Frames, Riesz Bases and Double Infinite Matrices

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(Received 01.11.2010, Accepted 15.11.2010)

Abstract. In this paper we have used double infinite matrix $A = (a_{iljk})$ of real numbers to define the $A$-frame. Some results on Riesz basis and $A$-frame also have been studied. This Work is motivated from the work of Moricz and Rhoades [7].

2001 AMS Classification. Primary 41A17, Secondary 42C15.

Keywords and phrases. Bessel sequence, moment sequence, double infinite matrix and $A$-frame.

1 Introduction

Let $U(F)$ and $V(F)$ be finite dimensional vector spaces over the field $F$ of dimension $n$. The elements $(y_1, \ldots, y_n) \in V$ and $(e_1, e_2, \cdots, e_n)$ is an ordered basis in $U$. Then there exists a unique linear transformation such that

$$Te_i = y_i, \quad i = 1, \cdots, n. \quad (1.1)$$

Let us extend the transformation $T$ to linear transformation of vectors from the basis such that

$$T \left( \sum_{i=1}^{n} \alpha_i e_i \right) = \sum_{i=1}^{n} \alpha_i y_i.$$ 

It is clear from (1.1) that $T$ is completely defined because any element in $U$ can be expressed as a linear combination of basis vectors uniquely. Also, if $U$ is $n$–dimensional and $V$ is $m$–dimensional then the class of all linear
transformation from $U \rightarrow V$ be $nm$–dimensional.

Let an ordered bases in $U$ and $V$ be $\{e_j\}_{j=1}^n$ and $\{e_i\}_{i=1}^m$ respectively. Then the set of all linearly independent $[a_{ij}].(i = 1, \ldots, m, j = 1, \ldots, n)$ i.e.,

$$
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & \ddots & \cdots & \\
  \vdots & \ddots & & \\
  a_{m1} & & \cdots & a_{mn}
\end{pmatrix}_{m \times n}
$$

be characterized by the mappings

$$
a_{ij}e_k = \delta_{jk}e_i \quad i = 1, \ldots, m, k, j = 1, \ldots, n.
$$

Now we have the following definitions

**Definition 1.1** Let $A = (a_{iljk}), (i, l, j, k = 1, 2, \cdots)$, be a double non-negative infinite matrix of real or complex numbers. Let $(X, Y)$ denote the class of all such matrices $A$ such that the series $A(x_{il}) = \sum_{j=k=0}^{\infty} a_{iljk}x_{jk}$ converges for all double sequences $x_{jk} \in X$ and the sequence $\{A(x_{il})\}$ will be called $A$–means or $A$–transform of $x_{il}$. Also $Ax = \lim_{i,l \rightarrow \infty} A(x_{il})$, whenever it exists.

**Definition 1.2.** A double matrix $A = (a_{iljk})$ is said to be regular if the matrix transformation $A : X \rightarrow Y$ is defined on a convergent sequence to a convergent sequence and limit is preserved i.e., $\lim_{i,l \rightarrow \infty} A(x_{il}) = \lim_{i,l \rightarrow \infty} x_{il}.$

**Definition 1.3.** [7] A double matrix $A = (a_{iljk})$ is said to be regular if the following conditions holds.

(I) $\lim_{i,l \rightarrow \infty} \sum_{j,k=0}^{\infty} a_{iljk} = 1,$

(II) $\lim_{i,l \rightarrow \infty} \sum_{j=0}^{\infty} |a_{iljk}| = 0, \quad (k = 0, 1, \cdots),$  

(III) $\lim_{i,l \rightarrow \infty} \sum_{j=0}^{\infty} |a_{iljk}| = 0, \quad (j = 0, 1, \cdots),$ 

(IV) $\|A\| = \sup_{i,l>0} \sum_{j,k=0}^{\infty} |a_{il}| < \infty.$

2 **Frames**

The theory for frames and bases has developed very fast over the last 15 years. The concept of frames were introduced by Duffin and Schaeffer [5] in
the context of non-harmonic Fourier series. A sequence in a Hilbert space $H$ is a frame if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|^2 \leq \sum_n |<x, x_n>|^2 \leq C_2\|x\|^2, \ \forall x \in H. \quad (2.1)$$

Any numbers $C_1, C_2$ for which (2.1) is valid are called frames bounds. They are not unique if we can choose $C_1 = C_2$, the frame is called tight and is said to be exact if it ceases to be a frame by removing any of its elements. The theory of frames are discussed in variety of sources, including [1,3,4,6,8]. The purpose of the present paper is to define $A$-frame for an infinite double non-negative regular matrix and to study some results on $A$-frame and Riesz basis.

Let $H$ be a separable Hilbert space with inner product $<..>$ and norm $\|..\|=<(..)^{1/2}$. In the sequel $z$, and $z^+$ denote the set of integers and strictly positive integers respectively.

**Definition 2.1.** A family of elements $\{x_n, n \in z^+\} \subseteq H$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum |<f, x_n>|^2 \leq B\|f\|^2, \forall f \in H. \quad (2.2)$$

It is given [1] that $\{x_n, n \in z^+\}$ is a Bessel sequence with bound $M$ if and only if, for every finite sequence of scalors $\{c_k\}$;

$$\| \sum_k c_k x_k \|^2 \leq M \sum_k |c_k|^2. \quad (2.3)$$

Chui and Shi’s [2] remarked that $\{x_n, n \in z^+\}$ is a Bessel sequence with bound $M$ if and only if (2.3) is satisfied for every sequence $\{c_k\} \in l^2$.

In the consequence of above discussion we have the following lemma.

**Lemma 2.1.** $\{x_n, n \in z^+\}$ is a Bessel sequence if and only if

$$T: \{c_n\} \rightarrow \sum_n c_n x_n$$

is well defined operator from $l^2$ into $H$. In that case $T$ is automatically bounded, and the adjoint of $T$ is given by

$$T^*: H \rightarrow l^2, \ T^* f = \{<f, x_n>\}.$$

An important consequence of above lemma 2.1 that if $\{x_n\}$ is a Bessel sequence, then $\sum_n c_n x_n$ converges unconditionally for all $\{c_n\} \in l^2$. When
Frames, Riesz Bases and double infinite matrices

\{x_n, n \in z^+\} \subset H is a frame, the operator T and T* are well defined, so we define the frame operator

\[ S : H \to H, \quad Sf = TT^*f = \sum_n <f, x_n>x_n. \]

Two sequences \( \{x_n, n \in z^+\} \) and \( \{y_n, n \in z^+\} \) in \( H \) are called biorthogonal if \( <x_n, y_n> = \delta_{m,n} \), where \( \delta_{m,n} \) is the Kronecker delta.

To prove that \( S \) is bounded, positive and surjective we have the following theorem from [1].

Theorem A. Let \( \{x_n, n \in z^+\} \subset H \)

(a) The following are equivalent

(i) \( \{x_n, n \in z^+\} \) is a frame for \( H \) with frame bounds \( C_1 \) and \( C_2 \).

(ii) \( S : H \to H \) is a topological isomorphism with norm bounds \( \|S\| \leq C_2 \) and \( \|S\| \leq C_1^{-1} \).

(b) In case of either condition in part (a), we obtain that

\[ C_1 I \leq S \leq C_2 I \quad C_2^{-1} I \leq S^{-1} \leq C_1^{-1} I, \]

\( \{S^{-1}x_n\} \) is a frame for \( H \) with frame bounds \( C_2^{-1} \) and \( C_1^{-1} \) and for all \( x \in H \),

\[ f = SS^{-1}f = \sum_n <x, S^{-1}x_n>x_n, \quad (2.4) \]

and

\[ f = \sum_n <x, x_n>S^{-1}x_n. \quad (2.5) \]

If \( \{x_n, n \in z^+\} \) is a frame, \( S \) is called frame operator, \( \{S^{-1}x_n\} \) is called dual frame of \( \{x_n\} \), (2.4) is the frame decomposition of \( x \) and (2.5) is the dual frame decomposition of \( x \). \( I \) is the identity map, \( S \leq C_2 I \) means that \( <(C_2I - S)x, x> \geq 0 \) for each \( x \in H \).

We also have

Theorem B.[1]. Let \( \{x_n, n \in z^+\} \subset H \) be a frame for \( H \) with frame bounds \( C_1 \) and \( C_2 \). Then for each sequence \( \{C_n\} \in l^2 \) such that \( x = \sum_n C_nx_n \) converges in \( H \) and \( \|x\|^2 \leq C_2\|C\|_2^2 \) and for any arbitrary vector \( v \) there exists a moment sequence \( \{y_n, n \in z^+\} \) such that \( v = \sum_{n=1}^\infty x_ny_n \) and \( C_2^{-1}\|v\|^2 \leq \sum_{n=1}^\infty |y_n|^2 \leq C_2\|v\|^2 \).

Theorem C.[1]. A sequence \( \{x_n, n \in z^+\} \) in a Hilbert space \( H \) is an exact frame for \( H \) if and only if it is bounded unconditional basis for \( H \).
3 Main Results

Theorem 3.1. Let $A = (a_{iljk})$ be a double non-negative regular infinite matrix. Then for any $f \in L^2(R)$ the frame condition for $A$--transform of $(a_{iljk})$ is

$$C_1\|f\|^2 \leq \sum_{i,l \in \mathbb{Z}} |<f, A(\phi_{i,l})>|^2 \leq C_2\|f\|^2,$$

where $A(\phi_{i,l}) = \sum_{j,k=0}^{\infty} a_{iljk}\phi_{j,k}, \{\phi_{i,l}\}$ is a sequence of vectors and $0 < C_1 \leq C_2 < \infty$ are frame bounds.

Since $A$ is regular matrix and by the definition of $A(\phi_{i,l})$, we get

$$\sum_{i,l \in \mathbb{Z}} |<f, A(\phi_{i,l})>|^2 = \sum_{i,l \in \mathbb{Z}} \int_{-\infty}^{\infty} |f(x)|^2 |A(\phi_{i,l})|^2 dx \leq \|f\|^2 \sum_{i,l \in \mathbb{Z}} |A(\phi_{i,l})|^2 = \|f\|^2 \|A\|^2 \sum_{i,l \in \mathbb{Z}} |\phi_{i,l}|^2.$$

Now for any $f \in L^2(R)$, let

$$\tilde{f} = \left[\sum_{i,l \in \mathbb{Z}} |<f, A(\phi_{i,l})>|^2\right]^{-1/2} f,$$

or

$$<\tilde{f}, A(\phi_{i,l})> = \left[\sum_{i,l \in \mathbb{Z}} |<f, A(\phi_{i,l})>|^2\right]^{-1/2} <f, A(\phi_{i,l})> \leq 1.$$

Hence, for positive constant $\alpha$, we get

$$\|\tilde{f}\|^2 \|\phi_{i,l}\|^2 \leq \alpha,$$

or

$$\left[\sum_{i,l \in \mathbb{Z}} |<f, A(\phi_{i,l})>|^2\right]^{-1} |<f, A(\phi_{i,l})>|^2 \leq \alpha.$$
Since $A$ is regular, it gives
\[ \sum_{i,l \in \mathbb{Z}} |< f, A(\phi_{i,l})>|^2 \|f\|^2 \leq \alpha 1. \]
Thus,
\[ C_1 \|f\|^2 \leq \sum_{i,l \in \mathbb{Z}} |< f, A(\phi_{i,l})>|^2. \] (3.3)

Combining (3.2) and (3.3) the proof of theorem is immediate.

**Theorem 3.2.** \{A(\phi_{i,l})\} is a frame for any $f \in L^2(\mathbb{R})$ if and only if the mapping
\[ T : \{\beta_{i,l}\} \to \sum_{i,l \in \mathbb{Z}} \beta_{i,l} A(\phi_{i,l}) \]
is a well defined mapping from $l^2$ into $L^2(\mathbb{R})$. Here $\beta_{i,l} = <f, A(\phi_{i,l})>$ is $A$–moment sequence of $f \in L(\mathbb{R})$ relative to the frame.

**Proof.** First we shall prove that if \{A(\phi_{i,l})\} is a frame and \{\beta_{i,l}\} $\in l^2$, then $\sum_{i,l \in \mathbb{Z}} \beta_{i,l} A(\phi_{i,l})$ converges, and
\[ \| \sum_{i,l \in \mathbb{Z}} \beta_{i,l} A(\phi_{i,l})\|^2 \leq C_2 \| \sum_{i,l \in \mathbb{Z}} |\beta_{i,l}|^2. \] (3.4)

To prove this let us assume
\[ f_{j,k} = \sum_{i,l=1}^{j,k} \beta_{i,l} A(\phi_{i,l}) \]
then for any $j, k \geq j_0, k_0$, using Schwartz inequality with the frame condition (3.1) we obtain
\[ \|f_{j,k} - f_{j_0,k_0}\|^2 = \{ \sum_{i,l=j_0+1,k_0+1}^{j,k} |\beta_{i,l}|^2 \}^{1/2} \{ C_2 \|f_{j,k} - f_{j_0,k_0}\|^2 \}^{1/2}. \]
Which gives
\[ \|f_{j,k} - f_{j_0,k_0}\|^2 \leq C_2 \sum_{i,l=j_0+1,k_0+1}^{j,k} |\beta_{i,l}|^2. \]

Now we assume that \{A(\phi_{i,l})\} is a frame. Since \{A(\phi_{i,l})\} is a Bessel sequence, $T$ is a bounded operator from $l^2$ into $L^2(\mathbb{R})$ by (3.4). Now for any $f \in L^2(\mathbb{R})$ we define a linear transformation $S$ by the relation
\[ Sf = \sum_{i,l \in \mathbb{Z}} <f, A(\phi_{i,l})> A(\phi_{i,l}). \]
The transformation is self adjoint and it gives with (3.1) that
\[ C_1 \|f\|^2 \leq \langle Sf, f \rangle \leq C_2 \|f\|^2. \]

This conclude that \( S \) is positive, bounded and surjective. Thus \( S = TT^* \) is surjective. Hence \( T \) is surjective.

Now suppose that \( T \) is a well defined operator from \( l^2 \) onto \( L^2(R) \). By (3.4) \( \{A(\phi_{i,l})\} \) satisfies the upper frame condition. Now consider that \( T \) be any bounded operator from a Hilbert space \( H_1 \) into a Hilbert space \( H \). Then the set \( C_T = H_1 \ominus N(T) \) i.e., the orthogonal complement of null space of \( T \) in \( H_1 \) is well defined, \( T \) is injective on \( C_T \) and ran \( T^* \) is dense in \( C_T \). We denote \( T^+ \) the inverse map from ran \( T \) to \( C_T \) i.e., \( T^+: H \rightarrow C_T \). By writing \( T^+ f = \{(T^+ f)_{i,l}\} \) for \( f \in H \), we get
\[ f = TT^+ f = \sum_{i,l \in \mathbb{Z}} (T^+ f)_{i,l} A(\phi_{i,l}). \]

We have
\[
\|f\|^4 = |\langle f, f \rangle|^2 = \left| \sum_{i,l \in \mathbb{Z}} \langle (T^+ f)_{i,l} A(\phi_{i,l}), f \rangle \right|^2 \\
\leq \sum_{i,l \in \mathbb{Z}} \| (T^+ f)_{i,l} \|^2 \sum_{i,l \in \mathbb{Z}} |\langle f, A(\phi_{i,l}) \rangle|^2 \\
\leq \|T^+\|^2 \|f\|^2 \sum_{i,l \in \mathbb{Z}} |\langle f, A(\phi_{i,l}) \rangle|^2.
\]

Thus, we obtain
\[
\sum_{i,l \in \mathbb{Z}} |\langle f, A(\phi_{i,l}) \rangle|^2 \geq \frac{1}{\|T^+\|^2} \|f\|^2, \forall f \in H.
\]

Taking \( H = L^2(R) \). The proof is completed in view of Theorem 3.1.

**Theorem 3.3** Let any sequence of numbers \( \{\beta_{i,l}\} \in l^2 \) is a moment sequence of any function \( f \in L^2(R) \) with respect to \( \{A(\phi_{i,l})\} \). If \( \{A(\phi_{i,l})\} \) is an exact \( A \)-frame then there exist constants \( C_1, C_2 > 0 \) such that
\[
C_1 \sum_{i,l \in \mathbb{Z}} |\beta_{i,l}|^2 \leq \| \sum_{i,l \in \mathbb{Z}} \beta_{i,l} A(\phi_{i,l}) \|^2 \leq C_2 \sum_{i,l \in \mathbb{Z}} |\beta_{i,l}|^2
\]

**Proof.** Since \( \{A(\phi_{i,l})\} \) is an exact \( A \)-frame therefore \( S^{-1} A(\phi_{i,l}) \) is a biorthogonal sequence. By Theorem 3.2 we conclude that for a given sequence \( \{\beta_{i,l}\} \in l^2 \) and for any \( f \in L^2(R) \), the series \( f = \sum_{i,l \in \mathbb{Z}} \beta_{i,l} A(\phi_{i,l}) \) has a finite norm. The proof is completed with (3.1) by using the fact that \( \{A(\phi_{i,l})\} \) is bounded.
References


