ if and only if $C$ commutes with $U$ and $V$.

\textbf{Proof.} Since $N(A)$ is invariant subspace for $T$ we observe that $PT = TP$ and $T^*P = PT^*$.

Let $T$ be $A$-quasinormal then

\begin{align*}
T(T^*T) &= (T^*T)T \\
T^*T^*T^* &= T^*T^*T^* \\
T^*PTPT^* &= T^*T^*PTP \\
PT^*PTT^* &= T^*PT^*PT \\
T^*TT^* &= T^*T^*T \\
\text{Hence} T^*TT^* &= T^*T^*T.
\end{align*}

Now it is easy to see that $C^2 = UC^2$. Since $C$ is non-negative definite, it follows that $CU = UC$. Similarly $CV = VC$.

Conversely, let $CU = UC$ and $CV = VC$. Then $C^2U = UC^2$ and $C^2V = VC^2$.

Hence $C^2T = TC^2$. Therefore $T^*T^2 = TT^*T$.

In the following theorem we give conditions under which an operator $T$ is $A$-quasinormal.

\textbf{Theorem 2.3} If $T$ is an operator such that (i) $B$ commutes with $U$ and $V$ (ii) $C^2T = TB^2$. Then $T$ is $A$-quasinormal.

\textbf{Proof.} Since $BU = UB$ and $BV = VB$ we have $B^2U = UB^2$ and $B^2V = VB^2$.

Then $B^2T + B^2T^* = TB^2 + T^*B^2$

$B^2T - B^2T^* = TB^2 - T^*B^2$

This gives $B^2T = TB^2 = C^2T$. Hence $T$ is $A$-quasinormal.
**Theorem 2.4** Let $T$ be $A$-quasi normal, $C^2 T = TB^2$ and $N(A)$ be an invariant subspace for $T$. Then $B$ commutes with $U$ and $V$.

**Proof.** Since $C^2 T = TB^2$ we have $T^* T^2 = T^2 T^*$. Hence $T^* T = TT^*$. Since $T$ is $A$-quasi normal we have

$$B^2 U = \frac{TT^* T + TT^* T^2}{2} = \frac{T^* T^2 + T^* T}{2} = \frac{T^2 T^* + T^* TT^*}{2} = \frac{T + T^*}{2} TT^* = UB^2.$$  

Hence $BU = UB$. Similarly $BV = VB$.

**Theorem 2.5** Let $S$ and $T$ be two $A$-quasinormal operators. Then their product $ST$ is $A$-quasinormal if the following conditions are satisfied (i) $ST = TS$ (ii) $ST^* = T^* S$.

**Proof.**

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Hence $ST$ is $A$-quasinormal.

**Theorem 2.6** Let $S$ and $T$ be two $A$-quasinormal operators such that $ST = TS = S^* T = T^* S = 0$. Then $S + T$ is $A$-quasinormal.

**Proof.**

$$(S + T)(S + T)^* (S + T) = (S + T)(S^* + T^*)(S + T) = (S + T)(S^* S + S^* T + T^* S + T^* T) = (S + T)(S^* S + T^* T) = SS^* S + ST^* T + T S^* + TT^* T = S^* S^2 + T^* T^2$$
Hence $S + T$ is $A$-quasi normal.

References


