Product of Statistical Manifolds with Doubly Warped Product

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Abstract

In this paper, we generalize the dualistic structures on warped product manifolds to the dualistic structures on doubly warped product. We extend some results related to the dualistic structure on doubly warped product studied in [8]. We also demonstrate that a dualistic structure on a doubly warped product manifold \((M_1 \times M_2, g_{f_1 f_2})\) induces dualistic structures on the manifolds \(M_1\) and \(M_2\) and conversely, in this case doubly warped product manifold \((M_1 \times M_2, g_{f_1 f_2})\) is a statistical manifold if and only if \((M_1, g_1)\) and \((M_2, g_2)\) are.

Keywords: Conjugate, doubly warped products, dual connection, product manifold.

1 Introduction

The warped product provides a way to construct new pseudo-riemannian manifolds from the given ones, see [13],[12] and [11]. This construction has useful applications in general relativity, in the study of cosmological models and black holes. It generalizes the direct product in the class of pseudo-Riemannian manifolds and it is defined as follows:

**Definition 1.1** Let \((M_1, g_1)\) and \((M_2, g_2)\) be two pseudo-Riemannian manifolds and let \(f_1 : M_1 \rightarrow \mathcal{R}^*\) be a positive smooth function on \(M_1\), the warped product of \((M_1, g_1)\) and \((M_2, g_2)\) is the product manifold \(M_1 \times M_2\) equipped
with the metric tensor \( g_{f_1} := \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2 \), where \( \pi_1 \) and \( \pi_2 \) are the projections of \( M_1 \times M_2 \) onto \( M_1 \) and \( M_2 \) respectively.

The manifold \( M_1 \) is called the base of \((M_1 \times M_2, g_{f_1})\) and \( M_2 \) is called the fiber. The function \( f_1 \) is called the warping function.

The doubly warped product construction in the class of pseudo-Riemannian manifolds generalized the warped product and the direct product. It is obtained by homothetically distorting the geometry of each base \( M_1 \times \{q\} \) and each fiber \( \{p\} \times M_2 \) to get a new ”doubly warped” metric tensor on the product manifold and defined as follows:

For \( i \in \{1, 2\} \), let \( M_i \) be a pseudo-Riemannian manifold equipped with metric \( g_i \), and \( f_i : M_i \to \mathcal{R}^* \) be a positive smooth function on \( M_i \). The well-know notion of doubly warped product manifold \( M_1 \times f_1^* f_2^* M_2 \) is defined as the product manifold \( M = M_1 \times M_2 \) equipped with pseudo-Riemannian metric which is denoted by \( g_{f_1 f_2} \), given by

\[
g_{f_1 f_2} = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2.
\]

In the cases \( f_1 = 1 \) or \( f_2 = 1 \) we obtain a warped product or a direct product.

Dualistic structures are closely related to statistical mathematics. They consist of pairs of affine connections on statistical manifolds, compatible with a pseudo-Riemannian metric [1]. Their importance in statistical physics was underlined by many authors; see [3],[4],[5] etc.

Let \( M \) be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric \( g \) and let \( \nabla, \nabla^* \) be the affine connections on \( M \). We say that a pair of affine connections \( \nabla \) and \( \nabla^* \) are compatible (or conjugate ) with respect to \( g \) if

\[
X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \quad \text{for all } X, Y, Z \in \Gamma(TM),
\]

where \( \Gamma(TM) \) is the set of all tangent vector fields on \( M \). Then the triplet \((g, \nabla, \nabla^*)\) is called the dualistic structure on \( M \).

We note that the notion of ”conjugate connection ” has been attributed to A.P. Norden in affine differential geometry literature (Simon, 2000) and was independently introduced by (Nagaoka and Amari, 1982) in information geometry, where it was called ” dual connection” (Lauritzen, 1987). The triplet \((M, \nabla, g)\) is called a statistical manifold if it admits another torsion-free connection \( \nabla^* \) satisfying the equation (1). We call \( \nabla \) and \( \nabla^* \) dual of each other with respect to \( g \).

In the notions of terms on statistical manifolds, for a torsion-free affine connection \( \nabla \) and a pseudo-Riemannian metric \( g \) on a manifold \( M \), the triple \((M, \nabla, g)\) is called a statistical manifold if \( \nabla g \) is symmetric. If the curvature
tensor $R$ of $\nabla$ vanishes, $(M, \nabla, g)$ is said to be flat.

This paper extends the study of dualistic structures on warped product and double warped product manifolds in the papers [7] and [8].

The paper is organized as follows. In section 2, we collect the basic material about Levi-Civita connection, the notion of conjugate, horizontal and vertical lifts. In section 3, we define the co-metric $\tilde{g}_{f_1f_2}$ of $g_{f_1f_2}$ the metric of the doubly warped products by using the musical isomorphisms, we calculate the gradient of the lift of $f_1$ (resp. $f_2$), it has been shown that its gradient is horizontal (resp. vertical) and $\pi_1$ related to gradient of $f_1$ on $M_1$ (resp. $\pi_2$ related to gradient of $f_2$ on $M_2$), we show that dualistic structures on manifolds $(M_1, g_1)$ and $(M_2, g_2)$ induce the dualistic structure on the doubly warped product manifold $(M_1 \times M_2, g_{f_1f_2})$ and conversely. Moreover $(M_1, g_1)$ and $(M_2, g_2)$ are statistical manifolds if and only if $(M_1 \times M_2, g_{f_1f_2})$ is a statistical manifold.

2 Preliminaries

2.1 Statistical Manifolds

We recall some standard facts about Levi-Civita connections and the dual statistical manifold. Many fundamental definitions and results about dualistic structure can be found in Amari’s monograph ([1],[2]).

Let $(M, g)$ be a pseudo-Riemannian manifold. The metric $g$ defines the musical isomorphisms

$$\sharp_g : \Gamma(T^*M) \rightarrow \Gamma(TM)$$

$$\alpha \mapsto \sharp_g(\alpha)$$

such that $g(\sharp_g(\alpha), Y) = \alpha(Y)$, and its inverse $\flat_g$. We can thus define the co-metric $\tilde{g}$ of the metric $g$ by:

$$\tilde{g}(\alpha, \beta) = g(\sharp_g(\alpha), \sharp_g(\beta)).$$

(2)

A fundamental theorem of pseudo-Riemannian geometry states that given a pseudo-Riemannian metric $g$ on the tangent bundle $TM$, there is a unique connection (among the class of torsion-free connection) that ”preserves” the metric; as long as the following condition is satisfied:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \text{for } X, Y, Z \in \Gamma(TM)$$

(3)

Such a connection, denoted as $\nabla$, is known as the Levi-Civita connection. Its component forms, called Christoffel symbols, are determined by the compo-
nents of pseudo-metric tensor as ("Christoffel symbols of the second Kink")

\[ \hat{\Gamma}^k_{ij} = \sum_l \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) \]

and ("Christoffel symbols of the first Kink")

\[ \hat{\Gamma}^i_{j,k} = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right). \]

The Levi-Civita connection is compatible with the pseudo metric, in the sense that it treats tangent vectors of the shortest curves on a manifold as being parallel.

It turns out that one can define a kind of "Compatibility" relation more generally than expressed by the (3), by introducing the notion of "Conjugate" (denoted by *) between two affine connections.

Let \((M, g)\) be a pseudo-Riemannian manifold and let \(\nabla, \nabla^*\) be affine connections on \(M\). A connection \(\nabla^*\) is said to be "conjugate" to \(\nabla\) with respect to \(g\) if

\[ X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \quad \text{for} \ X, Y, Z \in \Gamma(TM). \quad (4) \]

Clearly,

\[ (\nabla^*)^* = \nabla. \]

Otherwise, \(\hat{\nabla}\), which satisfies the (3), is special in the sense that it is self-conjugate

\[ (\hat{\nabla})^* = \hat{\nabla}. \]

Because pseudo-metric tensor \(g\) provides a one-to-one mapping between vectors in the tangent space and co-vectors in the cotangent space, the equation (1) can also be seen as characterizing how co-vector fields are to be parallel-transported in order to preserve their dual pairing \(<\cdot,\cdot>\) with vector fields.

Writing out the equation (1) explicitly,

\[ \frac{\partial g_{ij}}{\partial x^k} = \Gamma^i_{k,j} + \Gamma^*_j_{k,i}, \quad (5) \]

where

\[ \nabla^*_i \partial_j = \sum_l \Gamma^*_i_{j,l} \partial_l \]

so that

\[ \Gamma^*_j_{k,i} = g(\nabla^*_i \partial_k, \partial_i) = \sum_l g_{il} \Gamma^*_i_{j,k}. \]
In the following, a manifold $M$ with a pseudo-metric $g$ and a pair of conjugate connections $\nabla, \nabla^*$ with respect to $g$ is called a "pseudo-Riemannian manifold with dualistic structure" and denoted by $(M, g, \nabla, \nabla^*)$.

Obviously, $\nabla$ and $\nabla^*$ (or equivalently, $\Gamma$ and $\Gamma^*$) satisfy the relation

$$\hat{\nabla} = \frac{1}{2}(\nabla + \nabla^*) \quad (or \ equivalently, \ \hat{\Gamma} = \frac{1}{2}(\Gamma + \Gamma^*))$$

Thus an affine connection $\nabla$ on $(M, g)$ is metric if and only if $\nabla^* = \nabla$ (that is self-conjugate).

For a torsion-free affine connection $\nabla$ and a pseudo-Riemannian metric $g$ on a manifold $M$, the triplet $(M, \nabla, g)$ is called a statistical manifold if $\nabla g$ is symmetric. If the curvature tensor $\mathcal{R}$ of $\nabla$ vanishes, $(M, \nabla, g)$ is said to be flat.

For a statistical manifold $(M, \nabla, g)$, the conjugate connection $\nabla^*$ with respect to $g$ is torsion-free and $\nabla^* g$ symmetric. Then the triplet $(M, \nabla^*, g)$ is called the dual statistical manifold of $(M, \nabla, g)$ and $(\nabla, \nabla^*, g)$ the dualistic structure on $M$. The curvature tensor of $\nabla$ vanishes if and only if that of $\nabla^*$ does and in such a case, $(\nabla, \nabla^*, g)$ is called the dually flat structure [2].

It can be shown that for a pair of conjugate connections $\nabla, \nabla^*$, their curvature tensors $\mathcal{R}, \mathcal{R}^*$ satisfy

$$g(\mathcal{R}(X,Y)Z,W) + g(Z, \mathcal{R}^*(X,Y)W) = 0. \quad (6)$$

If the curvature tensor $\mathcal{R}$ of $\nabla$ vanishes, $\nabla$ is said to be flat.

So, $\nabla$ is flat if and only if $\nabla^*$ is flat. In this case, $(M, g, \nabla, \nabla^*)$ is said to be dually flat.

### 2.2 Horizontal and Vertical Lifts

Throughout this paper $M_1$ and $M_2$ will be respectively $m_1$ and $m_2$ dimensional manifolds, $M_1 \times M_2$ the product manifold with the natural product coordinate system and

$$\pi_1 : M_1 \times M_2 \rightarrow M_1 \quad , \quad \pi_2 : M_1 \times M_2 \rightarrow M_2$$

the usual projection maps. We recall briefly how the calculus on the product manifold $M_1 \times M_2$ derives from that of $M_1$ and $M_2$ separately. For details see [13].

Let $\varphi_1$ in $C^\infty(M_1)$. The horizontal lift of $\varphi_1$ to $M_1 \times M_2$ is $\varphi^h_1 = \varphi_1 \circ \pi_1$. One can define the horizontal lifts of tangent vectors as follows. Let $p \in M_1$ and $X_p \in T_pM_1$. For any $q \in M_2$ and $X_q \in T_qM_2$ the horizontal lift of $X_p$ to $T_{(p,q)}(M_1 \times M_2)$ is the unique tangent vector $X^h_{(p,q)}$ in $T_{(p,q)}(M_1 \times \{q\})$ such that

$$\begin{cases} d_{(p,q)}\pi_1(X^h_{(p,q)}) = X_p, \\ d_{(p,q)}\pi_2(X^h_{(p,q)}) = 0. \end{cases}$$
We can also define the horizontal lifts of vector fields as follows. Let \( X_1 \in \Gamma(TM_1) \). The horizontal lift of \( X_1 \) to \( \Gamma(T(M_1 \times M_2)) \) is the vector field \( X_1^h \in \Gamma(T(M_1 \times M_2)) \) whose value at each \((p,q)\) is the horizontal lift of the tangent vector \((X_1)_p \) to \( T_{(p,q)}(M_1 \times M_2) \). For \((p,q) \in M_1 \times M_2\), we will denote the set of the horizontal lifts to \( T_{(p,q)}(M_1 \times M_2) \) of all the tangent vectors of \( M_1 \) at \( p \) by \( \mathcal{L}^h_{(p,q)}(M_1) \). We will denote the set of the horizontal lifts of all vector fields on \( M_1 \) by \( \mathcal{L}^h(M_1) \).

The vertical lift \( \varphi^v \) of a function \( \varphi \in C^\infty(M_2) \) to \( M_1 \times M_2 \) and the vertical lift \( X^v \) of a vector field \( X \in \Gamma(TM_2) \) to \( \Gamma(T(M_1 \times M_2)) \) are defined in the same way using the projection map \( \pi_2 \). Note that the spaces \( \mathcal{L}^h(M_1) \) of the horizontal lifts and \( \mathcal{L}^v(M_2) \) of the vertical lifts are vector subspaces of \( \Gamma(T(M_1 \times M_2)) \) but neither is invariant under multiplication by arbitrary functions \( \varphi \in C^\infty(M_1 \times M_2) \).

We define the horizontal lift of a covariant tensor \( \omega_1 \) on \( M_1 \) to be its pullback \( \omega^h_1 \) to \( M_1 \times M_2 \) by the means of the projection map \( \pi_1 \), i.e. \( \omega^h_1 := \pi_1^*(\omega_1) \). In particular, for a 1-form \( \alpha_1 \) on \( M_1 \) and a vector field \( X \) on \( M_1 \times M_2 \), we have

\[
(\alpha^h_1)(X) = \alpha_1(d\pi_1(X)).
\]

Explicitly, if \( u \) is a tangent vector to \( M_1 \times M_2 \) at \((p,q)\), then

\[
(\alpha^h_1)_{(p,q)}(u) = (\alpha_1)_p(d_{(p,q)}\pi_1(u)).
\]

Similarly, we define the vertical lift of a covariant tensor \( w_2 \) on \( M_2 \) to be its pullback \( w^v_2 \) to \( M_1 \times M_2 \) by the means of the projection map \( \pi_2 \).

Observe that if \( \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\} \) is the local basis of the vector fields (resp. \( \{dx^1, \ldots, dx^m\} \) is the local basis of 1-forms) relative to a chart \((U, \Phi)\) of \( M_1 \) and \( \{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}\} \) is the local basis of the vector fields (resp. \( \{dy^1, \ldots, dy^m\} \) the local basis of the 1-forms) relative to a chart \((V, \Psi)\) of \( M_2 \), then \( \{(\frac{\partial}{\partial x^1})^h, \ldots, (\frac{\partial}{\partial x^m})^h, (\frac{\partial}{\partial y^1})^v, \ldots, (\frac{\partial}{\partial y^m})^v\} \) is the local basis of the vector fields (resp. \( \{(dx^1)^h, \ldots, (dx^m)^h, (dy^1)^v, \ldots, (dy^m)^v\} \) is the local basis of the 1-forms) relative to the chart \((U \times V, \Phi \times \Psi)\) of \( M_1 \times M_2 \).

The following lemma will be useful later for our computations.

**Lemma 2.1 [10]**

1. Let \( \varphi_i \in C^\infty(M_i), X_i, Y_i \in \Gamma(TM_i), \alpha_i \in \Gamma(T^*M_i), i = 1, 2 \), let \( \varphi = \varphi_1^h + \varphi_2^v, X = X_1^h + X_2^v \) and \( \alpha, \beta \in \Gamma(T^*(M_1 \times M_2)) \). Then

\[
i/ \text{ For all } (i, I) \in \{(1, h), (2, v)\} \text{ we have } X_i^I(\varphi) = X_i(\varphi_i)^I, \quad [X, Y_i]^I = [X_i, Y_i]^I \quad \text{and} \quad \alpha_i^I(X) = \alpha_i(X_i)^I.
\]
ii/ If for all \((i, I) \in \{(1, h), (2, v)\}\) we have \(\alpha(X_i^I) = \beta(X_i^I)\), then \(\alpha = \beta\).

2. Let \(\omega_i\) and \(\eta_i\) be \(r\)-forms on \(M_i\), \(i = 1, 2\), and set \(\omega = \omega_1^h + \omega_2^v\) and \(\eta = \eta_1^h + \eta_2^v\). Then we have
\[
d\omega = (d\omega_1)^h + (d\omega_2)^v \quad \text{and} \quad \omega \wedge \eta = (\omega_1 \wedge \eta_1)^h + (\omega_2 \wedge \eta_2)^v.
\]

**Remark 2.2** Let \(X\) be a vector field on \(M_1 \times M_2\), such that \(d\pi_1(X) = \varphi(X_1 \circ \pi_1)\) and \(d\pi_2(X) = \phi(X_2 \circ \pi_2)\). Then \(X = \varphi X_1^h + \phi X_2^v\).

### 3 About Doubly Warped Products

#### 3.1 The Doubly Warped Product

Let \(\psi : M \to N\) be a smooth map between smooth manifolds and \(g\) be a metric on \(k\)-vector bundle \((F, P_F)\) over \(N\). The metric \(g^{\psi} : \Gamma(\psi^{-1}F) \times \Gamma(\psi^{-1}F) \to C^\infty(M)\) on the pull-back \((\psi^{-1}F, P_{\psi^{-1}F})\) over \(M\) is defined by
\[
g^{\psi}(U, V)(p) = g_{\psi(p)}(U_p, V_p), \quad \forall U, V \in \Gamma(\psi^{-1}F), p \in M.
\]

Given a linear connection \(\nabla^N\) on \(k\)-vector bundle \((F, P_F)\) over \(N\), the pull-back connection \(\tilde{\nabla}\) is the unique linear connection on the pull-back \((\psi^{-1}F, P_{\psi^{-1}F})\) over \(M\) such that, for each \(W \in \Gamma(F)\), \(X \in \Gamma(TM)\)
\[
\tilde{\nabla}_X (W \circ \psi) = \nabla^N_{d\psi(X)} W.
\]

Further, let \(U \in \psi^{-1}F\), \(p \in M\) and \(X \in \Gamma(TM)\). Then
\[
(\tilde{\nabla}_X U)(p) = (\nabla^N_{d\psi(X)(p)} \tilde{U})(\psi(p)),
\]
where \(\tilde{U} \in \Gamma(F)\) with \(\tilde{U} \circ \psi = U\).

Now, let \(\pi_i\), \(i=1,2\), be the usual projection of \(M_1 \times M_2\) onto \(M_i\), given a linear connection \(\nabla_i\) on vector bundle \(\Gamma(TM_i)\), the pull-back connection \(\hat{\nabla}\) is the unique linear connection on the pull-back \(M_1 \times M_2 \to \pi_i^{-1}(TM_i)\) such that, for each \(Y_i \in \Gamma(TM_i)\), \(X \in \Gamma(TM_1 \times M_2)\)
\[
\hat{\nabla}_X Y_i \circ \pi_i = \nabla_i Y_i.
\]

Further, let \(U \in \Gamma(\pi_i^{-1}(TM_i))\), \((p, q) \in M_1 \times M_2\) and \(X \in \Gamma(T(M_1 \times M_2))\). Then
\[
(\hat{\nabla}_X U)(p, q) = (\hat{\nabla}_{d(p,q)(x(p,q))} \tilde{U}) \pi_i(p, q),
\]
where \(\tilde{U} \in \Gamma(TM)\) with \(\tilde{U} = \chi_0 \circ \psi \circ \tilde{U}\).
**Definition 3.1** Let \((M_1, g_1)\) and \((M_2, g_2)\) be pseudo-Riemannian manifolds and let \(f_1 : M_1 \to \mathbb{R}^*\) and \(f_2 : M_2 \to \mathbb{R}^*\) be a positive smooth functions. The doubly warped product is the product manifold \(M_1 \times M_2\) furnished with the metric tensor \(g_{f_1 f_2}\) defined by

\[
g_{f_1, f_2} = (f_2^h)^2 \pi_1^* g_1 + (f_1^2)^2 \pi_2^* g_2.
\]

Explicitly, if \(X, Y \in \Gamma(TM_1 \times M_2)\), then

\[
g_{f_1, f_2}(X, Y) = (f_2^v)^2 \pi_1^* (d\pi_1(X), d\pi_1(Y)) + (f_1^h)^2 \pi_2^* (d\pi_2(X), d\pi_2(Y)).
\]

By analogy with [6] we will denote this structure by \(M_1 \times_{f_1, f_2} M_2\). The function \(f_1 : M_1 \to \mathbb{R}^+ - \{0\} \ (i \in \{1, 2\})\) is called the warping function.

If \((M_1, g_1)\) and \((M_2, g_2)\) are both Riemannian manifolds, then \(M_1 \times_{f_1 f_2} M_2\) is also a Riemannian manifold. We call \(M_1 \times_{f_1 f_2} M_i\) as a Lorentzian doubly warped product if \((M_2, g_2)\) is Riemannian and either \((M_1, g_1)\) is Lorentzian or else \((M_1, g_1)\) is a one-dimensional manifold with a negative definite metric \(-dt^2\).

**Proposition 3.2** With the notation above, let \(X_i, Y_i \in \Gamma(TM_i), i = 1, 2\). Then the equation (11) is equivalent to

\[
\begin{align*}
g_{f_1, f_2}(X^h_1, Y^h_1) &= (f_2^v)^2 g_1(X_1, Y_1)^h; \\
g_{f_1, f_2}(X^h_1, Y^v_2) &= g_{f_1, f_2}(X_2^v, Y_1^h) = 0; \\
g_{f_1, f_2}(X^v_2, Y^v_2) &= (f_1^h)^2 g_2(X_2, Y_2)^v.
\end{align*}
\]

**Proof:** By definition of the doubly warped metric,

\[
g_{f_1, f_2}(X^h_1, Y^h_1)(p, q) = f_2^2(q) g_1(X_1, Y_1)(p) \quad \text{and} \quad g_{f_1, f_2}(X^h_1, Y^v_2)(p, q) = 0.
\]

Writing \(f_1^h\) for \(f_1 \circ \pi_1\) and \(f_2^v\) for \(f_2 \circ \pi_2\). Then it is easily seen that Equation (12) hold.

A direct computation using Proposition 3.2 and the definition of the musical isomorphism gives the following proposition.

**Proposition 3.3** ([9]) Let \((M_i, g_i)\) be a pseudo-Riemannian manifold and let \(f_i : M_i \to \mathbb{R}^*_+\), be a positive smooth function, \(i = 1, 2\). The co-metric \(\tilde{g}_{f_1, f_2}\) of \(g_{f_1, f_2}\) is characterized by the following identities

\[
\begin{align*}
\tilde{g}_{f_1, f_2}(\alpha^h_1, \beta^h_1) &= \frac{1}{(f_2^v)^2} \tilde{g}_1(\alpha_1, \beta_1)^h; \\
\tilde{g}_{f_1, f_2}(\alpha^h_1, \beta^v_2) &= \tilde{g}_{f_1, f_2}(\alpha^v_2, \beta^h_1) = 0; \\
\tilde{g}_{f_1, f_2}(\alpha^v_1, \beta^v_2) &= \frac{1}{(f_1^h)^2} \tilde{g}_2(\alpha_2, \beta_2)^v.
\end{align*}
\]

for any \(\alpha_i, \beta_i \in \Gamma(T^*M_i), i = 1, 2\). Where \(\tilde{g}_i\) is the co-metric of \(g_i\).
Lemma 3.4 If \( f_i \in C^\infty(M_i), i = 1, 2 \). Then the gradient of the lifts \( f_i^h \) of \( f_1 \) and \( f_2^v \) of \( f_2 \) to \( M_1 \times f_{1,2} M_2 \) w.r.t. \( g_{f_1f_2} \) is

\[
\text{grad}(f_i^h) = \frac{1}{(f_2^v)^2}\text{grad}(f_1^h) \quad \text{and} \quad \text{grad}(f_2^v) = \frac{1}{(f_1^h)^2}\text{grad}(f_2^v) \quad (14)
\]

Proof: Let \( Z_i \in \Gamma(TM_i), i = 1, 2 \). Then, for any \((i, I), (3-i, J) \in \{(1, h), (2, v)\}, \) we have

\[
g_{f_1f_2}(\text{grad}(f_i^I), Z_i^I) = (Z_i(f_i))^I = g_i(\text{grad} f_i, Z_i)^I = \frac{1}{(f_2^v)^2}g_{f_1f_2}((\text{grad} f_i)^I, Z_i^I),
\]

and

\[
g_{f_1f_2}(\text{grad}(f_i^I), Z_{3-i}^J) = 0.
\]

Therefore, from Equation (12), we get

\[
\text{grad}(f_i^I) = \frac{1}{(f_{3-i}^I)^2}(\text{grad} f_i)^I.
\]

3.2 Dualistic Structure on Doubly Warped Products

Proposition 3.5 Let \( (g_{f_1f_2}, \mathcal{V}, \mathcal{V}^*) \) be a dualistic structure on \( M_1 \times M_2 \). Then there exists an affine connections \( \mathcal{V}, \mathcal{V}^* \) on \( M_i \), such that \( (g_i, \mathcal{V}, \mathcal{V}^*) \) is a dualistic structure on \( M_i, i = 1, 2 \).

Proof: Taking the affine connections on \( M_i, i = 1, 2 \).

\[
\begin{align*}
    (\mathcal{V}_X Y_i) \circ \pi_i & = d\pi_i(\nabla_X Y_i^I), \quad \forall X, Y_i \in \Gamma(TM_i) \\
    (\mathcal{V}^*_X Y_i) \circ \pi_i & = d\pi_i(\nabla^*_X Y_i^I), \quad \forall (i, I) \in \{(1, h), (2, v)\}
\end{align*}
\]

Therefore, we have for all \( X, Y_i, Z_i \in \Gamma(TM_i) \).

\[
X_i^I(g_{f_1f_2}(Y_i^I, Z_i^I)) = g_{f_1f_2}(\nabla_X Y_i^I, Z_i^I) + g_{f_1f_2}(Y_i^I, \nabla_X^* Z_i^I). \quad (15)
\]

Since, \( d\pi((Z_i^I) = 0, X_i^I(f_{3-i}^I) = 0 \) and \( g_{f_1f_2}(X, Z_i^I) = (f_{3-i}^I)^2 g_i^*(d\pi(X), Z_i \circ \pi_i), \) for any \( X \in \Gamma(TM_1 \times M_2) \), then the equation (15) is equivalent to

\[
(f_{3-i}^I)^2(X_i(g_i(Y_i, Z_i)))^I = (f_{3-i}^I)^2\{g_i(\mathcal{V}_X Y_i, Z_i) + g_i(Y_i, \mathcal{V}^*_X Z_i)\}^I.
\]

Where \((i, I), (3-i, J) \in \{(1, h), (2, v)\} \). Hence, the pair of affine connections \( \mathcal{V} \) and \( \mathcal{V}^* \) are conjugate with respect to \( g_i \).

Proposition 3.6 Let \( (g_i, \mathcal{V}, \mathcal{V}^*) \) be a dualistic structure on \( M_i, i = 1, 2 \). Then there exists a dualistic structure on \( M_1 \times M_2 \) with respect to \( g_{f_1f_2} \).
Proof: Let $\nabla$ and $\nabla^*$ be the connections on $M_1 \times M_2$ given by
\[\begin{align*}
d\pi_i(\nabla_X Y) &= \nabla_X d\pi_i(Y) + Y(\ln f^I_{3-i})d\pi_i(X) + X(\ln f^I_{3-i})d\pi_i(Y) \\
&\quad - (f^I_{3-i})^{-2}f^I_i g^\pi_{3-i}(d\pi_{3-i}(X), d\pi_{3-i}(Y))((\text{grad}f_i) \circ \pi_i),
\end{align*}\]
(16)
for any $X, Y \in \Gamma(TM_1 \times M_2)$.

Or, for any $X_i, Y_i \in \Gamma(TM_i)$ we have
\[\begin{align*}
\nabla^*_X Y^I &= (\nabla_X Y)^I - (\frac{g(Y_i, Y_j)}{2f^I_i})^I (\text{grad}f^2_{3-i})^J; \\
\nabla^*_X Y^I &= (\nabla_X Y)^I - (\frac{g(Y_i, Y_j)}{2f^I_i})^I (\text{grad}f^2_{3-i})^J; \\
\nabla^*_X Y^I &= (\nabla_X Y)^I = (X_i(\ln f_i))^I Y^I_{3-i} + (Y^I_{3-i}(\ln f_i))^I X^I_i,
\end{align*}\]
(17)
where $(i, I), (3 - i, J) \in \{(1, h), (2, v)\}$. Let us assume that $(g_i, \nabla_i, \nabla^*_i)$ is a dualistic structure on $M_i$, $i = 1, 2$. Let $A$ be the tensor field of type $(0, 3)$ defined by
\[A(X, Y, Z) = X(g_{f_1, f_2}(Y, Z)) - g_{f_1, f_2}(\nabla_X Y, Z) - g_{f_1, f_2}(Y, \nabla_X Z),\]
for any $X, Y, Z \in \Gamma(TM_1 \times M_2)$, if $X_i, Y_i, Z_i \in \Gamma(TM_i)$, $i = 1, 2$. Then we have
\[X^I_i(g_{f_1, f_2}(Y^I_i, Z^I_i)) = X^I_i ((f^I_{3-i})^2 g_i(X_i, Y_i)^I).\]
Since $d\pi_{3-i}(X^I_i) = 0$, it follows that $d\pi_{3-i}(X^I_i)(f_{3-i}) = X^I_i(f_{3-i}) = 0$, and hence
\[X^I_i(g_{f_1, f_2}(Y^I_i, Z^I_i)) = (f^I_{3-i})^2(X(g_i(Y_i, Z_i))^I).\]
As $(g_i, \nabla_i, \nabla^*_i)$ is dualistic structure, we have thus
\[X^I_i(g_{f_1, f_2}(Y^I_i, Z^I_i)) = (f^I_{3-i})^2\{g_i(\nabla_X Y_i, Z_i)^I + g_i(Y_i, \nabla_X Z_i)^I\}.\]

From Proposition 3.2 and Equations (17), then it’s easily seen that the following equation holds
\[A(X^I_i, Y^I_i, Z^I_i) = 0.\]
In the different lifts, we have
\[X^I_{3-i}(g_{f_1, f_2}(Y^I_i, Z^I_i)) = 2g_i(Y_i, Z_i)^I(f^I_{3-i}X^I_{3-i}(f_{3-i}))^J,\]
\[g_{f_1, f_2}(\nabla^I_{X^I_{3-i}} Y^I_i, Z^I_i) = g_{f_1, f_2}((X^I_{3-i}(\ln f_{3-i}))^I Y^I_i, Z^I_i) = (f^I_{3-i}X^I_{3-i}(f_{3-i}))^J g_i(Y_i, Z_i)^I,\]
and
\[g_{f_1, f_2}(\nabla^I_{X^I_{3-i}} Z^I_i, Y^I_i) = g_{f_1, f_2}(\nabla^I_{X^I_{3-i}} Z^I_i, Y^I_i) = (f^I_{3-i}X^I_{3-i}(f_{3-i}))^J g_i(Y_i, Z_i)^I.\]
We add these equations and obtain
\[ A(X_3^I, Y_3^I, Z_3^I) = 0 \]
Hence the same applies for \( A(X_i^I, Y_i^I, Z_3^I) = A(X_3^I, Y_3^I, Z_3^I) = 0 \).
This proves that \( \nabla^* \) is conjugate to \( \nabla \) with respect to \( g_{f_1 f_2} \).

We recall that the connection \( \nabla \) on \( M_1 \times M_2 \) induced by \( \hat{\nabla} \) and \( \hat{\nabla} \) on \( M_1 \) and \( M_2 \) respectively, is given by the equations (10).

**Proposition 3.7** \((M_1, \nabla, g_1) \) and \((M_2, \nabla, g_2) \) are statistical manifolds if and only if \((M_1 \times M_2, g_{f_1 f_2}, \nabla) \) is a statistical manifold.

**Proof:** Let us assume that \((M_i, \nabla, g_i), (i = 1, 2) \) is a statistical manifold. Firstly, we show that \( \nabla \) is torsion-free. Indeed; by Equation (16), we have for any \( X, Y, X, Y, Z \in \Gamma(TM_1 \times M_2) \)
\[ d\pi_i(T(X, Y)) = \hat{\nabla}_X d\pi_i(Y) - \hat{\nabla}_Y d\pi_i(X) - d\pi_i([X, Y]) \]
Since for \( i = 1, 2, \hat{\nabla} \) is torsion-free, then
\[ \hat{\nabla}_X d\pi_i(Y) - \hat{\nabla}_Y d\pi_i(X) = d\pi_i([X, Y]) \]
Therefore, from Remark 2.2, the connection \( \nabla \) is torsion-free.

Secondly, we show that \( \nabla g_{f_1 f_2} \) is symmetric. In fact; for \( i = 1, 2, \)
\[ (\nabla g_{f_1 f_2})(X_i^I, Y_i^I, Z_i^I) = X_i^I(g_{f_1 f_2}(Y_i^I, Z_i^I)) - g_{f_1 f_2}(\nabla X_i^I Y_i^I, Z_i^I) - g_{f_1 f_2}(Y_i^I, \nabla X_i^I Z_i^I) \]
by Equations (12), (17) and since \((\nabla g_i), i = 1, 2, \) is symmetric, we have
\[ (\nabla g_{f_1 f_2})(X_i^I, Y_i^I, Z_i^I) = (f_{3-i}^I)^2((\nabla g_i)(X_i, Y_i, Z_i))^I \]
\[ = (f_{3-i}^I)^2((\nabla g_i)(Y_i, X_i, Z_i))^h \]
\[ = (\nabla g_{f_1 f_2})(Y_i^I, X_i^I, Z_i^I). \]

In the different lifts, we have
\[ (\nabla g_{f_1 f_2})(X_i^I, Y_i^I, Z_3^{-I}) = (\nabla g_{f_1 f_2})(X_3^I, Y_i^I, Z_i^I) = (\nabla g_{f_1 f_2})(X_i^I, Y_i^I, Z_i^I) = 0, \]
Therefore, \((\nabla g_{f_1 f_2}) \) is symmetric. Thus \((M_1 \times M_2, g_{f_1 f_2}, \nabla) \) is a statistical manifold.

Conversely, if \((M_1 \times M_2, g_{f_1 f_2}, \nabla) \) is statistical manifold, then \((\nabla g_{f_1 f_2}) \) is symmetric and \( \nabla \) is torsion-free, particularly, when \( X_i, Y_i, Z_i \in \Gamma(TM_i) \), we have
\[ \begin{cases} 
(\nabla g_{f_1 f_2})(X_i^I, Y_i^I, Z_i^I) = (\nabla g_{f_1 f_2})(Y_i^I, X_i^I, Z_i^I), \\
T(X_i^I, Y_i^I) = 0.
\end{cases} \quad \forall i = 1, 2, \]
Then, by Equations (12) and (17), we obtained, for \( i = 1, 2, \) \( \nabla g_i \), is symmetric and \( \nabla \), is torsion-free. Therefore, \((M_i, \nabla, g_i), (i = 1, 2)\) is statistical manifold.

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**References**


