A Common Fixed Point Theorem for
Asymptotically Regular Multi-Valued Three Maps

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Abstract
In this paper, we prove a common fixed point theorem for asymptotically regular multi valued three maps. Our result generalizes and extends some recent results in the literature.

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1 Introduction
In 2006, P.D. Proinov [12] obtained two types of generalizations of Banach fixed point theorem. The first type involves Meir-Keeler [9] type conditions (see, for instance, Cho et al., [3], Lim [8], Park and Rhoades [11]) and the second type involves contractive guage functions (see, for instance, Boyd and Wong [1] and Kim et al., [7]). Proinov [12] obtained equivalence between these two types of contractive conditions and also obtained a new fixed point theorem generalizing some fixed point theorems of Jachymski [6] have extended Proinov [12] Theorem
4.1 into multi valued maps. In this paper we extend Theorem 2.2 of S.L. Singh et al. [16] for three maps.

Asymptotic regularity for single-valued map is due to Browder and Petryshyn [2].

**Definition 1.1:** A self-map \( T \) on a metric space \( (X, d) \) is asymptotic regular if

\[
\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0 \quad \text{for each } x \in X.
\]

Rohades et al., [14] and Singh et al., [17] have extended this concept of asymptotic regularity to multi-valued maps as follows.

**Definition 1.2:** Let \( (X, d) \) be a metric space and \( S: Y \to \text{CL}(X) \). \( S \) is asymptotically regular at \( x_0 \in X \) if for any sequence \( \{x_n\} \) in \( Y \) and each sequence \( \{y_n\} \) in \( Y \) such that

\[
y_n \in S_{x_{n-1}} \quad \lim_{n \to \infty} (y_n, y_{n+1}) = 0.
\]

**Definition 1.3:** Let \( (X, d) \) be a metric space and \( S, T: Y \to \text{CL}(X) \). A pair \( (S, T) \) is said to be asymptotically regular at \( x_0 \in X \), if for any sequence \( \{x_n\} \) in \( X \) and each sequence \( \{y_n\} \) in \( X \) such that \( y_n \in S_{x_{n-1}} \cup T_{x_{n-1}} \),

\[
\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.
\]

**Definition 1.4:** Let \( f: Y \to Y \) and \( S: Y \to \mathcal{P}(Y) \) the collection of non-empty sub set of \( Y \). Then the hybrid pair \( (S, f) \) are \((IT)\)-Commuting on \( Y \) if \( fSz \subseteq Sz \) for all \( z \in Y \).

## 2 Common Fixed Point Theorem

The following theorem is extension and improves the Theorem of S.L. Singh et al., [16].

**Theorem 2.1:** Let \( (X, d) \) be a metric space and \( f: Y \to X \) and \( S, T: Y \to \text{CL}(X) \) such that

(C1). \( SY \cup TY \subseteq fY \).

(C2). \( H(Sx, Ty) \leq \varphi(h(x,y)) \) for all \( x, y \in Y \),

where \( h(x,y) = d(fx,fy)+\gamma[d(Sx,fx)+d(Ty,fy)] \), \( 0 \leq \gamma \leq 1 \) and \( \varphi \in \Phi \) is continuous.

If the pair \( (S, T) \) is asymptotically regular at \( x_0 \in X \) and either \( S(Y) \) or \( T(Y) \) or \( f(Y) \) is a complete sub space of \( X \). Then

(i). \( C(S, f) \) and

(ii). \( C(T, f) \) are non-empty. Further,
(iii). S and f have a common fixed point provided \( SSu = Su \) and S and f are (IT) Commuting at a point \( u \in C(S, f) \).

(iv). T and f have a common fixed point provided \( TTv = Tv \) and T and f are (IT) Commuting at a point \( v \in C(T, f) \).

(v). S, T and f have a common fixed point provided that (iii) and (iv) both are true.

**Proof:** We construct sequences \( \{y_n\} \) and \( \{x_n\} \) in \( Y \) in the following way.

Let \( y_1 \) be an element of \( Sx_0 \). Since \( Tx_1 \) is compact, we choose a point \( y_2 \in Y \) such that \( d(y_1, y_2) \leq H(Sx_0, Tx_1) \). Again, \( Tx_2 \) is compact we choose a point \( y_3 \in Y \) such that

\[
d(y_2, y_3) \leq H(Sx_1, Tx_2)
\]

continuing in the same manner we get

\[
d(y_n, y_{n+1}) \leq H(Sx_{n-1}, Tx_n)
\]

Since \( SY \cup TY \subseteq fY \), we may take

\[
y_n = fx_n \in Sx_{n-1} \cup Tx_{n-1}
\]

for \( n = 1, 2, \ldots \ldots \). The asymptotic regularity of the pair \( (S, T) \) at \( x_0 \) implies that

\[
\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.
\]

Fix \( \varepsilon > 0 \). Since \( \phi \in \Phi \) there exists \( \delta > \varepsilon \) such that for any \( t \in (0, \infty) \),

\[
\varepsilon < t < \delta \Rightarrow \phi(t) \leq \varepsilon.
\] (1)

Without loss of generality we may assume that \( \delta \leq 2 \varepsilon \). By the asymptotic regularity of the pair \( (S, T) \) at \( x_0 \),

\[
\lim_{n \to \infty} d(y_n, y_{n+1}) = 0.
\]

So, there exists an integer \( N_1 \geq 1 \) such that

\[
d(y_n, y_{n+1}) \leq H(Sx_{n-1}, Tx_n) < \frac{\delta - \varepsilon}{1 + 2\gamma}, \ m \geq N_1.
\] (2)

By the induction we show that

\[
d(y_n, y_m) \leq H(Sx_{n-1}, Tx_{m-1}) < \frac{\delta + 2\gamma \varepsilon}{1 + 2\gamma}, \ m \geq n \geq N_1.
\] (3)

Let \( n > N_1 \) be fixed. Then equation (3) holds for \( m = n + 1 \).

Assuming (3) to hold for an integer \( m \geq n \). We shall prove it for \( m + 1 \).

By the triangle inequality, we get

\[
d(y_n, y_{m+1}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{m+1}).
\]
A Common Fixed Point Theorem for

That is,

\[ d( y_n, y_{n+1} ) \leq d( y_n, y_{n+1} ) + H(Sx_n, Tx_m). \]  \hspace{1cm} (4)

We shall show that

\[ H(Sx_n, Tx_m) \leq \varepsilon. \]  \hspace{1cm} (5)

If \( H(Sx_n, Tx_m) \) not less than or equal \( \varepsilon \), then

\[ \varepsilon < H(Sx_n, Tx_m) < \varphi(h(x_n, x_m)) \leq h(x_n, x_m) < \delta. \]

\[ h(x_n, x_m) = d(x_n, x_m) + \gamma[d(x_n, Sx_n) + d(x_m, Tx_m)] \]

= \( d(x_n, x_m) + \gamma[d(x_n, x_{n+1}) + d(x_m, x_{m+1})] \).

Using (2) and (3) in this inequality yields,

\[ \frac{\delta + 2\gamma \varepsilon}{1 + 2\gamma} + \gamma \frac{\delta - \varepsilon}{1 + 2\gamma} + \gamma \frac{\delta - \varepsilon}{1 + 2\gamma} = \delta. \]

\[ \Rightarrow \varepsilon < h(x_n, x_m) \leq \varepsilon, \] which is a contradiction.

Therefore, \( H(Sx_n, Tx_m) \leq \varepsilon \). Hence (5).

(3) and (5) in (4), we get

\[ d( y_n, y_{n+1} ) \leq d( y_n, y_{n+1} ) + H(Sx_n, Tx_m) \]

\[ < \frac{\delta - \varepsilon}{1 + 2\gamma} + \varepsilon. \]

\[ = \frac{\delta - \varepsilon + 2\gamma \varepsilon}{1 + 2\gamma} = \frac{\delta + 2\gamma \varepsilon}{1 + 2\gamma}. \]

\[ d( y_n, y_{n+1} ) < \frac{\delta + 2\gamma \varepsilon}{1 + 2\gamma}. \]

This proves (3). Since \( \delta \leq 2\varepsilon \), then (3) implies that

\[ d( y_n, y_{m+1} ) < 2\varepsilon \] for all integers \( m \) and \( n \) with \( m \geq n \geq N_1 \) and hence \( \{ y_n \} \) is a Cauchy sequence.

Suppose \( f(Y) \) is complete subspace of \( X \), then there exists a point \( u \in Y \) such that \( fu = z \). To show that \( z = fu \in Su, \)
We suppose otherwise and use the condition (ii) we have
\[d(Su, Tx_n) \leq H(Su, Tx_n) \leq \varphi(h(u, x_n)) = \varphi(d(fu, fx_n)) + \gamma[d(Su, fu) + d(Tx_n, fx_n)]\).

Letting \(n \to \infty\), we get
\[d(Su, z) \leq \varphi(d(z, z)) + \gamma[d(Su, z) + d(z, z)] = \varphi(0 + \gamma[d(Su, z)]) = \varphi(\gamma d(Su, z)) < d(Su, z),\] (Since, \(\varphi(t) < t\))

Which is a contradiction.

Therefore, \(z = fu \in Su\).

Consequently, \(C(S, f)\) is non-empty. This proves (i).

Since \(SY \cup TY \subseteq fY\), there exists a point \(v \in Y\) such that \(z = fu = fv \in Tv\), so by (ii)
\[d(fv, Tv) = d(fu, Tv) \leq H(Su, Tv) \leq \varphi(h(u, v)) = \varphi(d(fu, fv)) + \gamma[d(Su, fu) + d(Tv, fv)] = \varphi(d(z, z)) + \gamma[d(z, z) + d(Tv, fv)]\]
\[d(fv, Tv) \leq \varphi(d(Tv, fv)) < d(Tv, fv),\] which is a contradiction.

Therefore, \(z = fu = fv \in Tv\).

Thus, \(C(T, f)\) is non-empty. This proves (ii).

Further, \(Su = SSu\) and The (IT)-Commutative of \(S\) and \(f\) at \(u \in C(S, f)\) implies that \(Su \in Sfu \subseteq fSu\). So, \(Su\) is a common fixed point of \(S\) and \(f\).

And \(Tv = TTv\) and the (IT)-Commutative of \(T\) and \(f\) at \(v \in C(S, f)\) implies that \(Tv \in Tfv \subseteq fTv\). So, \(Tv\) is a common fixed point of \(T\) and \(f\).

Since, \(z = fu \in Su\) and \(z = fu = fv \in Tv\).

Therefore, \(T, S\) and \(f\) have a common fixed point.

Analogous argument establishes the theorem when \(S(Y)\) or \(T(Y)\) is a complete sub space of \(X\). This completes the proof.
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References