On Pebbling Jahangir Graph

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Abstract

Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles off a vertex and placing one on an adjacent vertex. The pebbling number, f(G), of a graph G is the least number m such that, however m pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves. In this paper, we determine f(G) for Jahangir graph J_{2,m}(m ≥ 8).

Keywords: Pebbling, Jahangir graph, and graph parameters.

1 Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the
literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of pebbling results in [3]. There have been many developments since Hulbert’s survey appeared.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from V(G) to \( \mathbb{N} \cup \{0\} \) representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected.

A pebbling move (or pebbling step) is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. Define the pebbling number, \( f(G) \), to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex v, a pebble by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original configuration.

**Fact 1.1** [7,8]. For any vertex v of a graph G, \( f(v,G) \geq n \) where \( n=|V(G)| \).

**Proof.** Consider the configuration \( C:V-\{v\} \to \mathbb{N} \) defined by \( C(w)=1 \) for all \( w \in V-\{v\} \). Then the size \( |C| \) of the configuration is \( n-1 \). In this configuration we cannot move a pebble to v. Hence \( f(v,G) \geq |V(G)| \).

**Fact 1.2** [7]. The pebbling number of a graph G satisfies \( f(G) \geq \max \{2^{\text{diam}(G)}, |V(G)|\} \).

**Proof.** Let \( w \in V(G) \) be a vertex at a distance \( \text{diam}(G) \) from the target vertex v. Place \( 2^{\text{diam}(G)}-1 \) pebbles at w. Clearly we cannot move any pebble to v. Thus \( f(G) \geq \max \{2^{\text{diam}(G)}, |V(G)|\} \).

There are few other interesting results in the pebbling number of graphs. Hulbert [3] has written an excellent survey article on pebbling. We earnestly request the interested readers to refer to it for further study.

We also request the readers to read [2] in which Moews has studied the pebbling number of product of trees.

A. Lourdusamy, S. Samuel Jayaseelan and T. Mathivanan conjectured in [5] that the pebbling number of Jahangir graph \( J_{2,m} \) is \( 2m+10 \). We prove this result in this paper. For \( 3 \leq m \leq 7 \), we state the results from [5].

**Theorem 1.1**[5] *For the Jahangir graph \( J_{2,3} \), \( f(J_{2,3}) = 8 \).*

**Theorem 1.2**[5] *For the Jahangir graph \( J_{2,4} \), \( f(J_{2,4}) = 16 \).*

**Theorem 1.3**[5] *For the Jahangir graph \( J_{2,5} \), \( f(J_{2,5}) = 18 \).*

**Theorem 1.4**[5] *For the Jahangir graph \( J_{2,6} \), \( f(J_{2,6}) = 21 \).*

**Theorem 1.5**[5] *For the Jahangir graph \( J_{2,7} \), \( f(J_{2,7}) = 23 \).*
We now proceed to determine the pebbling number for $J_{2,m}$ ($m \geq 8$).

2  Pebbling Number of Jahangir Graph $J_{2,m}$ ($m \geq 8$)

**Definition 2.1** [6] Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at distance $n$ to each other on $C_{nm}$.

**Example 2.2** Figure 1 shows Jahangir graph $J_{2,8}$. The Figure 1 appears on Jahangir’s tomb in his mausoleum. It lies in 5 kilometer north-west of Lahore, Pakistan, across the River Ravi.

**Remark 2.3** Let $v_{2m+1}$ be the label of the center vertex and $v_1, v_2, \ldots, v_{2m}$ be the label of the vertices that are incident clockwise on cycle $C_{2m}$ so that $\deg(v_1) = 3$.

**Theorem 2.4** For the Jahangir graph $J_{2,m}$ ($m \geq 8$), $f(J_{2,m}) = 2m + 10$.

**Proof:** If $m$ is even, then consider the following configuration $C_1$ such that $C_1(v_2)=0$, $C_1(v_{m+2})=15$, $C_1(v_{m-2})=3$, $C_1(v_{m+6})=3$, and $C_1(x)=1$ where $x \not\in N[v_2]$, $x \not\in N[v_{m-2}]$, and $x \not\in N[v_{m+6}]$.

If $m$ is odd, then consider the following configuration $C_2$ such that $C_2(v_2)=0$, $C_2(v_{m+1})=15$, $C_2(v_{m-3})=3$, $C_2(v_{m+5})=3$, and $C_2(x)=1$ where $x \not\in N[v_2]$, $x \not\in N[v_{m+1}]$, $x \not\in N[v_{m-3}]$, and $x \not\in N[v_{m+5}]$.

Then, we cannot move a pebble to $v_2$. The total number of pebbles placed in both configurations is

$$15 + 2(3) + (m - 4)(1) + (m - 8)(1)$$

$$= 2m + 9.$$  

Therefore, $f(J_{2,m}) \geq 2m + 10$.
Now, consider the distribution of $2m + 10$ pebbles on the vertices of $J_{2,m}$. Consider the sets
\[ S_1 = \{ v_1, v_3, v_5, \ldots, v_{2m-3}, v_{2m-1} \} \quad \text{and} \quad S_2 = \{ v_2, v_4, v_6, \ldots, v_{2m-2}, v_{2m} \} \]

**Case (i)** Let the target vertex be $v_{2m+1}$.
Suppose we cannot move a pebble to the target vertex. Then the vertices of $S_1$ contain at most one pebble each and the vertices of $S_2$ contain at most three pebbles each. Also, note that, no two consecutive vertices of $S_2$ contain two or three pebbles each and if a vertex of $S_2$ contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Also, if a vertex of $S_1$ contains a pebble then the neighbors of that vertex contain at most one pebble each. Thus, in any distribution, $J_{2,m}$ contains at most $2m$ pebbles so that a pebble could not be moved to $v_{2m+1}$—a contradiction to the total number of pebbles placed over the vertices of $J_{2,m}$. Hence we can move a pebble to the target vertex.

**Case (ii)** Let the target vertex be a vertex of $S_1$. Without loss of generality, let $v_1$ be the target vertex.
Suppose we cannot move a pebble to the target vertex. Then the neighbors $v_2$, $v_{2m}$, and $v_{2m+1}$ of $v_1$ contain at most one pebble each. The vertices of $S_1 - \{ v_1 \}$ contain at most three pebbles each and the vertices of $S_2 - \{ v_2, v_{2m} \}$ contain at most seven pebbles each. Also, note that, no two vertices of $S_1 - \{ v_1 \}$ contain two or three pebbles each and if a vertex of $S_1 - \{ v_1 \}$ contains two or three pebbles then the neighbors of that vertex contain at most three pebbles each. Also, no two vertices of $S_2 - \{ v_2, v_{2m} \}$ contain four or more pebbles each and if a vertex of $S_2 - \{ v_2, v_{2m} \}$ contains four or more pebbles then the neighbors of that vertex contain at most one pebble each.

Suppose $v_{2m+1}$ has a pebble on it. Then clearly the vertices of $S_1 - \{ v_1 \}$ contain at most one pebble each and if a vertex of $S_1 - \{ v_1 \}$ contains a pebble then the neighbors of that vertex contain at most one pebble each. Also, the vertices of $S_2 - \{ v_2, v_{2m} \}$ contain at most three pebbles each and if a vertex of $S_2 - \{ v_2, v_{2m} \}$ contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Thus in any distribution, $J_{2,m}$ contains at most $(m-1)+(m-2)+1+1+1=2m$ pebbles so that a pebble could not be moved to $v_1$—a contradiction to the total number of pebbles placed over the vertices of $J_{2,m}$. So, assume that $v_{2m+1}$ has zero pebbles on it.

Suppose $v_2$ has a pebble on it. Then the vertex $v_3$ contains at most one pebble and the vertex $v_4$ contains at most three pebbles. If one of the vertices of $S_1 - \{ v_1, v_3 \}$, say $v_9$, contains two or three pebbles, then the vertices of $S_1 - \{ v_1, v_3, v_9 \}$ contain at most one pebble each. Thus, the path $v_8v_9v_{10}$ contains at most six pebbles. Also, note that, no two consecutive vertices of $S_2 - \{ v_2, v_{2m} \}$ contain two or three pebbles each. Also, the path $v_4v_3v_2v_1v_{2m}$ contains at most five pebbles. Clearly, the vertices of $S_1 - \{ v_1, v_3, v_7, v_9, v_{11}, v_{13} \}$ contains at most one pebble each and the vertices of $S_2 - \{ v_2, v_4, v_8, v_{10}, v_{12}, v_{2m} \}$ contain at most three pebbles each. This implies that the vertices of $S_2 - \{ v_2, v_4, v_8, v_{10}, v_{12}, v_{2m} \}$ contain at least $2m+10-(m-6+1+6+1+3+1)=m+4=(m-6)+10$. So we can move a pebble to $v_1$ easily. If we
cannot move a pebble to \( v \) then the graph \( J_{2,m} \) contains at most \( 2m+1 \) pebbles—a contradiction to the total number of pebbles placed. So assume that all the vertices of \( S_2^{1-\{v_1, v_3\}} \) contain at most one pebble each. If \( v_{10} \) contains four or more (at most seven) pebbles, then other vertices of \( S_2^{2-\{v_2, v_{10}, v_{2m}\}} \) contain at most three pebbles each and if a vertex of \( S_2^{2-\{v_2, v_{10}, v_{2m}\}} \) contains two or three pebbles then the neighbors of that vertex contain zero pebbles. Suppose, we cannot move a pebble to \( v \) then the graph \( J_{2,m} \) contains at most \( 1+3+1+7+3 = 12+2(m-4) = 2m-1 \) pebbles—a contradiction. So, assume that \( v_{10} \) contains at most three pebbles. Here also if we cannot move a pebble to \( v \) then the total number of pebbles placed over the vertices of the graph \( J_{2,m} \) is at most \( 3+3+3+3 = 2m+3 \). This is a contradiction to the total number of pebbles placed. Thus, we can always move a pebble to \( v \).
**Case (iii)** Let the target vertex be a vertex of $S_2$. Without loss of generality, let $v_2$ be the target vertex. Suppose we cannot move a pebble to the target vertex. Then, the neighbors $v_1$, and $v_3$ of $v_2$ contain at most one pebble each and the vertices $v_4$, $v_{2m}$, and $v_{2m+1}$ contain at most three pebbles each. The vertices of $S_1 - \{v_1, v_3\}$ contain at most seven pebbles each and the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ contain at most fifteen pebbles each. Also, note that, no two vertices of $S_1 - \{v_1, v_3\}$ contain four or more pebbles each and if a vertex of $S_1 - \{v_1, v_3\}$ contains four or more pebbles then the neighbors of that vertex contain at most seven pebbles each. Also, no two vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ contain eight or more pebbles each and if a vertex of $S_2 - \{v_2, v_4, v_{2m}\}$ contains eight or more pebbles then the neighbors of that vertex contain at most three pebbles each.

Suppose $v_3$ has a pebble on it. Clearly the neighbors $v_4$, and $v_{2m+1}$ of $v_3$ contain at most one pebble each and the path $v_3v_4v_5$ contains at most four pebbles. If $v_{2m+1}$ has a pebble on it, then the vertices of $S_1 - \{v_1, v_3\}$ contain at most one pebble each and the vertices of $S_2 - \{v_2, v_4\}$ contain at most three pebbles each. Also, note that, no two consecutive vertices of $S_2 - \{v_2, v_4\}$ contain two or three pebbles each and if a vertex of $S_2 - \{v_2, v_4\}$ contains two or three pebbles then the neighbors of that vertex contain zero pebbles. This implies that the vertices of $S_2 - \{v_2, v_4\}$ contain at least $2m+10-(m-3+3+1+1+4)=m-4+7$ pebbles. If we cannot move a pebble to $v_2$ easily. If we cannot move a pebble to $v_2$ then the total number of pebbles placed over the vertices of the graph $J_{2,m}$ is at most $4+1+3\left(\frac{m-2}{2}\right) + \frac{m-2}{2} = 2m+4$ - a contradiction. So, assume that $v_{2m+1}$ has zero pebbles on it. Suppose that, a vertex of $S_1 - \{v_1, v_3\}$, say $v_9$, contains two or three pebbles on it. Then, the path $v_8v_9v_{10}$ contains at most six pebbles. Clearly the vertices of $S_2 - \{v_2, v_4, v_8, v_{10}\}$ contain at least $2m+10-(m-3+3+1+1+4)=m-4+7$ pebbles. If one of the vertices from $S_2 - \{v_2, v_4, v_8, v_{10}\}$ contains four or more pebbles then we can move a pebble to $v_2$ through $v_{2m+1}$ and $v_3$. So assume that all the vertices of $S_2 - \{v_2, v_4, v_8, v_{10}\}$ contain at most three pebbles. Thus, we can move a pebble to $v_2$ easily. If not, then the total number of pebbles on the vertices of the graph $J_{2,m}$ is at most $6+3+3\left(\frac{m-4}{2}\right) + \frac{m-4}{2}=2m+1$ so that a pebble could not be moved to $v_2$-a contradiction to the total number of pebbles placed. So, assume that the vertices of $S_1 - \{v_1, v_3\}$ contain at most one pebble each. Suppose that, a vertex of $S_2 - \{v_2, v_4\}$, say $v_{10}$ has four or more pebbles on it. Then the path $v_9v_{10}v_{11}$ contains at most seven pebbles. Also, note that, no two vertices of $S_2 - \{v_2, v_4, v_{10}\}$ contain two or three pebbles each and if a vertex of $S_2 - \{v_2, v_4, v_{10}\}$ of contain two or three pebbles then the neighbors of that vertex contain zero pebbles. If we cannot move a pebble to $v_2$, then the total number of pebbles on the vertices of the graph $J_{2,m}$ is at most $7+1+3\left(\frac{m-2}{2}\right) + \frac{m-2}{2}=8+2m-4=2m+4$ -a contradiction. So, assume that the vertices of $S_2 - \{v_2, v_4\}$ contain at most three pebbles each. Clearly, we can move a pebble to $v_2$. Otherwise, the total number of
pebbles on the vertices of $J_{2,m}$ is at most $3+3+3+2+3+2m-10=2m+1$ - a contradiction. Thus, assume that $v_3$ has zero pebbles on it. In a similar way, we may assume that $v_1$ has zero pebbles on it.

Suppose $v_{2m+1}$ has two or three pebbles on it. Clearly, the vertices $v_4$ and $v_{2m}$ contain at most one pebble each and one of the vertices of $S_1-\{v_1, v_3\}$ contain two or three pebbles. If $v_7$ has two or three pebbles, then the path $v_6v_7v_8$ contains at most six pebbles. Also, all the vertices of $S_2-\{v_2, v_4, v_6, v_8, v_{2m}\}$ contain at most three pebbles each and no two consecutive vertices of $S_2-\{v_2, v_4, v_{2m}\}$ contain two or three pebbles each. If we cannot move a pebble to $v_2$, then the total number of pebbles on the vertices of $J_{2,m}$ is at most $1+1+3+3+2m-10=2m+1$ - a contradiction. So, assume that $v_{2m+1}$ contains at most one pebble. In a similar way, we may assume that $v_4$ and $v_{2m}$ contain at most one pebble each.

Suppose one of the vertices of $S_2-\{v_2, v_4, v_{2m}\}$, say $v_8$, contain four or more (at most seven) pebbles on it. Then the path $v_7v_8$ contains at most six pebbles. Also, all the vertices of $S_2-\{v_2, v_4, v_{2m}\}$ contain at most three pebbles each and no two consecutive vertices of $S_2-\{v_2, v_4, v_{2m}\}$ contain two or three pebbles each. If we cannot move a pebble to $v_2$, then the total number of pebbles on the vertices of $J_{2,m}$ is at most $1+1+6+3+2m-10=2m+4$ - a contradiction. So, assume that the vertices of $S_1-\{v_1, v_3\}$ contain at most one pebble each.

Suppose that, a vertex of $S_2-\{v_2, v_4, v_{2m}\}$, say $v_8$, has eight or more (at most fifteen) pebbles on it. Then the path $v_7v_8v_9$ contains at most fifteen pebbles. If any
two vertices of $S_2 - \{v_2, v_4, v_8, v_{2m}\}$ contain four or more pebbles (at most seven) or if a vertex of $S_2 - \{v_2, v_4, v_8, v_{2m}\}$ contains four or more (at most seven) and both the neighbors contain one pebble each then we can move a pebble to $v_2$ easily. If we cannot move a pebble to $v_2$, then the total number of pebbles on the vertices of $J_{2,m}$ is at most $15 + 1 + 1 + 3 \left( \frac{m-4}{2} \right) + \frac{m-4}{2} = 2m + 9$ - a contradiction. So, assume that the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ contain at most seven pebbles each. Clearly, at most three vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ contain four or more pebbles. If we cannot move a pebble to $v_2$, then the total number of pebbles on the vertices of $J_{2,m}$ contains at most $4 + 4 + 4 + 1 + 1 + 3 \left( \frac{m-5}{2} \right) + \left( \frac{m-5}{2} \right) = 14 + 2m - 10 = 2m + 4$ pebbles - a contradiction to the total number of pebbles placed. Thus, we can always move a pebble to $v_2$ while using $2m + 10$ pebbles on the vertices of $J_{2,m}$.

Thus, $f(J_{2,m}) \leq 2m + 10$.

Therefore, $f(J_{2,m}) = 2m + 10$.

References

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