On a Certain Subclass of Univalent Functions Defined by Differential Subordination Property

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Abstract

The object of the present paper is to investigate and study certain subclass of univalent functions defined by differential subordination by using the linear operator \( \mathcal{L}_{\lambda, \mu} \). Coefficient bounds, some properties of neighborhoods, convolution properties, Integral mean inequalities for the fractional integral for this certain subclass are given.
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1 Introduction

Let $G$ be the class of all functions of the form:

$$ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (n \in N), \quad (1.1) $$

which are analytic and univalent in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $A$ denote the subclass of $G$ containing functions of the form:

$$ f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, \quad n \in N). \quad (1.2) $$

The Hadamard product (or convolution) of two power series

$$ f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (1.3) $$

in $A$ is defined (as usual) by

$$ (f * g)(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \quad (1.4) $$

For positive real values of $\alpha_1, ..., \alpha_i$ and $\beta_1, ..., \beta_m (\beta_j \neq 0, -1, ..., j = 1, 2, ..., m)$, the generalized hypergeometric function $\,_{i}F_{m}(z)$ is defined by

$$ \,_{i}F_{m}(z) \equiv \,_{i}F_{m}(\alpha_1, ..., \alpha_i; \beta_1, ..., \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_i)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \quad (1.5) $$

($i \leq m + 1; i, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U$),

where $(a)_n$ is the Pochhammer symbol defined by

$$ (a)_n = \begin{cases} 1, & n = 0 \\ a(a + 1)(a + 2) \cdots (a + n - 1), & a \in N. \end{cases} \quad (1.6) $$

The notation $_{i}F_{m}$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel and Laguerre polynomial. Let
be a linear operator defined by

\[ H[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_m]: \mathcal{A} \to \mathcal{A} \]

\[ H[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_m]f(z) = z, F_m(\alpha_1, \alpha_2, \ldots, \alpha_i; \beta_1, \beta_2, \ldots, \beta_m; z) \ast f(z) \]

\[ = z - \sum_{n=2}^{\infty} w_n(\alpha_1; \iota; m) a_n \ z^n, \quad (1.7) \]

Where,

\[ w_n(\alpha_1; \iota; m) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_i)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{1}{(n-1)!} \cdot \quad (1.8) \]

For notational simplicity, we use shorter notation \( H_m[\alpha_1] \) for \( H[\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_m] \).

In the sequel. It follows from (1.7) that

\[ H_0^1[1]f(z) = f(z), \quad H_0^1[2]f(z) = zf'(z). \]

The linear operator \( H_m[\alpha_1] \) is called Dziok–Srivastava operator (see[3]) introduced by Dziok and Srivastava which was subsequently extended by Dziok and Raina [2] by using the generalized hypergeometric function, recently Srivastava et. al. ([10]) defined the linear operator \( \mathcal{L}_{\lambda, i, m}^{r, \alpha_1} \) as follows:-

\[ \mathcal{L}_{\lambda, i, m}^{1, \alpha_1} f(z) = f(z) \]

\[ \mathcal{L}_{\lambda, i, m}^{1, \alpha_1} f(z) = (1 - \lambda) H_m^{i \ast}[\alpha_1] f(z) + \lambda H_m^i[\alpha_1] f(z) \]

\[ = \mathcal{L}_{\lambda, i, m}^{\alpha_1} f(z), \quad (\lambda \geq 0), \quad (1.9) \]

\[ \mathcal{L}_{\lambda, i, m}^{2, \alpha_1} f(z) = \mathcal{L}_{\lambda, i, m}^{\alpha_1} \left( \mathcal{L}_{\lambda, i, m}^{1, \alpha_1} f(z) \right) \quad (1.10) \]

and in general,

\[ \mathcal{L}_{\lambda, i, m}^{r, \alpha_1} f(z) = \mathcal{L}_{\lambda, i, m}^{\alpha_1} \left( \mathcal{L}_{\lambda, i, m}^{r-1, \alpha_1} f(z) \right), \quad (i \leq m + 1; \ i, m \in N_0 = N \cup \{0\}; z \in U) \quad (1.11) \]

If the function \( f(z) \) is given by (1.2), then we see form (1.7), (1.8), (1.9) and (1.11) that
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\[ \mathcal{L}_{\lambda,\eta,m}^{\tau,a_1} f(z) = z - \sum_{n=2}^{\infty} w_n^{\tau}(a_1; \lambda; \eta; m) a_n z^n, \quad (1.12) \]

where,

\[ w_n^{\tau}(a_1; \lambda; \eta; m) = \left( \frac{(a_1)_{n-1} \cdots (a_1)_{n-1} \left[ 1 + \lambda (n - 1) \right]}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1} (n - 1)!} \right)^{\tau}, \]

\[ (n \in \mathbb{N} \setminus \{1\}, \tau \in \mathbb{N}_0). \quad (1.13) \]

Unless otherwise stated. We note that when \( \tau = 1 \) and \( \lambda = 0 \) the linear operator \( \mathcal{L}_{\lambda,\eta,m}^{\tau,a_1} \) would reduce to the familiar Dziok – Srivastava linear operator given by (see [3]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer[1], Owa[7] and Ruscheweyh[8].

For two analytic functions \( f, g \in \mathcal{A} \), we say that \( f \) is subordinate to \( g \), written \( f(z) \prec g(z) \) if there exists a Schwarz function \( w(z) \), which (by definition) is analytic in \( U \) with

\[ w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad \text{for all} \quad z \in U, \quad \text{such that} \quad f(z) = g(w(z)), z \in U. \]

Furthermore, if the function \( g(z) \) is univalent in \( U \), then we have the following equivalence (see [6]):

\[ f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

**Definition 1:** For any function \( f \in \mathcal{A} \) and \( \delta \geq 0 \), the \( \delta \) – neighborhood of \( f \) is defined as,

\[ N_\delta(f) = \left\{ \begin{array}{l} g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}: \sum_{n=2}^{\infty} n |a_n - b_n| \leq \delta \end{array} \right\}. \quad (1.14) \]

In particular, for the function \( e(z) = z \), we see that,

\[ N_\delta(e) = \left\{ \begin{array}{l} g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}: \sum_{n=2}^{\infty} n |b_n| \leq \delta \end{array} \right\}. \quad (1.15) \]

The concept of neighborhoods was first introduced by Goodman [4] and then generalized by Ruscheweyh [9].

**Definition 2:** For fixed parameters \( A \) and \( B \), with \(-1 \leq B < A \leq 1\), we say that \( f \in \mathcal{A} \) is in the class \( W(\tau, \theta, A) \) if it satisfies the following subordination condition:
\[
\frac{L^{r+\theta, \alpha_1}_\lambda f(z)}{L^{r, \alpha_1}_\lambda f(z)} < \frac{1 + Az}{1 + Bz}. \tag{1.16}
\]

In view of the definition of subordination, (1.16) is equivalent to the following condition:

\[
\left| \frac{L^{r+\theta, \alpha_1}_\lambda f(z)}{L^{r, \alpha_1}_\lambda f(z)} - 1 \right| < 1, \quad (z \in U).
\]

For convenience, we write
\[
W(\tau, \theta, \alpha_1, \lambda, \iota, m, 1 - 2\eta, -1) = W(\tau, \theta, \alpha_1, \lambda, \iota, m, \eta),
\]
where \( W(\tau, \theta, \alpha_1, \lambda, \iota, m, \eta) \) denotes the class of functions in \( \mathcal{A} \) satisfying the inequality:

\[
\text{Re} \left\{ \frac{L^{r+\theta, \alpha_1}_\lambda f(z)}{L^{r, \alpha_1}_\lambda f(z)} \right\} > \eta, \quad (0 \leq \eta < 1; \quad z \in U).
\]

2 Neighborhoods for the Class \( W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B) \)

**Theorem 2.1:** A function \( f \in \mathcal{A} \) belongs to the class \( W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B) \) if and only if

\[
\sum_{n=2}^{\infty} w_n^r(\alpha_1; \iota; m)(1 - B)w_n^\theta(\alpha_1; \iota; m) - (1 - A)a_n \leq A - B \tag{2.1}
\]

for \( \tau, \theta, \iota, m \in N_0, \iota \leq m + 1, \lambda \geq 0 \) and \( -1 \leq B < A \leq 1 \).

**Proof:** Let \( f \in W(\tau, \theta, \alpha_1, \lambda, t, m, A, B) \). Then,

\[
\frac{L^{r+\theta, \alpha_1}_\lambda f(z)}{L^{r, \alpha_1}_\lambda f(z)} < \frac{1 + Az}{1 + Bz} \quad z \in U. \tag{2.2}
\]

Therefore, there exists an analytic function \( w \) such that
\[ w(z) = \frac{L_{\alpha_1}^{r+\theta} f(z) - L_{\lambda,\alpha_1}^r f(z)}{BL_{\alpha_1}^{r+\theta} f(z) - AL_{\lambda,\alpha_1}^r f(z)}. \] (2.3)

Hence,

\[ |w(z)| = \left| \frac{L_{\alpha_1}^{r+\theta} f(z) - L_{\lambda,\alpha_1}^r f(z)}{BL_{\alpha_1}^{r+\theta} f(z) - AL_{\lambda,\alpha_1}^r f(z)} \right| < 1. \]

Taking \(|z| = r\), for sufficiently small \(r\) with \(0 < r < 1\), the denominator of (2.4) is positive and so it is positive for all \(r\) with \(0 < r < 1\), since \(w(z)\) is analytic for \(|z| < 1\). Then, the inequality (2.4) yields

\[ \sum_{n=2}^{\infty} w_n^r (\alpha_1; \lambda; \mu; m) \{w_n^\theta (\alpha_1; \lambda; \mu; m) - 1\} a_n r^n < (A - B)r + \sum_{n=2}^{\infty} w_n^r (\alpha_1; \lambda; \mu; m) \{Bw_n^\theta (\alpha_1; \lambda; \mu; m) - A\} a_n r^n. \]

Equivalently,

\[ \sum_{n=2}^{\infty} w_n^r (\alpha_1; \lambda; \mu; m) \{(1 - B)w_n^\theta (\alpha_1; \lambda; \mu; m) - 1 - A\} a_n r^n \leq (A - B)r, \]

and (2.1) follows upon letting \(r \to 1\).

Conversely, for \(|z| = r\), \(0 < r < 1\), we have \(r^n < r\). That is,

\[ \sum_{n=2}^{\infty} w_n^r (\alpha_1; \lambda; \mu; m) \{(1 - B)w_n^\theta (\alpha_1; \lambda; \mu; m) - (1 - A)\} a_n r^n \]
\[
\leq \sum_{n=2}^{\infty} w_n^r (\alpha_1; \lambda; t; m) \{ (1 - B) w_n^\theta (\alpha_1; \lambda; t; m) - (1 - A) \} a_n r \leq (A - B) r.
\]

From (2.1), we have
\[
\left| \sum_{n=2}^{\infty} w_n^r (\alpha_1; \lambda; t; m) \{ w_n^\theta (\alpha_1; \lambda; t; m) - 1 \} a_n z^n \right| \leq \sum_{n=2}^{\infty} w_n^r (\alpha_1; \lambda; t; m) \{ w_n^\theta (\alpha_1; \lambda; t; m) - 1 \} a_n r^n
\]
\[
< (A - B) r + \sum_{n=2}^{\infty} \left\{ B w_n^\theta (\alpha_1; \lambda; t; m) - A \right\} w_n^r (\alpha_1; \lambda; t; m) a_n r^n
\]
\[
< (A - B) z + \sum_{n=2}^{\infty} \left\{ B w_n^\theta (\alpha_1; \lambda; t; m) - A \right\} w_n^r (\alpha_1; \lambda; t; m) a_n z^n.
\]

This proves that
\[
\frac{L_{r, \alpha_1, t}^{r_1, \alpha_1} f(z)}{L_{r, \alpha_1, t}^{r_1, \alpha_1} f(z)} \leq \frac{1 + Az}{1 + Bz}, \quad z \in U
\]

and hence \( f \in W(\tau, \theta, \alpha_1, \lambda, t, m, A, B) \).

**Theorem 2.2** If
\[
\delta = \frac{(A - B)}{\left( \frac{(\alpha_1)_1 \cdots (\alpha_1)_1}{(\beta_1)_1 \cdots (\beta_1)_1} (1 + \lambda) \right)^{r - 1} \left( \frac{(\alpha_1)_1 \cdots (\alpha_1)_1}{(\beta_1)_1 \cdots (\beta_1)_1} (1 + \lambda) \right)^{\theta} - (1 - \lambda)}{((\alpha_1)_1 \cdots (\alpha_1)_1) (1 + \lambda)^{r - 1} (1 - \lambda)}, \quad (2.5)
\]
then \( W(\tau, \theta, \alpha_1, \lambda, t, m, A, B) \subset N_{\delta}(\epsilon) \).

**Proof:** It follows from (2.1), that if \( f \in W(\tau, \theta, \alpha_1, \lambda, t, m, A, B) \), then
\[
w_2^{\tau - 1} (\alpha_1; \lambda; t; m) \{ (1 - B) w_2^{\theta} (\alpha_1; \lambda; t; m) - (1 - A) \} \sum_{n=2}^{\infty} a_n \leq (A - B),
\]
Hence
\[
\left( \frac{(\alpha_1)_1 \cdots (\alpha_1)_1}{(\beta_1)_1 \cdots (\beta_1)_1} (1 + \lambda) \right)^{r - 1} \left( \frac{(\alpha_1)_1 \cdots (\alpha_1)_1}{(\beta_1)_1 \cdots (\beta_1)_1} (1 + \lambda) \right)^{\theta} - (1 - A) \sum_{n=2}^{\infty} a_n \leq (A - B). \quad (2.6)
\]
Which implies,

\[ \sum_{n=2}^{\infty} n a_n \leq \frac{(A - B)}{\left( \sum_{i=1}^{m} \frac{(\alpha_i)_1 \ldots (\alpha_i)_1}{(\beta)_1 \ldots (\beta)_1} (1 + \lambda) \right)^{r-1}} \left( 1 - B \right) \left( \sum_{i=1}^{k} \frac{(\alpha_i)_1 \ldots (\alpha_i)_1}{(\beta)_1 \ldots (\beta)_1} (1 + \lambda) \right)^{\theta} - (1 - A) \]

\[ = \delta. \]  

(2.7)

Using (1.15), we get the result.

**Definition (2.1):** The function \( g \) defined by

\[ g(z) = z - \sum_{n=2}^{\infty} b_n z^n \]

is said to be a member of the class \( W_\varphi(\tau, \theta, \alpha, \lambda, \iota, m, A, B) \) if there exists a function \( f \in W(\tau, \theta, \alpha, \lambda, \iota, m, A, B) \) such that

\[ \left| \frac{g(z)}{f(z)} - 1 \right| \leq 1 - y, \quad (z \in U, 0 \leq y < 1). \]  

(2.8)

**Theorem (2.3):** If \( f \in W(\tau, \theta, \alpha, \lambda, \iota, m, A, B) \) and

\[ y = 1 - \frac{\delta w^2(\alpha, \lambda; m)[(1 - B)w^2(\alpha, \lambda; m) - (1 - A)]}{2(w^2(\alpha, \lambda; m)[(1 - B)w^2(\alpha, \lambda; m) - (1 - A)] - (A - B))}, \]  

then \( N_\varphi(f) \subset W_\varphi(\tau, \theta, \alpha, \lambda, \iota, m, A, B) \).

**Proof:** Let \( g \in N_\varphi(f) \). Then we have from (1.14) that

\[ \sum_{n=2}^{\infty} n \left| a_n - b_n \right| \leq \delta, \]

which implies the coefficient inequality

\[ \sum_{n=2}^{\infty} \left| a_n - b_n \right| \leq \frac{\delta}{2}. \]

Also since \( f \in W(\tau, \theta, \alpha, \lambda, \iota, m, A, B) \), we have from (2.1)

\[ \sum_{n=2}^{\infty} a_n \leq \frac{(A - B)}{w^2(\alpha; \lambda; m)[(1 - B)w^2(\alpha; \lambda; m) - (1 - A)]}, \]
where

\[ w_2^r(\alpha_1; \lambda; \iota; m) = \left( \frac{(\alpha_1)_1 \ldots (\alpha_1)_1}{(\beta_1)_1 \ldots (\beta_m)_1} (1 + \lambda) \right)^r, \]

\[ w_2^\theta(\alpha_1; \lambda; \iota; m) = \left( \frac{(\alpha_1)_1 \ldots (\alpha_1)_1}{(\beta_1)_1 \ldots (\beta_m)_1} (1 + \lambda) \right)^\theta. \]

So that

\[
\left| \frac{g(z)}{f(z)} - 1 \right| = \left| \sum_{n=2}^{\infty} (a_n - b_n) z^n \right| < \sum_{n=2}^{\infty} \left| a_n - b_n \right| \frac{1}{1 - \sum_{n=2}^{\infty} a_n} \leq \frac{\delta}{2} \cdot \frac{w_2^r(\alpha_1; \lambda; \iota; m)[(1 - B)w_2^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}{w_n^r(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)] - (A - B)} = 1 - y.
\]

Thus by Definition (2.1), \( g \in W_\gamma(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B) \) for \( y \) given by (2.9). This completes the proof.

3 Convolution Properties:

**Theorem 3.1:** Let the functions \( f_j \ (j = 1, 2) \) defined by

\[ f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \geq 0, j = 1, 2) \]

be in the class \( W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B) \).

Then \( f_1 * f_2 \in W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, \sigma) \), where

\[
\sigma \leq \frac{w_n^r(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]^2 A - (A - B)^2 (w_n^r(\alpha_1; \lambda; \iota; m) - (1 - A))}{w_n^r(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]^2 - w_n^r(\alpha_1; \lambda; \iota; m)(A - B)^2}.
\]

**Proof:** We must find the largest \( \sigma \) such that

\[
\sum_{n=2}^{\infty} \frac{w_n^r(\alpha_1; \lambda; \iota; m)[(1 - \sigma)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - \sigma} a_{n,1}a_{n,2} \leq 1.
\]

Since \( f_j \in W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B) \) (\( j = 1, 2 \), then
\[ \sum_{n=2}^{\infty} w_n^r(\alpha_1; \lambda; \iota; m) \frac{[(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - B} a_{n,j} \leq 1, \quad (j = 1, 2). \quad (3.2) \]

By Cauchy-Schwarz inequality, we get
\[
\sum_{n=2}^{\infty} w_n^r(\alpha_1; \lambda; \iota; m) \frac{[(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - B} \frac{\sqrt{a_{n,1}a_{n,2}}}{\sqrt{a_{n,1}a_{n,2}}} \leq 1. \quad (3.3)
\]

We want only to show that
\[
\frac{w_n^r(\alpha_1; \lambda; \iota; m) [(1 - \sigma)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - \sigma} a_{n,1}a_{n,2} \leq \frac{\sqrt{a_{n,1}a_{n,2}}}{\sqrt{a_{n,1}a_{n,2}}}. \]

This equivalently to
\[
\sqrt{a_{n,1}a_{n,2}} \leq \frac{(A - \sigma)[(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}{(A - B)[(1 - \sigma)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}. \]

From (3.3), we have
\[
\sqrt{a_{n,1}a_{n,2}} \leq \frac{A - B}{w_n^r(\alpha_1; \lambda; \iota; m) [(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}. \]

Thus, it is sufficient to show that
\[
\frac{A - B}{w_n^r(\alpha_1; \lambda; \iota; m) [(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]} \leq \frac{(A - \sigma)[(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}{(A - B)[(1 - \sigma)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]}. \]

Which implies to
\[
\sigma \leq \frac{w_n^r(\alpha_1; \lambda; \iota; m) [(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]^2 - (A - B)^2(A - \sigma w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A))}{w_n^r(\alpha_1; \lambda; \iota; m) [(1 - B)w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A)]^2 - w_n^\theta(\alpha_1; \lambda; \iota; m)(A - B)^2}. \]
**Theorem (3.2):** Let the functions $f_j$ ($j = 1,2$) defined by (3.1) be in the class $W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B)$. Then the function $k$ defined by

$$k(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2)z^n$$  \hspace{1cm} (3.4)

belong to the class $W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B)$, where

$$\varepsilon \leq \frac{A(w_n^{\alpha}(\alpha_1; \lambda; \iota; m) - (1 - A]^2 - 2(\lambda - B)^2w_n^{\alpha,\varepsilon}(\alpha_1; \lambda; \iota; m) + 2(\lambda - B)^2(1 - A)w_n^{\alpha}(\alpha_1; \lambda; \iota; m)}{(w_n^{\alpha}(\alpha_1; \lambda; \iota; m))^2(1 - B)w_n^{\alpha}(\alpha_1; \lambda; \iota; m) - (1 - A)^2 - 2(\lambda - B)^2w_n^{\alpha,\varepsilon}(\alpha_1; \lambda; \iota; m)}$$

**Proof:** We must find the largest $\varepsilon$ such that

$$\sum_{n=2}^{\infty} \frac{w_n^{\alpha}(\alpha_1; \lambda; \iota; m)[(1 - \varepsilon)w_n^{\varepsilon}(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - \varepsilon} (a_{n,1}^2 + a_{n,2}^2) \leq 1.$$  \hspace{1cm} (3.5)

Since $f_j \in W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B)$ ($j = 1,2$), we get

$$\sum_{n=2}^{\infty} \left( \frac{w_n^{\alpha}(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^{\alpha}(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - B} \right)^2 a_{n,1}^2 \leq \left( \sum_{n=2}^{\infty} \frac{w_n^{\alpha}(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^{\alpha}(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - B} a_{n,1} \right)^2 \leq 1,$$  \hspace{1cm} (3.6)

and

$$\sum_{n=2}^{\infty} \left( \frac{w_n^{\alpha}(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^{\alpha}(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - B} \right)^2 a_{n,2}^2 \leq \left( \sum_{n=2}^{\infty} \frac{w_n^{\alpha}(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^{\alpha}(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - B} a_{n,2} \right)^2 \leq 1.$$  \hspace{1cm} (3.6)

Combining the inequalities (3.5) and (3.6), gives

$$\sum_{n=2}^{\infty} \frac{1}{2} \left( \frac{w_n^{\alpha}(\alpha_1; \lambda; \iota; m)[(1 - B)w_n^{\alpha}(\alpha_1; \lambda; \iota; m) - (1 - A)]}{A - B} \right)^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1.$$  \hspace{1cm} (3.7)
But, \( k \in W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, \varepsilon) \), if and only if
\[
\sum_{n=2}^{\infty} w_n^s(\alpha_1; \lambda; \iota; m) \left[ (1 - \varepsilon) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A) \right] \frac{A - \varepsilon}{\alpha_{n,1}^2 + \alpha_{n,2}^2} \leq 1.
\]
\[\text{(3.8)}\]

The inequality (3.8) will be satisfied if
\[
\frac{w_n^s(\alpha_1; \lambda; \iota; m) \left[ (1 - \varepsilon) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A) \right]}{A - \varepsilon} \leq \frac{(w_n^s(\alpha_1; \lambda; \iota; m))^2 \left[ (1 - B) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1 - A) \right]^2}{2(A - B)^2},
\]
\[\text{(n=2,3,...)}\]

4 Integral Mean Inequalities for the Fractional Integral:

**Definition (4.1) [6]:** The fractional integral of order \( s (s > 0) \) is defined for a function \( f \) by
\[
D_z^{-s} f(z) = \frac{1}{\Gamma(s)} \int_0^z \frac{f(t)}{(z - t)^{1-s}} \, dt,
\]
where the function \( f \) is an analytic in a simply-connected region of the complex \( z \)-plane containing the origin, and multiplicity of \( (z - t)^{s-1} \) is removed by requiring \( \log (z - t) \) to be real, when \( z - t > 0 \).

In 1925, Littlewood [5] proved the following subordination theorem:

**Theorem 4.1 (Littlewood [5]):** If \( f \) and \( g \) are analytic in \( U \) with \( f < g \), then for \( \mu > 0 \) and \( z = re^{i\theta} (0 < r < 1) \)
\[
\int_0^{2\pi} |f(z)|^\mu \, d\theta \leq \int_0^{2\pi} |g(z)|^\mu \, d\theta.
\]

**Theorem 4.2:** Let \( f \in W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B) \) and suppose that \( f_n \) is defined by
\[ f_n(z) = z - \frac{A-B}{w_n^*(\alpha_1;\lambda;z;m)(1-B)w_n^*(\alpha_1;\lambda;z;m) - (1-A)} \] \[ z^n, \quad (n \geq 2). \] (4.2)

Also let
\[
\begin{align*}
\sum_{i=2}^{\infty} (i-\eta)_{\eta+1} a_i \\
\leq \frac{(A-B)\Gamma(n+1)\Gamma(s+\eta+3)}{w_n^*(\alpha_1;\lambda;z;m)[(1-B)w_n^*(\alpha_1;\lambda;z;m) - (1-A)]\Gamma(n + s + \eta + 1)\Gamma(2-\eta)},
\end{align*}
\] (4.3)

for \( 0 \leq \eta \leq i \), \( s > 0 \), where \((i - \eta)_{\eta+1}\) denote the pochhammer symbol defined by \((i - \eta)(i - \eta + 1) \ldots i\).

If there exists an analytic function \( q \) defined by
\[
(q(z))^n^{-1} = \frac{w_n^*(\alpha_1;\lambda;z;m)[(1-B)w_n^*(\alpha_1;\lambda;z;m) - (1-A)]\Gamma(n + s + \eta + 1)}{(A-B)\Gamma(n+1)} \sum_{i=p+1}^{\infty} (i-\eta)_{\eta+1} H(i) a_i z^{i-1},
\] (4.5)

where \( i \geq \eta \) and
\[
H(i) = \frac{\Gamma(i-\eta)}{\Gamma(i+s+\eta+1)}, \quad (s > 0, \ i \geq 2),
\] (4.6)

then, for \( z = re^{iy} \) and \( 0 < r < 1 \)
\[
\int_0^{2\pi} |D_z^{-s-\eta} f(z)|^\mu dy \leq \int_0^{2\pi} |D_z^{-s-\eta} f_n(z)|^\mu dy, \quad (s > 0, \mu > 0). \] (4.7)

**Proof:** Let
\[ f(z) = z - \sum_{i=2}^{\infty} a_i z^i. \]

For \( \eta \geq 0 \) and Definition(4.1), we get
\[
D_z^{-s-\eta} f(z) = \frac{\Gamma(2)z^{s+\eta+1}}{\Gamma(s+\eta+2)} \left( 1 - \sum_{i=2}^{\infty} \frac{\Gamma(i+1)\Gamma(s+\eta+2)}{\Gamma(2)\Gamma(i+s+\eta+1)} a_i z^{i-1} \right)
\]
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\[
= \frac{\Gamma(2)z^{5+\eta+1}}{\Gamma(s + \eta + 2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \eta + 2)}{\Gamma(2)} (i - \eta)_{\eta+1} H(i) a_i z^{i-1}\right),
\]

where

\[
H(i) = \frac{\Gamma(i - \eta)}{\Gamma(i + s + \eta + 1)}, \quad (s > 0, i \geq 2).
\]

Since \( H \) is a decreasing function of \( i \), we have

\[
0 < H(i) \leq H(2) = \frac{\Gamma(2 - \eta)}{\Gamma(s + \eta + 3)}.
\]

Similarly, from (4.2) and Definition 4.1, we get

\[
D_z^{-s-\eta} f_n (z) = \frac{\Gamma(2)z^{5+\eta+1}}{\Gamma(s + \eta + 2)} \left(1 - \frac{(A - B)\Gamma(n + 1)\Gamma(s + \eta + 2)}{w_n^\tau(\alpha_1; \lambda; \iota; m)\Gamma(n + s + \eta + 1)} z^{-n-1}\right).
\]

For \( \mu > 0 \) and \( z = re^{iy} \) (0 < \( r < 1 \)), we must show that

\[
\int_0^{2\pi} \left| 1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \eta + 2)}{\Gamma(2)} (i - \eta)_{\eta+1} H(i) a_i z^{i-1}\right|^\mu dy.
\]

By applying Littlewood’s subordination theorem, it would suffice to show that

\[
1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \eta + 2)}{\Gamma(2)} (i - \eta)_{\eta+1} H(i) a_i z^{i-1}
\]

\[
< 1 - \frac{(A - B)\Gamma(n + 1)\Gamma(s + \eta + 2)}{w_n^\tau(\alpha_1; \lambda; \iota; m)\Gamma(n + s + \eta + 1)} z^{-n-1}.
\]

By setting

\[
1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \eta + 2)}{\Gamma(2)} (i - \eta)_{\eta+1} H(i) a_i z^{i-1}
\]
we find that
\[
(q(z))^n - 1 = \frac{w_n^R(\alpha_1; \lambda; \nu; m)(1 - B)w_n^B(\alpha_1; \lambda; \nu; m) - (1 - A)\Gamma(n + s + \eta + 1)}{(A - B)\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1}H(i)a_i z^{i-1},
\]
which readily yields \( w(0) = 0 \). For such a function \( q \), we obtain

\[
|w(z)|^n - 1 \leq \frac{w_n^R(\alpha_1; \lambda; \nu; m)(1 - B)w_n^B(\alpha_1; \lambda; \nu; m) - (1 - A)\Gamma(n + s + \eta + 1)}{(A - B)\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1}H(i)a_i |z|^{i-1}
\]

\[
\leq \frac{w_n^R(\alpha_1; \lambda; \nu; m)(1 - B)w_n^B(\alpha_1; \lambda; \nu; m) - (1 - A)\Gamma(n + s + \eta + 1)}{(A - B)\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1}H(i)a_i |z|^{i-1} \leq |z| < 1.
\]

This completes the proof of the theorem.

By taking \( \eta = 0 \) in the Theorem 4.2, we have the following corollary:

**Corollary 4.1:** Let \( f \in W(\tau, \theta, \alpha, \lambda, \nu, m, A, B) \) and suppose that \( f_n \) is defined by (4.2). Also let

\[
\sum_{i=2}^{\infty} i^\alpha_i \leq \frac{w_n^R(\alpha_1; \lambda; \nu; m)(1 - B)w_n^B(\alpha_1; \lambda; \nu; m) - (1 - A)\Gamma(n + s + \eta + 1)}{(A - B)\Gamma(n + 1)\Gamma(s + 3)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1}a_i \geq 2.
\]

If there exists an analytic function \( q \) defined by
\[(q(z))^{n-1} = \frac{\wp_n^\tau(\alpha_1; \lambda; \tau; m)}{(A - B)\Gamma(n + 1)} \{ (1 - B)w_n^\theta(\alpha_1; \lambda; \tau; m) - (1 - A) \} \Gamma(n + s + 1) \sum_{i=2}^\infty iH(i) a_i z^{i-1},\]

where
\[
H(i) = \frac{\Gamma(i)}{\Gamma(i + s + 1)}, \quad (s > 0, i \geq 2),
\]

then, for \( z = re^{iy} \) and \( 0 < r < 1 \)

\[
\int_0^{2\pi} |D_z^{-s} f(z)|^\mu \, dy \leq \int_0^{2\pi} |D_z^{-s} f_n(z)|^\mu \, dy, \quad (s > 0, \mu > 0)
\]

**References**


