Module Amenability and Tensor Product of Banach Algebras

A. Sahleh\(^1\) and S. Grailo Tanha\(^2\)

\(^1\)\(^2\)Department of Mathematics, Faculty of Science University of Guilan, Rasht 1914, Iran
\(^1\)E-mail: sahlehj@guilan.ac.ir
\(^2\)E-mail: so.grailo@guilan.ac.ir

(Received: 11-3-14 / Accepted: 27-4-14)

Abstract

Let Banach algebra \(A\) is a \(U\)-module with compatible actions. In this paper we show that \(\hat{A} \otimes A\) is module amenable when \(A\) is module amenable. In particular, we investigate module amenability of unitization of \(A\).

Keywords: module amenability, module derivation, commutative action, Banach algebra.

1 Introduction

A Banach algebra \(A\) is amenable if every bounded derivation from \(A\) into any dual Banach \(A\)-module inner. This concept was introduced by Barry Johnson in [4]. He proved that if \(A\) and \(B\) are amenable Banach algebra, then so is \(\hat{A} \otimes B\)(see also [3]).

M. Amini in [1] introduced the concept of module amenability for a Banach algebra which is a Banach module on another Banach algebra with compatible actions. This could be considered as a generalization of the Johnson’s amenability.

In this paper we prove that module amenability of \(A\) implies module amenability of \(\hat{A} \otimes A\) as a \(U\)-module, when \(A\) has a unite and it is a commutative \(U\)-module. Also we show, when \(A\) is a \(U\)-module with the compatible actions, also \(\hat{A}^2\) is a \(U\)-module if and only if the actions are trivial, where \(\hat{A}^2\)
is the unitization of $\mathcal{A}$.

## 2 Notation and Preliminaries

Throughout this paper, $\mathcal{A}$ and $\mathcal{U}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module with compatible action, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathcal{U}).$$

Moreover, if $\alpha \cdot a = a \cdot \alpha$, we say that $\mathcal{A}$ is commutative $\mathcal{U}$-module.

The Banach algebra $\mathcal{U}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathcal{U}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$ ($a \cdot \alpha = f(\alpha)a$), where $f$ is a continuous character on $\mathcal{U}$.

Let $X$ be a Banach $\mathcal{A}$-module and a Banach $\mathcal{U}$-module with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$

$$(\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathcal{U}, x \in X);$$

and the same for right or two-sided actions, then we say that $X$ is a Banach $\mathcal{A}$-$\mathcal{U}$-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathcal{U}, x \in X)$$

then $X$ is called a commutative $\mathcal{A}$-$\mathcal{U}$-module. If $X$ is a (commutative) Banach $\mathcal{A}$-$\mathcal{U}$-module then so is $X^*$, where the actions of $\mathcal{A}$ and $\mathcal{U}$ on $X^*$ are defined by

$$(\alpha \cdot f)(x) = f(x \cdot \alpha), \quad (a \cdot f)(x) = f(x \cdot a) \quad (a \in \mathcal{A}, \alpha \in \mathcal{U}, x \in X, f \in X^*)$$

and the same for the right actions.

It is well known that the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach algebra with respect to the canonical multiplication defined by

$$(a \otimes b)(c \otimes d) = (ac \otimes bd)$$

and extended by bi-linearity and continuity. Then $\mathcal{A} \hat{\otimes} \mathcal{A}$ is a Banach $\mathcal{U}$-module by the following canonical actions:

$$\alpha \cdot (a \otimes b) = (\alpha \cdot a) \otimes b, \quad (a \otimes b) \cdot \alpha = (a \otimes b \cdot \alpha) \quad (\alpha \in \mathcal{U}, a, b \in \mathcal{A})$$

similarity, for the right actions.

Let $I$ be the closed ideal of the projective tensor product $\mathcal{A} \hat{\otimes} \mathcal{A}$ generated by elements of the form $\alpha \cdot a \otimes b - a \otimes b \cdot \alpha$ for $\alpha \in \mathcal{U}, a, b \in \mathcal{A}$. Also we consider $J$, the closed ideal of $\mathcal{A}$ generated by elements of the form $(\alpha \cdot a)b - a(b \cdot \alpha)$ for $\alpha \in \mathcal{U}, a, b \in \mathcal{A}[5]$. 
Let $A$ and $U$ be as in the above and $X$ be a Banach $A - U$-module. A bounded map $D : A \to X$ is called a module derivation if

$$
D(a \pm b) = D(a) \pm D(b), \quad D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A)
$$

$$
D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in A, \alpha \in U)
$$

although $D$ is not necessary linear, but still boundedness implies its norm continuity (since it preserves subtraction). When $X$ is commutative, each $x \in X$ defines a module derivation $\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A)$

which is called inner module derivations.

The Banach algebra $A$ is called module amenable (as an $U$-module) if for any commutative Banach $A - U$-module $X$, each module derivation $D : A \to X^*$ is inner.

When the Banach algebra $U$ acts trivially on $A$ from left(right) and $A$ is module amenable we say that $A$ is module amenable with the trivial left(right) action. Bodaghi in [2] proved that if $A$ is a Banach $U$-module with trivial left action and $A \hat{\otimes} A$ is module amenable, then $A/J \hat{\otimes} A/J$ is amenable.

3 Module Amenability of Tensor Product of Banach Algebras

Suppose that $A$ has a unit, we consider the projective tensor products $A \hat{\otimes} e$ and $e \hat{\otimes} A$ which are Banach $U$-modules by the following usual actions:

$$
\alpha \cdot (a \otimes e) = \alpha \cdot a \otimes e, \quad (a \otimes e) \circ \alpha = a \cdot \alpha \otimes e \quad (\alpha \in U, a \in A),
$$

$$
\alpha \bullet (e \otimes a) = e \otimes \alpha \cdot a, \quad (e \otimes a) \cdot \alpha = e \otimes a \cdot \alpha \quad (\alpha \in U, a \in A).
$$

Note that although $A \hat{\otimes} e$ and $e \hat{\otimes} A$ are subalgebra of $A \hat{\otimes} A$ but by the module actions defined on $A \hat{\otimes} A$ they are not $U$-modules.

**Lemma 3.1.** If $A$ is module amenable, then $A \hat{\otimes} e$ and $e \hat{\otimes} A$ are module amenable.

**Proof.** Consider a module homomorphism $\varphi : A \to A \hat{\otimes} e$ defined by $\varphi(a) = a \hat{\otimes} e$. By Proposition 2.5 of [1], the module amenability of $A$ implies the module amenability of $A \hat{\otimes} e$. Similarly $e \hat{\otimes} A$ is module amenable.

**Theorem 3.2.** Let $A$ be a commutative $U$-module with a unit. If $A$ is module amenable, then so is $A \hat{\otimes} A$. 

Proof. Let $X$ be a commutative $A \hat{\otimes} A - \mathcal{U}$-module and $D : A \hat{\otimes} A \rightarrow X^*$ be a module derivation. Since $X$ is a commutative $\mathcal{U}$-module, we have

$$(a \otimes e) \cdot (\alpha \cdot x) = (a \otimes e) \cdot (x \cdot \alpha)$$

$$= ((a \otimes e) \cdot x) \cdot \alpha$$

$$= \alpha \cdot ((a \otimes e) \cdot x)$$

$$= (\alpha \cdot (a \otimes e)) \cdot x$$

$$= (a \cdot a \otimes e) \cdot x$$

$$= (a \cdot a \otimes e) \cdot x$$

$$= ((a \otimes e) \circ \alpha) \cdot x.$$

Clearly $\alpha \cdot ((a \otimes e) \cdot x) = (\alpha \cdot (a \otimes e)) \cdot x$, $(\alpha \cdot x) \cdot (a \otimes e) = \alpha \cdot (x \cdot (a \otimes e))$. And the same for right or two-sided actions. Thus $X$ is a commutative $A \hat{\otimes} A - \mathcal{U}$-module with the compatible actions. Consider $\tilde{D} : A \hat{\otimes} e \rightarrow X^*$ defined by $\tilde{D}(a \otimes e) = D(a \otimes e)$ for all $a \in A$. By previous lemma, there exist $x^* \in X^*$ such that $\tilde{D} = \delta_{x^*}$. Hence $D|_{A \hat{\otimes} e} = \delta_{x^*}$. Now consider $\tilde{D} := D - \delta_{x^*}$. Thus $\tilde{D}|_{A \hat{\otimes} e} = 0$. Let $F$ be the closed linear span of elements of the form

$$\{(a \otimes e) \cdot x - x \cdot (a \otimes e) : a \in A, x \in X\}.$$

We prove in three step that $F$ is a $e \hat{\otimes} A - \mathcal{U}$-module with module actions given by

$$(e \otimes b) \ast ((a \otimes e) \cdot x - x \cdot (a \otimes e)) := (a \otimes e) \cdot y - y \cdot (a \otimes e)$$

such that $y = (e \otimes b) \cdot x$ and

$$\alpha \diamond ((a \otimes e) \cdot x - x \cdot (a \otimes e)) = (\alpha \cdot a \otimes e) \cdot x - x \cdot (\alpha \cdot a \otimes e),$$

and the same for right or two-sided actions.

Step 1. we show that

$$\alpha \diamond ((e \otimes b) \ast ((a \otimes e) \cdot x - x \cdot (a \otimes e))) = (a \bullet (e \otimes b)) \ast (a \otimes e) \cdot x - x \cdot (a \otimes e)).$$
We have

\[
\alpha \odot ((e \otimes b) \star ((a \otimes e) \cdot x - x \cdot (a \otimes e))) = \alpha \odot ((a \otimes e) \cdot ((e \otimes b) \cdot x) \\
- ((e \otimes b) \cdot (a \otimes e)) \\
= (\alpha \cdot a \otimes e) \cdot ((e \otimes b) \cdot x) \\
- ((e \otimes b) \cdot (a \cdot a \otimes e)) \\
= (\alpha \cdot (a \otimes e)) \cdot ((e \otimes b) \cdot x) \\
- ((e \otimes b) \cdot (a \cdot a \otimes e)) \\
= \alpha \cdot ((a \otimes e) \cdot ((e \otimes b) \cdot x)) \\
- ((e \otimes b) \cdot (x \cdot a)) \cdot (a \otimes e) \\
= ((a \otimes e) \cdot ((e \otimes b) \cdot x)) \cdot \alpha \\
- ((e \otimes b) \cdot (a \cdot x)) \cdot (a \otimes e) \\
= (a \otimes e) \cdot ((e \otimes b) \cdot (x \cdot a)) \\
- ((e \otimes b \cdot a) \cdot x) \cdot (a \otimes e) \\
= (a \otimes e) \cdot ((e \otimes b) \cdot (a \cdot x)) \\
- ((e \otimes a \cdot b) \cdot x) \cdot (a \otimes e) \\
= (a \otimes e) \cdot ((a \cdot (e \otimes b)) \cdot x) \\
- ((\alpha \odot (e \otimes b)) \cdot x) \cdot (a \otimes e)
\]

and proof of step 1 is complete.

**Step 2.** Now we prove that

\[
(e \otimes b) \star (\alpha \odot ((a \otimes e) \cdot x - x \cdot (a \otimes e))) = ((e \otimes b) \cdot \alpha) \star ((a \otimes e) \cdot x - x \cdot (a \otimes e)).
\]
We have
\[(e \otimes b) \ast (\alpha \circ ((a \otimes e) \cdot x - x \cdot (a \otimes e))) = (e \otimes b) \ast ((\alpha \cdot a \otimes e) \cdot x - x \cdot (\alpha \cdot a \otimes e)) = (e \otimes b) \ast ((\alpha \cdot a \otimes e) \cdot x) - x \cdot (\alpha \cdot a \otimes e) = (\alpha \cdot a \otimes e) \cdot ((e \otimes b) \cdot x) - ((e \otimes b) \cdot x) \cdot (\alpha \cdot a \otimes e) = ((\alpha \cdot a \otimes e) \cdot ((e \otimes b) \cdot x)) - ((e \otimes b) \cdot x) \cdot (\alpha \cdot a \otimes e) = (\alpha \cdot ((a \otimes e) \cdot ((e \otimes b) \cdot x))) - ((e \otimes b) \cdot x) \cdot (\alpha \cdot a \otimes e) = (a \otimes e) \cdot ((e \otimes b) \cdot (\alpha \cdot x)) - ((e \otimes b) \cdot (\alpha \cdot x)) \cdot (a \otimes e) = (a \otimes e) \cdot ((e \otimes b) \cdot (\alpha \cdot x)) - (\alpha \cdot (e \otimes b) \cdot (\alpha \cdot x)) = (a \otimes e) \cdot ((e \otimes b) \cdot (\alpha \cdot x)) - (\alpha \cdot ((e \otimes b) \cdot (\alpha \cdot x))) = (a \otimes e) \cdot (\alpha \cdot (e \otimes b) \cdot (\alpha \cdot x)) - \alpha \cdot ((e \otimes b) \cdot (\alpha \cdot x)) = (a \otimes e) \cdot (\alpha \otimes (e \otimes b) \cdot (\alpha \cdot x)) - \alpha \cdot ((e \otimes b) \cdot (\alpha \cdot x)) = (a \otimes e) \cdot (\alpha \otimes (e \otimes b) \cdot (\alpha \cdot x)) - \alpha \cdot ((e \otimes b) \cdot (\alpha \cdot x)) = (a \otimes e) \cdot (\alpha \otimes (e \otimes b) \cdot ) \cdot (\alpha \otimes a \otimes e).
\]

**Step 3.** Finally, we show that
\[(\alpha \circ ((a \otimes e) \cdot x - x \cdot (a \otimes e))) \ast (e \otimes b) = \alpha \circ ((a \otimes e) \cdot x - x \cdot (a \otimes e)) \ast (e \otimes b).
\]

Similarly for the right and two sided actions, these relations are correct.

For each \(a, b \in \mathcal{A}\) and \(x \in X\) we have
\[\tilde{D}(e \otimes b)((a \otimes e) \cdot x - x \cdot (a \otimes e)) = \tilde{D}(e \otimes b) \cdot (a \otimes e) - (a \otimes e) \cdot \tilde{D}(e \otimes b)(x).
\]

Since \(\tilde{D}_{|A \otimes e} = 0\),
\[\tilde{D}(a \otimes b) = (a \otimes e) \cdot \tilde{D}(e \otimes b) = \tilde{D}(e \otimes b) \cdot (a \otimes e).
\]

Therefore \(\tilde{D}(e \hat{\otimes} A) \subseteq F^\perp = (X/F)^*\) and \(\tilde{D}|_{e \hat{\otimes} A} : e \hat{\otimes} A \to (X/F)^*\) is a module derivation. By the previous lemma, there exists \(f^* \in F^\perp\) such that \(\tilde{D}|_{e \hat{\otimes} A} = \delta_{f^*}\). Then we have \(D - \delta_x|_{e \hat{\otimes} A} = \delta_{f^*}\). Since \(\delta_{f^*}|_{A \otimes e} = 0\), \(D - \delta_x = \delta_{f^*}\) and proof is complete.
Corollary 3.3. Let $A$ be a $U$-module and $A/J$ has a unit. Then $A/J \hat{\otimes} A/J$ is module amenable.

References


