Determinant and Permanent of Hessenberg Matrix and Fibonacci Type Numbers

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Abstract
In this paper, we obtain determinants and permanents of some Hessenberg matrices that give the terms of k sequences of generalized order-k Fibonacci numbers for k = 2. These results are important, since k sequences of generalized order-k Fibonacci numbers for k = 2 are general form of ordinary Fibonacci sequence, Pell sequence and Jacobsthal sequence.

Keywords: Fibonacci Numbers, Jacobsthal Numbers, k sequences of generalized order-k Fibonacci numbers, Pell Numbers, Hessenberg Matrix.

1 Introduction
Fibonacci numbers \(F_n\), Pell numbers \(P_n\) and Jacobsthal numbers \(J_n\) are defined by

\[
F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2 \quad \text{and} \quad F_1 = F_2 = 1,
\]
\[
P_n = 2P_{n-1} + P_{n-2} \quad \text{for } n > 1 \quad \text{and} \quad P_0 = 0, \quad P_1 = 1,
\]
\[
J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n > 2 \quad \text{and} \quad J_1 = J_2 = 1,
\]

respectively.

Generalizations of these sequences have been studied by many researchers.
Er [3] defined $k$ sequences of generalized order-$k$ Fibonacci numbers ($k$SO$k$F) as, for $n > 0$, $1 \leq i \leq k$

\[
f_{k,n}^i = \sum_{j=1}^{k} c_j f_{k,n-j}^i \tag{1}
\]

with boundary conditions for $1 - k \leq n \leq 0$,

\[
f_{k,n}^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,}
\end{cases}
\]

where $c_j$ ($1 \leq j \leq k$) are constant coefficients, $f_{k,n}^i$ is the $n$-th term of $i$-th sequence of order $k$ generalization.

**Example 1.1** $f_{k,n}^1$ and $f_{k,n}^2$ sequences are

\[
\begin{align*}
0, & \quad 1, \quad c_1, \quad c_2 + c_1^2, \quad 2c_1c_2 + c_1^3, \quad c_2^2 + c_1^2 + 3c_1c_2 + 4c_1^3c_2, \\
& \quad c_1^3 + 5c_1^5c_2 + 6c_1^2c_2^3, \quad c_1^4 + 4c_1^3c_2 + 6c_1^5c_2^2 + 10c_1^3c_2^3, \\
& \quad c_1^4 + 7c_1^6c_2 + 10c_1^4c_2^3 + 15c_1^6c_2^3,
\end{align*}
\]

and

\[
\begin{align*}
1, & \quad 0, \quad c_2, \quad c_1c_2, \quad c_2^2 + c_1^2c_2, \quad 2c_1c_2^2 + c_1^3c_2, \quad c_2^3 + c_1^4c_2 + 3c_1^2c_2^2, \\
& \quad 3c_1c_2^3 + c_1^5c_2 + 4c_1^3c_2^3, \quad c_2^4 + c_1^6c_2^2 + 6c_1^2c_2^2 + 5c_1^4c_2^2, \\
& \quad 4c_1^4c_2^4 + c_1^7c_2 + 10c_1^3c_2^3 + 6c_1^5c_2^2,
\end{align*}
\]

respectively.

A direct consequence of (1) is

\[
f_{k,n}^2 = c_2 f_{k,n-1}^1, \quad \text{for } n \geq 0. \tag{2}
\]

**Remark 1.2** Let $f_{k,n}^i$, $F_n$, $P_n$ and $J_n$ be $k$SO$k$F (1), Fibonacci sequence, Pell sequence and Jacobsthal sequence, respectively. Then,

(i) Substituting $c_1 = c_2 = 1$ for $k = 2$ in (1), we obtain $f_{k,n-1}^1 = F_n$.

(ii) Substituting $c_1 = 2$ and $c_2 = 1$ for $k = 2$ in (1), we obtain $f_{k,n-1}^1 = P_n$.

(iii) Substituting $c_1 = 1$ and $c_2 = 2$ for $k = 2$ in (1), we obtain $f_{k,n-1}^1 = J_n$. 
Remark 1.2 shows that \( f_{k,n}^1 \) is a general form of Fibonacci sequence, Pell sequence and Jacobsthal sequence. Therefore, any result obtained from \( f_{k,n}^1 \) holds for other sequences mentioned above.

Many researchers studied on determinantal and permanental representations of \( k \) sequences of generalized order-\( k \) Fibonacci and Lucas numbers. For example, Minc [7] defined an \( n \times n \) (0,1)-matrix \( F(n,k) \), and showed that the permanents of \( F(n,k) \) is equal to the generalized order-\( k \) Fibonacci numbers.

The author of [5, 6] defined two \((0,1)\)-matrices and showed that the permanents of these matrices are the generalized Fibonacci and Lucas numbers. Öcal et al. [8] gave some determinantal and permanental representations of \( k \)-generalized Fibonacci and Lucas numbers, and obtained Binet’s formula for these sequences. Yilmaz and Bozkurt [9] derived some relationships between Pell and Perrin sequences, and permanents and determinants of a type of Hessenberg matrices.

In this paper, we give some determinantal and permanental representations of \( k \) sequences of generalized order-\( k \) Fibonacci numbers for \( k = 2 \) by using various Hessenberg matrices. These results are general form of determinantal and permanental representations of ordinary Fibonacci numbers, Pell numbers and Jacobsthal numbers.

\section{The Determinantal Representations}

An \( n \times n \) matrix \( A_n = (a_{ij}) \) is called lower Hessenberg matrix if \( a_{ij} = 0 \) when \( j - i > 1 \), i.e.,

\[
A_n = \begin{bmatrix}
a_{11} & a_{12} & 0 & \cdots & 0 \\
a_{21} & a_{22} & a_{23} & \cdots & 0 \\
a_{31} & a_{32} & a_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n}
\end{bmatrix}.
\]  

\( (3) \)

\textbf{Theorem 2.1} [2] Let \( A_n \) be an \( n \times n \) lower Hessenberg matrix for all \( n \geq 1 \) and \( \det(A_0) = 1 \). Then,

\[
\det(A_1) = a_{11}
\]

and for \( n \geq 2 \)

\[
\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} (-1)^{n-r} a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} \det(A_{r-1}) \quad (4)
\]

An n x n matrix A_n = (a_{ij}) is called lower Hessenberg matrix if a_{ij} = 0 when j - i > 1, i.e.,
Theorem 2.2 Let \( f_{2,n} \) be the first sequence of 2SO2F and \( Q_n = (q_{rs})_{n \times n} \) be a Hessenberg matrix defined by

\[
q_{rs} = \begin{cases} 
i^{r-s} \cdot \frac{c_{r-s+1}}{c_2}, & \text{if } -1 \leq r - s < 2, \\ 0, & \text{otherwise,} \end{cases}
\]

that is

\[
Q_n = \begin{bmatrix}
c_1 & ic_2 & 0 & 0 & \cdots & 0 \\
i & c_1 & ic_2 & 0 & \cdots & 0 \\
0 & i & c_1 & ic_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & c_1 & ic_2 \\
0 & 0 & 0 & 0 & i & c_1
\end{bmatrix}.
\] (5)

Then,

\[
\det(Q_n) = f_{2,n}^1,
\] (6)

where \( c_0 = 1 \) and \( i = \sqrt{-1} \).

Proof. To prove (6), we use the mathematical induction on \( m \). The result is true for \( m = 1 \) by hypothesis.

Assume that it is true for all positive integers less than or equal to \( m \), namely \( \det(Q_m) = f_{2,m}^1 \). Then, we have

\[
\begin{align*}
\det(Q_{m+1}) &= q_{m+1,m+1} \det(Q_m) + \sum_{r=1}^{m-1} (-1)^{m+1-r} q_{m+1,r} \prod_{j=r}^{m} q_{j,j+1} \det(Q_{r-1}) \\
&= c_1 \det(Q_m) + \sum_{r=1}^{m-1} (-1)^{m+1-r} q_{m+1,r} \prod_{j=r}^{m} q_{j,j+1} \det(Q_{k,r-1}) \\
& \quad + (-1)^{q_{m+1,m} q_{m,m+1} \det(Q_{k,m-1})} \\
&= c_1 \det(Q_m) + (-1)ic_2i \det(Q_{k,m-1}) \\
&= c_1 \det(Q_m) + c_2 \det(Q_{k,m-1})
\end{align*}
\]

by using Theorem 2.1. From the hypothesis of induction and the definition of 2SO2F, we obtain

\[
\det(Q_{m+1}) = c_1 f_{2,m}^1 + c_2 f_{2,m-1}^1 = f_{2,m+1}^1.
\]

Therefore, (6) holds for all positive integers \( n \).
Example 2.3 We obtain \( f_{2,6} \), by using Theorem 2.2

\[
\text{det}(Q_6) = \text{det} \begin{bmatrix}
  c_1 & ic_2 & 0 & 0 & 0 & 0 \\
  i & c_1 & ic_2 & 0 & 0 & 0 \\
  0 & i & c_1 & ic_2 & 0 & 0 \\
  0 & 0 & i & c_1 & ic_2 & 0 \\
  0 & 0 & 0 & i & c_1 & ic_2 \\
  0 & 0 & 0 & 0 & i & c_1
\end{bmatrix}
\]
\[
= c_2^3 + c_2^6 + 5c_1^4c_2 + 6c_1^2c_2^2
\]
\[
= f_{2,6}.
\]

Theorem 2.4 Let \( f_{2,n} \) be the first sequence of 2SO2F and \( B_n = (b_{ij})_{n \times n} \) be a Hessenberg matrix, where

\[
b_{ij} = \begin{cases} 
-c_2, & \text{if } j = i + 1, \\
\frac{c_{i+j+1}}{c_2^{i+j}}, & \text{if } 0 \leq i - j < 2, \\
0, & \text{otherwise},
\end{cases}
\]

that is

\[
B_n = \begin{bmatrix}
  c_1 & -c_2 & 0 & \cdots & 0 & 0 \\
  1 & c_1 & -c_2 & \cdots & 0 & 0 \\
  0 & 1 & c_1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & c_1 & -c_2 \\
  0 & 0 & 0 & \cdots & 1 & c_1
\end{bmatrix}
\]

Then,

\[
\text{det}(B_n) = f_{2,n}^1,
\]

where \( c_0 = 1 \).

Proof. Since the proof is similar to the proof of Theorem 2.2 by using Theorem 2.1, we omit the detail.

Example 2.5 We obtain \( f_{2,4} \) by using Theorem 2.4 that

\[
\text{det}(B_5) = \text{det} \begin{bmatrix}
  c_1 & -c_2 & 0 & 0 \\
  1 & c_1 & -c_2 & 0 \\
  0 & 1 & c_1 & -c_2 \\
  0 & 0 & 1 & c_1
\end{bmatrix}
\]
\[
= c_2^2 + c_1^4 + 3c_1^2c_2
\]
\[
= f_{2,4}.
\]
Corollary 2.6 If we rewrite Theorem 2.2 and Theorem 2.4 for \( c_i = 1 \), then we obtain \( \det(Q_n) = F_{n+1} \) and \( \det(B_n) = F_{n+1} \), respectively, where \( F_n \)'s are the ordinary Fibonacci numbers.

Corollary 2.7 If we rewrite Theorem 2.2 and Theorem 2.4 for \( c_1 = 2 \) and \( c_2 = 1 \), then we obtain \( \det(Q_n) = P_{n+1} \) and \( \det(B_n) = P_{n+1} \), respectively, where \( P_n \)'s are the Pell numbers.

Corollary 2.8 If we rewrite Theorem 2.2 and Theorem 2.4 for \( c_1 = 1 \) and \( c_2 = 2 \), then we obtain \( \det(Q_n) = J_{n+1} \) and \( \det(B_n) = J_{n+1} \), respectively, where \( J_n \)'s are the Jacobsthal numbers.

3 The Permanent Representations

Let \( A = (a_{i,j}) \) be an \( n \times n \) matrix over a ring. Then, the permanent of \( A \) is defined by

\[
\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},
\]

where \( S_n \) denotes the symmetric group on \( n \) letters.

Theorem 3.1 \([8]\) Let \( A_n \) be an \( n \times n \) lower Hessenberg matrix for all \( n \geq 1 \) and \( \text{per}(A_0) = 1 \). Then, \( \text{per}(A_1) = a_{11} \) and for \( n \geq 2 \)

\[
\text{per}(A_n) = a_{n,n} \text{per}(A_{n-1}) + \sum_{r=1}^{n-1} a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} \text{per}(A_{r-1}).
\] (7)

Theorem 3.2 Let \( f_{2,n}^{1} \) be the first sequence of SO2F and \( H_n = (h_{rs}) \) be an \( n \times n \) Hessenberg matrix, where

\[
h_{rs} = \begin{cases} i^{(r-s)} \frac{c_{r-s+1}}{c_2}, & \text{if } -1 \leq r - s < 2, \\ 0, & \text{otherwise}, \end{cases}
\]

that is

\[
H_n = \begin{bmatrix} c_1 & -ic_2 & 0 & \cdots & 0 & 0 \\ i & c_1 & -ic_2 & 0 & 0 \\ 0 & i & c_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 & -ic_2 \\ 0 & 0 & 0 & \cdots & i & c_1 \end{bmatrix}.
\] (8)

Then,

\[
\text{per}(H_n) = f_{2,n}^{1},
\]

where \( c_0 = 1 \) and \( i = \sqrt{-1} \).
Proof. This is similar to the proof of Theorem 2.2 using Theorem 3.1.

Example 3.3 We obtain $f_{2,3}^1$ by using Theorem 3.2 that

$$\begin{align*}
\text{per}(H_{4,3}) &= \text{per} \begin{bmatrix}
c_1 & -ic_2 & 0 \\
i & c_1 & -ic_2 \\
0 & i & c_1
\end{bmatrix} \\
&= 2c_1c_2 + c_1^3 \\
&= f_{2,3}^1.
\end{align*}$$

Theorem 3.4 Let $f_{2,n}^1$ be the first sequence of $2SO2F$ and $L_n = (l_{ij})$ be an $n \times n$ lower Hessenberg matrix given by

$$l_{ij} = \begin{cases}
c_{i-j+1} / c_2^{i-j}, & \text{if } 0 \leq i - j < 2, \\
0, & \text{otherwise,}
\end{cases}$$

that is

$$L_n = \begin{bmatrix}
c_1 & c_2 & 0 & \cdots & 0 & 0 \\
1 & c_1 & c_2 & \cdots & 0 & 0 \\
0 & 1 & c_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_1 & c_2 \\
0 & 0 & 0 & \cdots & 1 & c_1
\end{bmatrix}.$$ 

Then,

$$\text{per}(L_n) = f_{2,n}^1,$$

where $c_0 = 1$.

Proof. This is similar to the proof of Theorem 2.2 by using Theorem 3.1.

Corollary 3.5 If we rewrite Theorem 3.2 and Theorem 3.4 for $c_i = 1$, we obtain $\text{per}(H_n) = F_{n+1}$ and $\text{per}(L_n) = F_{n+1}$, respectively, where $F_n$’s are the Fibonacci numbers.

Corollary 3.6 If we rewrite Theorem 3.2 and Theorem 3.4 for $c_1 = 2$ and $c_2 = 1$, we obtain $\text{per}(H_n) = P_{n+1}$ and $\text{per}(L_n) = P_{n+1}$, respectively, where $P_n$’s are the Pell numbers.

Corollary 3.7 If we rewrite Theorem 3.2 and Theorem 3.4 with $c_1 = 1$ and $c_2 = 2$, then we obtain $\text{per}(H_n) = J_{n+1}$ and $\text{per}(L_n) = J_{n+1}$, respectively, where $J_n$’s are the Jacobsthal numbers.
3.1 Binet’s formula for 2 sequences of generalized order−2 Fibonacci numbers (2SO2F)

Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series of the analytical function $f$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ when } f(0) \neq 0$$

and

$$A_n = \begin{bmatrix}
a_1 & a_0 & 0 & \cdots & 0 \\
a_2 & a_1 & a_0 & \cdots & 0 \\
a_3 & a_2 & a_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_1
\end{bmatrix}_{n \times n}.$$

Then, the reciprocal of $f(z)$ can be written in the following form

$$g(z) = \frac{1}{f(z)} = \sum_{n=0}^{\infty} (-1)^n \det(A_n) z^n,$$

whose radius of converge is $\inf\{|\lambda| : f(\lambda) = 0\}$, [1].

Let

$$p_k(z) = 1 + a_1 z + \cdots + a_k z^k.$$  \hspace{1cm} (9)

Then, the reciprocal of $p_k(z)$ is

$$\frac{1}{p_k(z)} = \sum_{n=0}^{\infty} (-1)^n \det(A_{k,n}) z^n,$$

where

$$A_{k,n} = \begin{bmatrix}
a_1 & 1 & 0 & \cdots & 0 \\
a_2 & a_1 & 1 & \cdots & 0 \\
a_3 & a_2 & a_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_k & a_{k-1} & a_{k-2} & \cdots & 0 \\
0 & a_k & a_{k-1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_k & \cdots & a_1
\end{bmatrix}_{n \times n}.$$  \hspace{1cm} (10)

Inselberg [4] showed that

$$\det(A_{k,n}) = \sum_{j=1}^{k} \frac{1}{p_k'(\lambda_j)} \left(\frac{-1}{\lambda_j}\right)^{n+1} \quad (n \geq k)$$  \hspace{1cm} (11)
if \( p_k(z) \) has distinct zeros \( \lambda_j \) for \( j \in \{1, 2, \ldots, k\} \); where \( p'_k(z) \) is the derivative of polynomial \( p_k(z) \) in (9).

**Theorem 3.8** Let \( f_{2,n}^1 \) be the first sequence of \( 2SO2F \). Then, for \( n \geq 2 \) and \((c_1)^2 + 4c_1c_2 > 0\),

\[
f_{2,n}^1 = \sum_{j=1}^{k} \frac{1}{p'(\lambda_j)} \left(\frac{-1}{\lambda_j} \right)^{n+1},
\]

where \( p(z) = 1 + c_1 z - c_2 z^2 \) and \( p'(z) \) denotes the derivative of polynomial \( p(z) \).

**Proof.** This is immediate from Theorems 2.4 and (11).

**Corollary 3.9** Let \( f_{2,n}^2 \) be the second sequences of \( 2SO2F \). Then,

\[
f_{2,n+1}^2 = c_2 \sum_{j=1}^{k} \frac{-1}{p'(\lambda_j)} \left(\frac{1}{\lambda_j} \right)^{n+1}
\]

for \( n \geq 2 \).

**Proof.** One can easily obtain the proof from (2) and Theorem 3.8.

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**References**


