On the Generalized Hyers-Ulam Stability of an Euler-Lagrange-Rassias Functional Equation

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Abstract

In this paper, the general solution and the generalized Hyers-Ulam-Rassias stability of the following Euler-Lagrange type quadratic functional equation

\[ f(x+ky) + f(y+kz) + f(z+kx) - kf(x+y+z) = (k^2 - k + 1)(f(x) + f(y) + f(z)), \]

for all \( k \in \mathbb{N} \), is investigated.

Keywords: Quadratic functional equation, Hyers-Ulam-Rassias stability.

1 Introduction

The stability problem for the functional equations was first raised by S. M. Ulam [21]. He proposed the following famous question concerning the stability of homomorphisms:

Let \( G \) be a group and let \( G' \) be a metric group with metric \( d \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if \( f : G \to G' \) satisfies

\[ d(f(xy), f(x)f(y)) < \delta \quad \text{for all} \quad x, y \in G, \]

then there exists a homomorphism \( F : G \to G' \) with

\[ d(f(x), F(x)) < \varepsilon \quad \text{for all} \quad x \in G. \]

In 1941, Hyers [6] considered the case of approximately additive mappings \( f : X \to Y \), where \( X \) and \( Y \) are Banach spaces and \( f \) satisfies

\[ \|f(x+y) - f(x) - f(y)\| \leq \varepsilon \]
for all \( x, y \in X \). It was shown that the limit

\[
F(x) = \lim_{n \to \infty} 2^{-n} f(2^n x),
\]

exists for all \( x \in X \) and that \( F : X \to Y \) is the unique additive mapping satisfying

\[
\|f(x) - F(x)\| \leq \varepsilon.
\]


The quadratic function \( f(x) = cx^2 \) satisfies the functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y),
\]

and therefore the above equation is called the quadratic functional equation.

In 1982-1994, J. M. Rassias (see [11-18]) solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J. M. Rassias considered the mixed product-sum of powers of norms control function [20]. In 1994, a generalization of the Rassias’ theorem was obtained by Gavruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach. For more details about the results concerning such problems the reader is referred to [2, 3, 4, 9, 10] and [22].

Consider the following functional equations:

\[
f(x + y) + f(y + z) + f(z + x) = f(x + y + z) + f(x) + f(y) + f(z), \quad (1)
\]

and

\[
f(x + 2y) + f(y + 2z) + f(z + 2x) = 2f(x + y + z) + 3(f(x) + f(y) + f(z)). \quad (2)
\]

The functional equation (1) was solved by Pl. Kannappan in [8]. Recently, the author investigated in his paper [22] the general solution and generalized Hyers-Ulam stability of the equation (2).

In the present paper we consider the quadratic functional equation

\[
f(x + ky) + f(y + kz) + f(z + kx) - kf(x + y + z) = (k^2 - k + 1)(f(x) + f(y) + f(z)),
\]

for all \( k \in \mathbb{N} \), which is a generalization of equations (1) and (2), and determine the general solution and generalized Hyers-Ulam stability of this functional equation.
The General Solution and Hyers-Ulam Stability

The following theorem provide the general solution of the proposed functional equation by establishing a connection with the classical quadratic functional equation.

For convenience, we use the following abbreviations:

\[ Df(x, y, z) = f(x + ky) + f(y + kz) + f(z + kx) \]

\[ -kf(x + y + z) - (k^2 - k + 1)(f(x) + f(y) + f(z)). \] (3)

**Theorem 2.1** Let \( X \) and \( Y \) be real vector spaces. A function \( f : X \rightarrow Y \) satisfies the functional equation

\[ Df(x, y, z) = 0, \] (4)

for all \( x, y, z \in X \) and all \( k \in \mathbb{N} \) if and only if it satisfies

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad (x, y \in X). \] (5)

**Proof:** The result is proved for the case \( k = 1 \) and \( k = 2 \), in [8] and [22], respectively. So we give the proof for \( k \geq 3 \). Assume that a function \( f : X \rightarrow Y \) satisfies (4). Letting \( x = y = z \) in (4), we get

\[ 3f((k + 1)x) - kf(3x) = 3(k^2 - k + 1)f(x) \]

for all \( x \in X \), which implies that \( f(0) = 0 \). Letting \( y = z = 0 \) in (4), we have

\[ f(x) + f(kx) = kf(x) + (k^2 - k + 1)f(x), \]

which yields

\[ f(kx) = k^2 f(x) \quad (\dagger), \]

for all \( x \in X \) and all \( k \in \mathbb{N} \). Letting \( z = 0 \) in (4), we obtain

\[ f(x + ky) + f(y) + f(kx) = kf(x + y) + (k^2 - k + 1)(f(x) + f(y)). \]

Applying Eq. \((\dagger)\), then we have

\[ f(x + ky) - kf(x + y) = (1 - k)f(x) + (k^2 - k)f(y). \] (6)

Replacing \( x \) by \( y \) and \( y \) by \( x \) in (6), so

\[ f(y + kx) - kf(y + x) = (1 - k)f(y) + (k^2 - k)f(x). \] (7)
Letting $y = z$ in (4), we get
\[ f(x + ky) + f((k+1)y) + f(y + kx) = kf(x + 2y) + (k^2 - k + 1)(f(x) + 2f(y)). \]

Using Eq. (‡) for $k + 1$, the above equation simplifies to
\[ f(x + ky) + f(y + kx) - kf(x + 2y) = 
\]
\[ k^2(f(x) + f(y)) + (1 - k)f(x) + (1 - 4k)f(y). \] (8)

Eliminating $f(x + ky)$ and $f(y + kx)$ from (8) by applying (6) and (7), we get
\[ 2kf(x + y) + 2kf(y) = kf(x) + kf(x + 2y). \] (9)

Replacing $x$ by $x - y$ in above equation, thus the classical quadratic functional equation (5) follows.

Conversely, assume that a function $f : X \rightarrow Y$ satisfies (5), and suppose the result is establish for each $s < k$, where $k \geq 3$. Replacing $x$ by $x + (k-1)y$ and all cyclic permutations of the variables in (5), then
\[ f(x + ky) + f(x + (k-2)y) = 2f(x + (k-1)y) + 2f(y), \]
\[ f(y + kz) + f(y + (k-2)z) = 2f(y + (k-1)z) + 2f(z), \]
\[ f(z + kx) + f(z + (k-2)x) = 2f(z + (k-1)x) + 2f(x). \] (10)

By ammunitions we have
\[ f(x + (k-1)y) + f(y + (k-1)z) + f(z + (k-1)x) = 
\]
\[ (k - 1)f(x + y + z) + (k^2 - 3k + 3)(f(x) + f(y) + f(z)). \] (11)

and
\[ f(x + (k-2)y) + f(y + (k-2)z) + f(z + (k-2)x) = 
\]
\[ (k - 2)f(x + y + z) + (k^2 - 5k + 7)(f(x) + f(y) + f(z)). \] (12)

Applying Eq. (11) and (12), to eliminate $f(x + (k-1)y)$, $f(x + (k-2)y)$ and all cyclic permutations of the variables in the sum of all equations in (10), then the quadratic functional equation (4) follows, so the induction argument finishes the proof.

**Theorem 2.2** Suppose $X$ is a real vector space and $Y$ is a Banach space. Let $k \geq 3$ and $\varphi : X^3 \rightarrow [0, \infty)$ be a function such that
\[
\sum_{n=0}^{\infty} k^{-2n} \varphi(k^n x, k^n y, k^n z)
\] (13)
be convergent. Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and
\[
\|Df(x, y, z)\| \leq \varphi(x, y, z)
\] (14)
for all \( x, y, z \in X \), then there exists a unique function \( F : X \to Y \) which satisfies (4) and
\[
\|f(x) - F(x)\| \leq \frac{1}{k^2} \sum_{n=0}^{\infty} k^{-2n} \varphi(k^n x, 0, 0) \quad (x \in X).
\] (15)

Proof: Letting \( y = z = 0 \) in (14), we get
\[
\|f(kx) - k^2 f(x)\| \leq \varphi(x, 0, 0).
\]
Dividing the above inequality by \( k^2 \), we obtain
\[
\left\| \frac{f(kx)}{k^2} - f(x) \right\| \leq \frac{1}{k^2} \varphi(x, 0, 0). \tag{16}
\]
Make the induction hypothesis
\[
\| \frac{f(k^n x)}{k^{2n}} - f(x) \| \leq \frac{1}{k^2} \sum_{i=0}^{n-1} k^{-2i} \varphi(k^i x, 0, 0), \tag{17}
\]
which is true for \( n = 1 \) by (16). Replacing \( x \) by \( k^m x \) in (17) and divide the result by \( k^{2m} \), then we have
\[
\left\| \frac{f(k^{n+m} x)}{k^{2(n+m)}} - \frac{f(k^m x)}{k^{2m}} \right\| \leq \frac{1}{k^2} \sum_{i=m}^{n+m-1} k^{-2i} \varphi(k^i x, 0, 0) \quad (x \in X).
\]
It follows that the sequence \( \{ \frac{1}{k^{2n}} f(k^n x) \} \) is Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, we may define a function \( F : X \to Y \) by
\[
F(x) := \lim_{n \to \infty} \frac{1}{k^{2n}} f(k^n x), \quad (x \in X).
\]
Then by the definition of \( F \), we can see that (15) holds. To show that \( F \) satisfies in (4), replacing \( x, y \) and \( z \) in (14) by \( k^n x, k^n y \) and \( k^n z \), respectively, and divide the result by \( k^{2n} \), we get
\[
\| \frac{1}{k^{2n}} Df(k^n x, k^n y, k^n z) \| \leq \frac{\varphi(k^n x, k^n y, k^n z)}{k^{2n}} \to 0, \quad \text{as} \quad n \to \infty,
\]
which implies \( F \) satisfies (4). The uniqueness of \( F \) follows from Theorem 2.1.
Corollary 2.3 Let \( k \geq 3 \) and \( f : X \rightarrow Y \) be a function such that
\[
\|Df(x,y,z)\| \leq \varepsilon
\]
for some \( \varepsilon > 0 \) and for all \( x, y, z \in X \). Then there exists a unique function \( F : X \rightarrow Y \) which satisfies (4), and
\[
\|f(x) - F(x)\| \leq \frac{\varepsilon}{k^2 - 1} \quad (x \in X).
\]

Proof: Apply Theorem 2.2 for \( \varphi(x,y,z) = \varepsilon \).

Corollary 2.4 Let \( k \geq 3 \) and \( f : X \rightarrow Y \) be a function such that satisfies
\[
\|Df(x,y,z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),
\]
with \( p < 2 \) and for some \( \varepsilon > 0 \) and for all \( x, y, z \in X \). Then there exists a unique quadratic function \( F : X \rightarrow Y \) which satisfies (4), and
\[
\|f(x) - F(x)\| \leq \frac{\varepsilon}{|k^2 - k^p|}\|x\|^p \quad (x \in X).
\]

Proof: Apply Theorem 2.2 for \( \varphi(x,y,z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \).

3 Conclusion

This paper generalized some well-known results in the area of Hyers-Ulam stability of the Euler-Lagrange-Rassias type quadratic functional equation in three variables, in fact, the proposed quadratic functional equations which are given in [8] and [22], can be obtained of the proposed functional equation in the present paper, for \( k = 1 \) and \( k = 2 \), respectively.

Concluding remarks, the results of the paper is also true for all \( k \in \mathbb{Z} \), but the paper discussed for the case \( k \in \mathbb{N} \).

If we take \( k = -1 \) in the proposed quadratic functional equation we get
\[
f(x - y) + f(y - z) + f(z - x) + f(x + y + z) = 3(f(x) + f(y) + f(z)),
\]
that Hyers-Ulam stability of it investigated by Jung in [7]. Thus, the paper is also generalized the Jung’s work.

References


