Unique Fixed Point Theorem for Weakly S-Contractive Mappings

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(Received: 2-2-11 /Accepted: 5-4-11)

Abstract

In this paper, we have unique fixed point theorem using S-contractive mappings in complete metric space. We supported our result by some examples.

Keywords: Complete metric space, Fixed point, Weak S-contraction.

1 Introduction

It is well known that Banach's contraction mapping theorem is one of the pivotal results of functional analysis. A mapping $T:X \rightarrow X$ where $(X,d)$ is a metric space, is said to be a contraction if there exist $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq k d(x, y) \text{ for all } x, y, \in X$$  \hspace{1cm} (1.1)

If the metric space $(X, d)$ is complete then the mapping satisfying (1.1) has a unique fixed point which established by Banach (1922). The contractive definition (1.1) implies that. $T$ is uniformly continuous. It is natural to ask if there is
contractive definition which do not force T to be continuous. It was answered in affirmative by Kannan [5] who establish a fixed point theorem for mapping satisfying

\[ d(Tx, Ty) \leq k [d(x, Tx) + d(y, Ty)] \] (1.2)

for all \( x, y \in X \) and \( 0 \leq k < \frac{1}{2} \)

The mapping satisfying (1.2) are called Kannan type mapping. It is clear that contractions are always continuous and Kannan mapping are not necessarily continuous.

There is a large literature dealing with Kannan type mapping and generalization some of which are noted in [2, 4, 6, 7].

A similar contractive condition has been introduced by Shukla's we call this contraction a S-contraction.

**Definition 1.1. S-contraction**

Let \( T : X \rightarrow X \) where \( (X, d) \) is a complete metric space is called a S-contraction if there exist \( 0 \leq k < \frac{1}{3} \) such that for all \( x, y \in X \) the following inequality holds:

\[ d(Tx, Ty) \leq k [d(x, Ty) + d(Tx, y) + d(x, y)] \] (1.3)

A weaker contraction has been introduced in Hilbert spaces in [1].

**Definition 1.2. Weakly contractive mapping**

A mapping \( T : X \rightarrow X \) where \( (X, d) \) is a complete metric space is said to be weakly contractive [3] if

\[ d(Tx, Ty) \leq d(x, y) - \psi [d(x, y)] \] (1.4)

where \( x, y \in X \), \( \psi : [0, \infty) \rightarrow [0, \infty) \) is continuous and non decreasing

\[ \psi(x) = 0 \text{ iff } x = 0 \text{ and } \lim_{x \to \infty} \psi(x) = \infty \]

If we take \( \psi(x) = kx \) where \( 0 \leq k < 1 \) then (1.4) reduces to (1.1)

**Definition 1.3. Weak S-contraction**

A mapping \( T : X \rightarrow X \) where \( (X, d) \) is a complete metric space is said to be weakly S-contractive or a weak S-Contraction if for all \( x, y \in X \) such that

\[ d(Tx, Ty) \leq \frac{1}{3}[d(x, Ty)+d(Tx, y)+d(x,y)] - \psi [d(x, Ty),d(Tx, y),d(x,y)] \] (1.5)

where \( \psi : [0, \infty)^3 \rightarrow [0, \infty) \) is a continuous mapping such that
If we take $\psi(x, y, z) = k(x + y + z)$ where $0 \leq k < 1/3$ then (1.5) reduces to (1.3).

i.e. weak S-contractions are generalization of S-contraction. The next section we established that in a complete metric space a weak S-contraction has a unique fixed point. At the end of the next section we supported some examples.

2 Main Results

Theorem 2.1. Let $T : X \rightarrow X$, where $(X, d)$ is a complete metric space be a weak S-contraction. Then $T$ has a unique fixed point.

Proof. Let $x_0 \in X$ and $n \geq 1$, $x_{n+1} = Tx_n$. (2.1)

If $x_n = x_{n+1}$ then $x_n$ is a fixed point of $T$.

So we assume $x_n \neq x_{n+1}$.

Putting $x = x_{n-1}$ and $y = x_n$ in (1.5) we have for all $n = 0, 1, 2, \ldots$

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \frac{1}{3} [d(x_{n-1}, Tx_n) + d(Tx_{n-1}, x_n) + d(x_{n-1}, x_n)] - \psi[d(x_{n-1}, x_n), d(Tx_{n-1}, x_n), d(x_{n-1}, x_n)]$$

$$\leq \frac{1}{3} \left[d(x_{n-1} - x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right] \leq \frac{1}{3} [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] - \psi[d(x_{n-1}, x_n), d(Tx_{n-1}, x_n), d(x_{n-1}, x_n)]$$

$$\leq \frac{2}{3} d(x_n, x_{n+1}) \leq \frac{2}{3} d(x_{n-1}, x_n) - \psi[d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)] \leq \frac{2}{3} d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$$

i.e. $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of (2.3) decreasing sequence of non-negative real numbers and hence is convergent.

i.e. $\lim_{n \to \infty} d(x_n, x_{n+1})$ exist.

let $d(x_n, x_{n+1}) \to r$ as $n \to \infty$ (2.4)

We next prove that $r = 0$.

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \frac{1}{3} [d(x_{n-1}, Tx_n) + d(Tx_{n-1}, x_n) + d(x_{n-1}, x_n)] - \psi[d(x_{n-1}, x_n), d(Tx_{n-1}, x_n), d(x_{n-1}, x_n)]$$

$$\leq \frac{1}{3} \left[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\right] \leq \frac{1}{3} [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] - \psi[d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)]$$

$$\leq \frac{2}{3} d(x_n, x_{n+1}) \leq \frac{2}{3} d(x_{n-1}, x_n) - \psi[d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)] \leq \frac{2}{3} d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$$

(2.5)
Unique Fixed Point Theorem for Weakly...

31

taking \( n \to \infty \) in (2.5) we have by (2.4).

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) \leq 1/3 \lim_{n \to \infty} [d(x_{n-1}, x_n) + d(x_{n-1}, x_n)]
\]

\[r \leq 1/3 [ \lim_{n \to \infty} d(x_{n-1}, x_{n+1}) + r]
\]

\[2r \leq \lim_{n \to \infty} d(x_{n-1}, x_{n+1}) \tag{2.6}\]

Since \( d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \)

taking limit as \( n \to \infty \) in above we have by (2.4)

\[
\lim_{n \to \infty} d(x_{n-1}, x_{n+1}) \leq 2r \tag{2.7}\]

from (2.6) and (2.7)

\[
\lim_{n \to \infty} d(x_{n-1}, x_{n+1}) = 2r
\]

Again taking \( n \to \infty \) in (2.2)

\[
\lim_{n \to \infty} d(x_n, x_{n-1}) \leq 1/3 [ \lim_{n \to \infty} d(x_{n-1}, x_n) + d(x_{n-1}, x_n) + d(x_{n-1}, x_n)]
\]

\[- \psi [ \lim_{n \to \infty} \{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}, \lim_{n \to \infty} d(x_{n-1}, x_n)]
\]

\[r \leq 1/3 [r + r + r] - \psi (2r, r, 0)
\]

\[r \leq r - \psi (2r, r, 0)
\]

or \( \psi (2r, r, 0) \leq 0 \) which is contraction unless \( r = 0 \)

Thus we have established that

\[
d(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty \tag{2.9}\]

Next we show that \( \{x_n\} \) is a Cauchy sequence. If otherwise, then there exist \( \varepsilon > 0 \) and increasing sequences of integers \( \{m(k)\} \) and \( \{n(k)\} \) such that for all integers 'k',

\[n(k) > m(k) > k,
\]

\[d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{2.10}\]

and

\[d(x_{m(k)}, x_{n(k)-1}) < \varepsilon \tag{2.11}\]

Then,

\[
\varepsilon \leq d(x_{m(k)}, x_{n(k)}) = d(Tx_{m(k)-1} Tx_{n(k)-1})
\]

\[\leq 1/3 [d(x_{m(k)-1}, Tx_{n(k)-1}) + d (Tx_{m(k)-1}, x_{n(k)-1}) + (d(x_{m(k)-1}, x_{n(k)-1})]
\]

\[- \psi [d(x_{m(k)-1}, Tx_{n(k)-1}) + d(Tx_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{n(k)-1})]
\]

\[= 1/3 [d(x_{m(k)-1}, x_{m(k)}) + d (x_{m(k)}, x_{n(k)-1}) + (d(x_{m(k)-1}, x_{n(k)-1})]
\]
\[ - \psi \left[ d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)}, x_{n(k)-1}), d(x_{m(k)-1}, x_{n(k)-1}) \right] \] (2.12)

Again
\[ \varepsilon \leq d(x_{m(k)}, x_{n(k)}) \]
\[ \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \text{ by (2.11),} \]
\[ \rightarrow \varepsilon + d(x_{n(k)-1}, x_{n(k)}) \]

Taking \( k \to \infty \) is an above inequality and using (2.9) we obtain
\[ \varepsilon \leq \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) \leq \varepsilon \]

and
\[ \varepsilon \leq \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)-1}) + \lim_{k \to \infty} d(x_{n(k)-1}, x_{n(k)}) \leq \varepsilon \]

we have
\[ \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon \] (2.13)

And
\[ \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon \] (2.14)

Similarly
\[ \lim_{k \to \infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon \] (2.15)

Taking \( k \to \infty \) in (2.12) and using (2.9), (2.13), (2.14) and (2.15)
we obtain
\[ \varepsilon \leq 1/3 \left[ \varepsilon + \varepsilon + \varepsilon \right] - \psi (\varepsilon, \varepsilon, \varepsilon) \]
\[ \varepsilon \leq \varepsilon - \psi (\varepsilon, \varepsilon, \varepsilon) \]
\[ \psi (\varepsilon, \varepsilon, \varepsilon) \leq 0 \text{ which is contraction since } \varepsilon > 0 \]

Hence \( \{x_n\} \) is a Cauchy sequence and therefore is convergent in the complete metric space \((X, d)\)

Let \( x_n \to z \) and \( n \to \infty \). (2.16)

Then
\[ d(z, Tz) \leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \]
\[ = d(z, x_{n+1}) + d(Tx_n, Tz). \]
\[ \leq d(z, x_{n+1}) + 1/3 \left[ d(x_n, Tz) + d(Tx_n, z) + d(x_n, z) \right] \]
\[ - \psi \left[ d(x_n, Tz), d(Tx_n, z), d(x_n, z) \right] \]
\[ = d(z, Tz) + 1/3 \left[ d(x_n, Tz) + d(Tx_n, z) + d(x_n, z) \right] \]
\[ - \psi \left[ d(x_n, Tz), d(Tx_n, z), d(x_n, z) \right] \]
\[ = 2d(z, Tz) - \psi \left[ d(z, Tz), d(Tz, z), d(z, z) \right] \]
\[ < 2d(z, Tz) \]
\[ - d(z, Tz) < 0 \]
\[ d(z, Tz) \geq 0 \]

Hence \( Tz = z \)
Next we establish that the fixed point $z$ is unique.

Let $z_1$ and $z_2$ be two fixed points of $T$, then

\[ d(z_1, z_2) = d(Tz_1, Tz_2) \]
\[ \leq \frac{1}{3} [d(z_1, Tz_2) + d(Tz_1, z_2) + d(z_1, z_2)] \]
\[ - \psi(d(z_1, Tz_2), d(z_1, z_2), d(z_1, z_2)) \]

i.e.

\[ d(z_1, z_2) \leq d(z_1, z_2) - \psi(d(z_1, z_2), d(z_1, z_2), d(z_1, z_2)) \]

which by property of $\psi$ is a contradiction unless $d(z_1, z_2) = 0$, that is $z_1 = z_2$. Hence fixed point is unique in S-contraction.

Example 2.1. Let $x = \{p, q, r\}$ and $d$ is a metric defined on $X$ as follows.

\begin{align*}
(i) & \quad d(p, q) = 2 \quad d(q, r) = 4 \quad d(r, p) = 3 \\
and & \quad T(p) = q \quad T(q) = q \quad T(r) = p \\
(ii) & \quad d(q, r) = 2 \quad d(r, p) = 4 \quad d(p,q) = 3 \\
& \quad T(q) = r \quad T(r) = r \quad T(p) = q \\
(iii) & \quad d(r, p) = 2 \quad d(p, q) = 4 \quad d(q, r) = 3 \\
& \quad T(r) = p \quad T(p) = p \quad T(q) = r
\end{align*}

where $T: X \rightarrow X$ is a mapping defined as (i) (ii) and (iii) respectively.

Then $(X, d)$ is a complete metric space.

Let $\psi(a, b, c) = \frac{1}{3} \min \{a, b, c\}$

Then $T$ is a weak S-contraction and conditions of theorem are satisfied. Hence $T$ must have a unique fixed point.

It is clear that $q$, $r$ and $p$ are fixed point of $T$.

Corresponding mapping of $T$.

and if $x$ replace $p$ or $q$ and $y$ replace $r$ then inequality (1.3) does not holds by definition of $T$ in (i)

Similarly $x$ replace $q$ and $r$ and $y$ replace $p$ then inequality (1.3) does not holds by definition of $T$ in (ii)

and $x$ replace $r$ and $p$ and $y$ replace $q$ then inequality (1.3) does not holds by definition of $T$ in (iii)

Acknowledgements

The author's express grateful thanks to Prof. S.K. Chatterjee and Binayak S. Choudhury for there valuable research papers for the improvement for our paper.

References


