**δ̂g-Closed Sets in Topological Spaces**

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**Abstract**

In this paper a new class of sets, namely δ̂g-closed sets is introduced in topological spaces. We prove that this class lies between the class of δ-closed sets and the class of δg-closed sets. Also we find some basic properties and applications of δ̂g-closed sets. We also introduce and study a new class of space namely T̂_{3\frac{1}{4}}-space.

**Keywords:** generalized closed sets, δ̂g-closed sets, δ-closure, ̂g-open sets and T̂_{3\frac{1}{4}}-space.

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1 **Introduction**

Levine [4], Mashhour et al.[8], Njastad[10] and Velicko[13] introduced semi-open sets, pre-open sets, α-open sets and δ-closed sets respectively. Levine[5] introduced generalized closed (briefly g-closed) sets and studied their basic properties. Bhattacharya and Lahiri[2], Arya and Nour[1], Maki et a [6,7], Dontchev and Ganster[3] introduced semi-generalized closed (briefly sg-closed) sets, generalized semi-closed (briefly gs-closed) sets, generalized α-closed (briefly ga-closed) sets, α-generalized closed (briefly ag-closed) sets and δ-generalized closed (briefly δg-closed) sets respectively. Veera Kumar [12] introduced ̂g-closed sets in topological spaces. The purpose of this present paper is to define a new class of closed sets called δ̂g-closed sets and also we obtain some basic properties of δ̂g-closed sets in topological spaces. Applying these sets, we obtain a new space which is called T̂_{3\frac{1}{4}}-space.
2 Preliminaries

Throughout this paper (X, τ) (or simply X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively. Let us recall the following definitions, which are useful in the sequel.

Definition 2.1 A subset A of a space (X, τ) is called a
1. semi-open set [4] if A ⊆ cl(int(A)).
2. pre-open set [8] if A ⊆ int(cl(A)).
3. α-open set [10] if A ⊆ int(cl(int(A))).
4. regular open set [11] if A = int(cl(A)).

The complement of a semi-open(resp. pre-open, α-open, regular open) set is called semi-closed (resp. semi-closed, α-closed, regular closed).

Definition 2.2 The δ-interior[13] of a subset A of X is the union of all regular open set of X contained in A and is denoted by Int_δ(A). The subset A is called δ-open[13] if A = Int_δ(A), i.e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open is called δ-closed. Alternatively, a set A ⊆ (X, τ) is called δ-closed [13] if A = cl_δ(A), where cl_δ(A) = { x ∈ X: int(cl(U)) ∩ A ≠ φ, U ∈ τ and x ∈ U }.

Definition 2.3 A subset A of (X, τ) is called
1. generalized closed (briefly g-closed) set[5] if cl(A) ⊆ U whenever A ⊆ U and U is open set in (X, τ).
2. semi-generalized closed (briefly sg-closed) set [2] if scl(A) ⊆ U whenever A ⊆ U and U is a semi-open set in (X, τ).
3. generalized semi-closed (briefly gs-closed) set [1] if scl(A) ⊆ U whenever A ⊆ U and U is open set in (X, τ).
4. α- generalized closed (briefly αg-closed) set [7] if αcl(A) ⊆ U whenever A ⊆ U and U is open set in (X, τ).
5. generalizedα-closed (briefly αα-closed) set [6] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open set in (X, τ).
6. δ-generalized closed (briefly δg-closed) set [3] if cl_δ(A) ⊆ U whenever A ⊆ U and U is open set in (X, τ).
7. ĝ-closed set [12] if cl(A) ⊆ U whenever A ⊆ U and U is a semi-open set in (X, τ).
(viii) $\alpha$-$\hat{g}$-closed (briefly $\alpha\hat{g}$-closed) set [9] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\hat{g}$-open set in $(X,\tau)$.

The complement of a $g$-closed (resp. $sg$-closed, $gs$-closed, $g\alpha$-closed, $\delta g$-closed and $\hat{g}$-closed and $\alpha\hat{g}$-closed) set is called $g$-open (resp. $sg$-open, $gs$-open, $g\alpha$-open, $\delta g$-open, $\hat{g}$-open and $\alpha\hat{g}$-open).

**Theorem 2.4** Every open set is $\hat{g}$-open.

*Proof*: Let $A$ be an open set in $X$. Then $A^c$ is closed. Therefore, $\text{Cl}(A^c) = A^c \subseteq X$ whenever $A^c \subseteq X$ and $X$ is semi-open. This implies $A^c$ is $\hat{g}$-closed. Hence $A$ is $\hat{g}$-open.

**Definition 2.5** A space $(X,\tau)$ is called

(i) $T_{1/2}$-space [5] if every $g$-closed set in it is closed.

(ii) $T_{3/4}$-space [3] if every $\delta g$-closed set in it is $\delta$-closed.

(iii) $T_{\alpha\hat{g}}$-space [9] if every $\alpha\hat{g}$-closed set in it is $\alpha$-closed.

## 3 $\delta\hat{g}$-Closed Sets

We introduce the following definition.

**Definition 3.1** A subset $A$ of a space $(X,\tau)$ is called $\delta\hat{g}$-closed if $\text{cl}_{\delta\hat{g}}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\hat{g}$-open set in $(X,\tau)$.

**Proposition 3.2** Every $\delta$-closed set is $\delta\hat{g}$-closed set.

*Proof*: Let $A$ be an $\delta$-closed set and $U$ be any $\hat{g}$-open set containing $A$. Since $A$ is $\delta$-closed, $\text{cl}_{\delta}(A) = A$ for every subset $A$ of $X$. Therefore $\text{cl}_{\delta}(A) \subseteq U$ and hence $A$ is $\delta\hat{g}$-closed set.

**Remark 3.3** The converse of the above theorem is not true as shown in the following example.

**Example 3.4** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$

$\delta$-closed $= \{\phi, X, \{b\}, \{a, c\}\}$; $\delta\hat{g}$-closed $= \{\phi, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$

Here $\{b, c\}$ is $\delta\hat{g}$-closed but not $\delta$-closed in $(X,\tau)$.

**Proposition 3.5** Every $\delta\hat{g}$-closed set is $g$-closed.

*Proof*: Let $A$ be an $\delta\hat{g}$-closed set and $U$ be an any open set containing $A$ in $(X,\tau)$. Since every open set is $\hat{g}$-open and $A$ is $\delta\hat{g}$-closed, $\text{cl}_{\delta}(A) \subseteq U$ for every subset $A$ of $X$. Since $\text{cl}(A) \subseteq \text{cl}_{\delta}(A) \subseteq U, \text{cl}(A) \subseteq U$ and hence $A$ is $g$-closed.
Remark 3.6 An $g$-closed set need not be $\hat{\delta}g$-closed set as shown in the following example.

Example 3.7 Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, X, \{b\}, \{a, c\}\}$. Then the set $\{a\}$ is $g$-closed but not $\hat{\delta}g$-closed in $(X, \tau)$.

Proposition 3.8 Every $\hat{\delta}g$-closed set is $gs$-closed.

proof: Let $A$ be an $\hat{\delta}g$-closed and $U$ be any open set containing $A$ in $(X, \tau)$. Since every open set is $\hat{g}$-open, $cl_\delta(A) \subseteq U$ for every subset $A$ of $X$. Since $scl(A) \subseteq cl_\delta(A) \subseteq U$, $scl(A) \subseteq U$ and hence $A$ is $gs$-closed.

Remark 3.9 A $gs$-closed set need not be $\hat{\delta}g$-closed as shown in the following example.

Example 3.10 Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then the set $\{c\}$ is $gs$-closed but not $\hat{\delta}g$-closed in $(X, \tau)$.

Proposition 3.11 Every $\hat{\delta}g$-closed set is $\alpha g$-closed.

proof: It is true that $\alpha cl(A) \subseteq cl_\delta(A)$ for every subset $A$ of $(X, \tau)$.

Remark 3.12 A $\alpha g$-closed set need not be $\hat{\delta}g$-closed as shown in the following example.

Example 3.13 Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then the set $\{b\}$ is $\alpha g$-closed but not $\hat{\delta}g$-closed in $(X, \tau)$.

Proposition 3.14 Every $\hat{\delta}g$-closed set is $\delta g$-closed.

proof: Let $A$ be an $\hat{\delta}g$-closed set and $U$ be any open set containing $A$. Since every open set is $\hat{g}$-open, $cl_\delta(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\hat{g}$-open. Therefore $cl_\delta(A) \subseteq U$ and $U$ is open. Hence $A$ is $\delta g$-closed.

Remark 3.15 A $\delta g$-closed set need not be $\hat{\delta}g$-closed as shown in the following example.

Example 3.16 Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, X, \{c\}, \{a, b\}\}$. Then the set $\{a\}$ is $\delta g$-closed but not $\hat{\delta}g$-closed in $(X, \tau)$.

Remark 3.17 The class of $\delta g$-closed sets is properly placed between the classes of $\delta$-closed and $\delta g$-closed sets.

Proposition 3.18 Every $\hat{\delta}g$-closed set is $\alpha g$-closed.

proof: It is true that $acl(A) \subseteq cl_\delta(A)$ for every subset $A$ of $(X, \tau)$. 
Remark 3.19 A $\alpha g$-closed set need not be $\delta g$-closed as shown in the following example.

Example 3.20 Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. Then the set $\{a\}$ is $\alpha g$-closed but not $\delta g$-closed in $(X, \tau)$.

Remark 3.21 The following examples show that $\delta g$-closeness is independent from $\hat{g}$-closeness, $sg$-closeness, $g\alpha$-closeness and $\alpha$-closeness.

Example 3.22 Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$. Then the set $\{a, b\}$ is $\delta g$-closed but neither $\hat{g}$-closed nor $sg$-closed and the set $\{a, c\}$ is $\delta g$-closed but neither $g\alpha$-closed nor $\alpha$-closed.

Also the another example Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. Then the set $\{c\}$ is $\hat{g}$-closed, $sg$-closed and $g\alpha$-closed but not $\delta g$-closed.

Example 3.23 Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\}$. Then the set $\{a\}$ is $\alpha$-closed but not $\delta g$-closed in $(X, \tau)$.

Remark 3.24 The following diagram shows the relationships of $\delta g$-closed sets with other known existing sets. $A \rightarrow B$ represents $A$ implies $B$ but not conversely.

Fig. 1

1. $\delta g$-Closed 2.$\delta$-Closed 3.$\delta g$-Closed 4. $\hat{g}$-closed 5.$g$-closed 6.$\alpha g$-closed 7.$gs$-closed 8.$sg$-closed 9.$g\alpha$-closed 10.$\alpha$-closed 11.$\alpha\hat{g}$-closed 12.closed.
4 Characterisation

Theorem 4.1 The finite union of $\delta g$-Closed sets is $\delta g$-Closed.

*proof*: Let $\{A_i/i = 1, 2, \ldots, n\}$ be a finite class of $\delta g$-Closed subsets of a space $(X, \tau)$. Then for each $\hat{g}$-open set $U_i$ in $X$ containing $A_i$, $cl_\delta(A_i) \subseteq U_i$ i.e. $\{1, 2, \ldots, n\}$. Hence $\bigcup_i A_i \subseteq \bigcup_i U_i = V$. Since arbitrary union of $\hat{g}$-open sets in $(X, \tau)$ is also $\hat{g}$-open set in $(X, \tau)$, $V$ is $\hat{g}$-open in $(X, \tau)$. Also $\bigcup_i cl_\delta(A_i) = cl_\delta(\bigcup_i A_i) \subseteq V$. Therefore $\bigcup_i A_i$ is $\delta g$-Closed in $(X, \tau)$.

**Remark 4.2** Intersection of any two $\delta g$-Closed sets in $(X, \tau)$ need not be $\delta g$-Closed since, in Example 3.22, $\{a, b\}$ and $\{a, c\}$ are $\delta g$-Closed sets but their intersection $\{a\}$ is not $\delta g$-Closed.

**Proposition 4.3** Let $A$ be a $\delta g$-Closed set of $(X, \tau)$. Then $cl_\delta(A) - A$ does not contain a non-empty $\hat{g}$-closed set.

*proof*: Suppose that $A$ is $\delta g$-Closed, let $F$ be a $\hat{g}$-closed set contained in $cl_\delta(A) - A$. Now $F^c$ is $\hat{g}$-open set of $(X, \tau)$ such that $A \subseteq F^c$. Since $A$ is $\delta g$-Closed set of $(X, \tau)$, then $cl_\delta(A) \subseteq F^c$. Thus $F \subseteq (cl_\delta(A))^c$. Also $F \subseteq cl_\delta(A) - A$. Therefore $F \subseteq (cl_\delta(A))^c \cap (cl_\delta(A)) = \phi$. Hence $F = \phi$.

**Proposition 4.4** If $A$ is $\hat{g}$-open and $\delta g$-Closed subset of $(X, \tau)$ then $A$ is an $\delta$-closed subset of $(X, \tau)$.

*proof*: Since $A$ is $\hat{g}$-open and $\delta g$-Closed, $cl_\delta(A) \subseteq A$. Hence $A$ is $\delta$-closed.

**Theorem 4.5** The intersection of a $\delta g$-Closed set and a $\delta$-closed set is always $\delta g$-Closed.

*proof*: Let $A$ be $\delta g$-Closed and let $F$ be $\delta$-closed. If $U$ is an $\hat{g}$-open set with $A \cap F \subseteq U$, then $A \subseteq U \cup F^c$ and so $cl_\delta(A) \subseteq U \cup F^c$.

Now $cl_\delta(A \cap F) \subseteq cl_\delta(A) \cap F \subseteq U$. Hence $A \cap F$ is $\delta g$-Closed.

**Theorem 4.6** In a $T_{3\frac{1}{4}}$-space every $\delta g$-Closed set is $\delta$-closed.

*proof*: Let $X$ be $T_{3\frac{1}{4}}$-space. Let $A$ be $\delta g$-Closed set of $X$. We know that every $\delta g$-Closed set is $\delta g$-closed. Since $X$ is $T_{3\frac{1}{4}}$-space, $A$ is $\delta$-closed.

**Proposition 4.7** If $A$ is a $\delta g$-Closed set in a space $(X, \tau)$ and $A \subseteq B \subseteq cl_\delta(A)$, then $B$ is also a $\delta g$-Closed set.

*proof*: Let $U$ be a $\hat{g}$-open set of $(X, \tau)$ such that $B \subseteq U$. Then $A \subseteq U$. Since $A$ is $\delta g$-Closed set, $cl_\delta(A) \subseteq U$. Also since $B \subseteq cl_\delta(A)$, $cl_\delta(B) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A)$. Hence $cl_\delta(B) \subseteq U$. Therefore $B$ is also a $\delta g$-Closed set.
Theorem 4.8  Let A be $\delta\hat{g}$-Closed of $(X,\tau)$. Then A is $\delta$-closed iff $cl_\delta(A)-A$ is $\hat{g}$-closed.

proof: Necessity. Let A be a $\delta$-closed subset of X. Then $cl_\delta(A)=A$ and so $cl_\delta(A)-A=\phi$ which is $\hat{g}$-closed.

Sufficiency. Since A is $\delta\hat{g}$-Closed, by proposition 4.4, $cl_\delta(A)-A$ does not contain a non-empty $\hat{g}$-closed set. But $cl_\delta(A)-A=\phi$. That is $cl_\delta(A)=A$. Hence A is $\delta$-closed.

5 Applications

We introduce the following definition.

Definition 5.1 A space $(X,\tau)$ is called $\hat{T}_{3/4}$-space if every $\delta\hat{g}$-Closed set in it is $\delta$-closed.

Theorem 5.2 For a topological space $(X,\tau)$, the following conditions are equivalent.

(i) $(X,\tau)$ is a $\hat{T}_{3/4}$-space.

(ii) Every singleton $\{x\}$ is either $\hat{g}$-closed or $\delta$-open.

proof: (i) $\Rightarrow$ (ii) Let $x\in X$. Suppose $\{x\}$ is not a $\hat{g}$-closed set of $(X,\tau)$. Then $X-\{x\}$ is not a $\hat{g}$-open set. Thus $X-\{x\}$ is an $\delta\hat{g}$-Closed set of $(X,\tau)$. Since $(X,\tau)$ is $\hat{T}_{3/4}$-space, $X-\{x\}$ is a $\delta$-closed set of $(X,\tau)$, i.e. $\{x\}$ is $\delta$-open set of $(X,\tau)$.

(ii) $\Rightarrow$ (i) Let A be an $\delta\hat{g}$-Closed set of $(X,\tau)$. Let $x\in cl_\delta(A)$. By (ii), $\{x\}$ is either $\hat{g}$-closed or $\delta$-open.

Case(i). Let $\{x\}$ be $\hat{g}$-closed. If we assume that $x\notin A$, then we would have $x\in cl_\delta(A)-A$, which cannot happen according to proposition 4.4. Hence $x\in A$.

Case(ii) Let $\{x\}$ be $\delta$-open. Since $x\in cl_\delta(A)$, then $\{x\}\cap A\neq\phi$. This shows that $x\in A$.

So in both cases we have $cl_\delta(A)\subseteq A$. Trivially $A\subseteq cl_\delta(A)$. Therefore $A=cl_\delta(A)$ or equivalently A is $\delta$-closed. Hence $(X,\tau)$ is a $\hat{T}_{3/4}$-space.

Theorem 5.3 Every $T_{3/4}$-space is a $\hat{T}_{3/4}$-space.

proof: The proof is straight forward since every $\delta\hat{g}$-Closed set is $\delta g$-closed set.

Remark 5.4 The converse of the above theorem is not true as it can be seen from the following example.

Example 5.5 Let $X =\{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. $(X,\tau)$ is a $\hat{T}_{3/4}$-space but not a $T_{3/4}$-space.
**Theorem 5.6** Every $\hat{T}_{3/4}$-space is a $T_{\alpha\hat{g}}$-space.

**proof:** Let $(X,\tau)$ be a $\hat{T}_{3/4}$-space, then every singleton is either $\hat{g}$-closed or $\delta$-open. Since every $\delta$-open is $\alpha$-open, then every singleton is either $\hat{g}$-closed or $\alpha$-open. Hence $(X,\tau)$ is a $T_{\alpha\hat{g}}$-space.

**Remark 5.7** The following example supports that the converse of the above theorem is not true.

**Example 5.8** Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. $(X,\tau)$ is a $T_{\alpha\hat{g}}$-space but not a $\hat{T}_{3/4}$-space.

**Remark 5.9** $\hat{T}_{3/4}$-space and $T_{1/2}$-space are independent of one another as the following examples show.

**Example 5.10** Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. $(X,\tau)$ is a $\hat{T}_{3/4}$-space but is not a $T_{1/2}$-space.

**Example 5.11** Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$. $(X,\tau)$ is a $T_{1/2}$-space but not a $\hat{T}_{3/4}$-space.

**Remark 5.12** The following diagram shows the relationships $\hat{T}_{3/4}$-space with other known existing spaces. $A \rightarrow B$ represents $A$ implies $B$ but not conversely.

![Fig. 2](image-url)

1. $\hat{T}_{3/4}$-space 2. $T_{3/4}$-space 3. $T_{\alpha\hat{g}}$-space 4. $T_{1/2}$-space


References


