A Two-Sided Multiplication Operator Norm

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Abstract

Let \( A \) be a \( C^\ast \)-algebra and define an elementary operator \( T_{a,b} : A \to A \) by \( T_{a,b}(x) = \sum_{i=1}^{n} a_i x b_i, \; \forall \; x \in A \) where \( a_i \) and \( b_i \) are fixed in \( A \) or multiplier algebra \( M(A) \) of \( A \). Here, we determine the norm of a two-sided multiplication operator.

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1 Introduction

Let \( H \) be a complex Hilbert space and \( B(H) \) the algebra of all bounded linear operators on \( H \). Then \( T : B(H) \to B(H) \) is an elementary operator if \( T \) has a representation \( T_{a,b}(x) = \sum_{i=1}^{n} a_i x b_i, \; \forall \; x \in B(H) \), where \( a_i \) and \( b_i \) are fixed in \( B(H) \). Some examples of elementary operators are the left multiplication \( L_a(x) = ax; \) the right multiplication \( R_b(x) = xb; \) the generalized derivation \( \delta_{a,b} = L_a - R_b; \) the inner derivation, the two-sided multiplication operator
$M_{a,b} = L_a R_b$ and the Jordan elementary operator $U_{a,b} = M_{a,b} + M_{b,a}$. Determining the lower estimate of the norm of elementary operators has attracted a lot of interest from many mathematicians (see [1-5, 7-18]). Clearly, every elementary operator is bounded. For the lower estimates of the norms, there have been several results obtained by different mathematicians. For example, Mathieu [6] proved that for a prime $C^*$- algebra $A$, $\|U_{a,b}|A\| \geq \frac{2}{3}\|a\|\|b\|$, Cabrera and Rodriguez [4] proved that for JB* algebras, $\|U_{a,b}|A\| \geq \frac{1}{20412}\|a\|\|b\|$, while Stacho and Zalar [12] obtained results for standard operator algebras on Hilbert spaces i.e. they showed that $\|U_{a,b}|A\| \geq 2(\sqrt{2} - 1)\|a\|\|b\|$. Recently, Timoney [15, 16] demonstrated that $\|U_{a,b}|A\| \geq \|a\|\|b\|$. He [18] also gave a formula for the norm of an elementary operator on a $C^*$-algebra using the notion of matrix valued numerical ranges and a kind of tracial geometric mean.

**Theorem 1.1.** For $a = [a_1, \ldots, a_n] \in B(H)^n$ (a row matrix of operators $a_i \in B(H)$), $b = [b_1, \ldots, b_n] \in B(H)^n$ (a column matrix of operators $b_i \in B(H)$) and $T_{a,b}(x) = \sum_{i=1}^n a_i x b_i$, $\forall x \in B(H)$, an elementary operator, we have

$$\|T\| = \sup\{tgm(Q(a^*,\xi), Q(b,\eta)) : \xi, \eta \in H, \|\xi\| = 1, \|\eta\| = 1\}.$$  

For proof, see [18, Theorem 1.4].

Interestingly, for Calkin algebras, it has been easy to calculate the norms of elementary operators as shown by Mathieu [7]. Considering a two-sided multiplication operator $M_{a,b}$, it has been shown in [2], the necessary and sufficient conditions for any pair of operators $a, b \in B(H)$ to satisfy the equation $\|I + M_{a,b}\| = 1 + \|a\|\|b\|$.

**Definition 1.2.** Let $T \in B(H)$. The maximal numerical range of $T$ is defined by $W_0(T) = \{\lambda : \langle Tx_n, x_n \rangle \to \lambda$, where $\|x_n\| = 1$ and $\|Tx_n\| \to \|T\|\}$ and the normalized maximal numerical range is given by

$$W_N(T) = \begin{cases} W_0(\frac{T}{\|T\|}), & \text{if } T \neq 0, \\ 0, & \text{if } T = 0. \end{cases}$$

The set $W_0(T)$ is nonempty, closed, convex and contained in the closure of the numerical range, see [14].

**Theorem 1.3.** For $a, b \in B(H)$ the following are equivalent:

1. $\|I + M_{a,b}\| = 1 + \|a\|\|b\|$,  
2. $W_N(a^*) \cap W_N(b) \neq \emptyset$.


**Conjecture 1.4.** Let $A$ be a standard operator subalgebra of $B(H)$. The estimate of $M$, such that $\|M_{a,b}x\| = \|a\|\|b\|$ holds for every $a, b \in A$.
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This conjecture was verified in the following cases:
(i) for \(a, b \in B(H)\) such that \(\inf_{\lambda \in C} \|a + \lambda b\| = \|a\|\) or \(\inf_{\lambda \in C} \|b + \lambda a\| = \|b\|\),
(ii) in the Jordan algebra of symmetric operators. See [1, 13].

Nyamwala and Agure [8] used the spectral resolution theorem to calculate the norm of an elementary operator induced by normal operators in a finite dimensional Hilbert space. They gave the following result.

**Theorem 1.5.** Let \(T_{a,b} : B(H) \to B(H)\) be an elementary operator defined by \(T_{a,b}(x) = \sum_{i=1}^{k} a_i x b_i\) where \(a_i\) and \(b_i\) are normal operators and \(H\) a finite \(m\)-dimensional Hilbert space then
\[
\|T\| = \left(\sum_{j=1}^{k} \left(\sum_{j=1}^{m} |\alpha_{i,j}|^2 |\beta_{i,j}|^2\right)\right)^{\frac{1}{2}}
\]
where \(\alpha_{i,j}\) and \(\beta_{i,j}\) are distinct eigenvalues of \(a_i\) and \(b_i\) respectively.

A specific example in [8, Example 2.3] shows that \(\|T\| = 2\). In the next section, we determine the norm of a two-sided multiplication operator.

## 2 Two-sided Multiplication Operator Norm

In this section we concentrate on a complex Hilbert space over the field \(K\). We show that for a two-sided multiplication operator \(M\), \(\|M_{a,b}x\| = \|a\|\|b\|\).

**Definition 2.1.** Let \(\phi \in H^*\) and \(\xi \in H\). We define \(\phi \otimes \xi \in B(H)\) by
\[
(\phi \otimes \xi)\eta = \phi(\eta)\xi, \ \forall \ \eta \in H.
\]

**Theorem 2.2.** Let \(H\) be a complex Hilbert space, \(B(H)\) the algebra of all bounded linear operators on \(H\). Let \(M_{a,b} : B(H) \to B(H)\) be defined by \(M_{a,b}(x) = axb\), \(\forall x \in B(H)\) where \(a, b\) are fixed in \(B(H)\). Then \(\|M_{a,b}x\| = \|a\|\|b\|\).

**Proof.** By definition, \(\|M_{a,b}|B(H)|\| = \sup \{\|M_{a,b}(x)\| : x \in B(H), \|x\| = 1\}\).

This implies that \(\|M_{a,b}|B(H)|\| \geq \|M_{a,b}(x)\|, \forall x \in B(H), \|x\| = 1\).

So \(\forall \epsilon > 0, \|M_{a,b}|B(H)|\| - \epsilon < \|M_{a,b}(x)\|, \forall x \in B(H), \|x\| = 1\).

But, \(\|M_{a,b}|B(H)|\| - \epsilon < \|axb\| \leq \|a\|\|x\|\|b\| = \|a\|\|b\|\).

Since \(\epsilon\) is arbitrary, this implies that
\[
\|M_{a,b}|B(H)|\| \leq \|a\|\|b\|.
\] (1)

On the other hand, let \(\xi, \eta \in H, \|\xi\| = \|\eta\| = 1, \phi \in H^*\).

Now,
\[
\|M_{a,b}|B(H)|\| \geq \|M_{a,b}(x)\|, \forall x \in B(H), \|x\| = 1.
\]
But,

\[ \| M_{a,b}(x) \| = \sup \{ \| (M_{a,b}(x))\eta \| : \forall \eta \in H, \| \eta \| = 1 \} \]

\[ = \sup \{ \| (axb)\eta \| : \eta \in H, \| \eta \| = 1 \}. \]

Setting \( a = (\phi \otimes \xi_1), \forall \xi_1 \in H, \| \xi_1 \| = 1 \) and \( b = (\varphi \otimes \xi_2), \forall \xi_2 \in H, \| \xi_2 \| = 1 \), we have,

\[ \| M_{a,b}|_B(H) \| \geq \| a \| \| b \|. \]

Hence by inequalities (1) and (2),

\[ \| M_{a,b}|_B(H) \| = \| a \| \| b \|. \]

This completes the proof. \( \square \)

### 3 The Jordan Elementary Operator

**Theorem 3.1.** Let \( H \) be a 2-dimensional complex Hilbert space, \( B(H) \) the algebra of bounded linear operators on \( H \). Let \( T_{a,b} : B(H) \to B(H) \) be defined by \( T_{a,b}(x) = axb + bxa, \forall x \in B(H) \) where \( a, b \) are fixed in \( B(H) \) and \( \{ e_1, e_2 \} \) an orthonormal basis for \( H \). Then for a constant \( C > 0 \) such that \( \| T_{a,b} \| \geq C\|a\|\|b\|, C = 1 \).

**Proof.** The proof of this theorem follows immediately from the results obtained in [3]. \( \square \)

**Remark 3.2.** From [13], we see that \( C = 1 \) is also true for symmetric operators (in this case, \( a \) and \( b \) are self adjoint).

**Theorem 3.3.** Let \( a, b \in \text{Symm}(H) \). Then \( \| U_{a,b}|_A \| \geq \| a \| \| b \|. \)

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References


