Lower $k$-Hessenberg Matrices and $k$-Fibonacci, Fibonacci-$p$ and Pell $(p, i)$ Numbers

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Abstract

In this work, we define a family of sparse Hessenberg matrices whose permanents lead us to $k$-Fibonacci, Fibonacci-$p$ and Pell $(p, i)$ numbers. Furthermore, we show that it contains some well-known general number sequences in it. We provide a Maple 13 source code describing the contraction steps.

Keywords: Determinant, Fibonacci-$p$ and Pell $(p, i)$ numbers, Hessenberg matrix, $k$-Fibonacci numbers, Permanent.

1 Introduction

Matrix theory combines linear algebra, graph theory, algebra, combinatorics and statistics. Some special type of matrices are very important in these areas. In this paper, we consider lower $k$-Hessenberg matrices which have the
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pattern

$$H_n(k) = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix}$$

which will be defined more precisely later.

Most of the well-known number sequences are defined as a result of a natural event or a mathematical modelling of an occurrence in nature. Fibonacci numbers are one of the most famous number sequences defined on modelling for proliferating of rabbits. In literature, there is a huge number of papers on Fibonacci numbers and their generalizations. For example, Lee et al. [7] investigated the $k$-generalized Fibonacci sequence $(g^{(k)}_n)$ with initial conditions

$$g^{(k)}_1 = \cdots = g^{(k)}_{k-2} = 0, \quad g^{(k)}_{k-1} = g^{(k)}_k = 1,$$

and, for $n > k \geq 2$,

$$g^{(k)}_n = g^{(k)}_{n-1} + g^{(k)}_{n-2} + \cdots + g^{(k)}_{n-k}.$$  \hfill (1)

Then, Lee [6] introduced $k$-Lucas numbers, which has similar recurrence but for different initial conditions.

Kılıç and Stakhov [3] considered certain generalizations of well-known Fibonacci and Lucas numbers and the generalized Fibonacci and Lucas $p$-numbers defined by the following recurrence relation for $p = 1, 2, 3, \ldots$, and $n > p + 1$

$$F_p(n) = F_p(n-1) + F_p(n-p-1),$$
$$L_p(n) = L_p(n-1) + L_p(n-p-1),$$

where $F_p(0) = 0, F_p(1) = \cdots = F_p(p) = F_p(p+1) = 1$ and $L_p(0) = p + 1, L_p(1) = \cdots = L_p(p) = L_p(p+1) = 1$, respectively. Furthermore they defined $n$-square $(0, 1)$-matrix as below

$$M(n, p) = \begin{cases} 1, & \text{for } m_{i+1, i} = m_{i, i} = m_{i, i+p} \\ 0, & \text{for } j = i + 1 \end{cases} \quad (2)$$

for a fixed integer $p$, which corresponds to the adjacency matrix of the bipartite graph $G(M(n, p))$. Then they showed that permanents of $M(n, p)$ are the number of 1-factors of $G(M(n, p))$ that is the $(n + 1)$th generalized Fibonacci $p$-number. Moreover Yilmaz et al. [4, 9] considered Hessenberg matrices and the Fibonacci, Lucas, Pell and Perrin numbers. Öcal et al. [8] gave some determinantal and permanental representations for $k$-generalized Fibonacci and
Lucas numbers. On the other hand, Kılıç [2] studied the generalized Pell $(p, i)$-numbers for $p = 1, 2, 3, \ldots, n > p + 1$, and $0 \leq i \leq p$ 

$$P_p^{(i)}(n) = 2P_p^{(i)}(n - 1) + P_p^{(i)}(n - p - 1)$$

with initial conditions $P_p^{(i)}(1) = P_p^{(i)}(2) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i + 1) = P_p^{(i)}(i + 2) = \cdots = P_p^{(i)}(p + 1) = 1$. Moreover, the author defined $n$-square integer matrix $M(n, p) = (m_{ij})$ as below:

$$M(n, p) = \begin{cases} 1, & \text{for } m_{i+1,i} = m_{i,i+p} \\ 2, & \text{for } m_{i,i} \\ 0, & \text{for } j = i + 1 \end{cases}$$

(3)

for a fixed integer $p$, then showed

$$\text{per } M(n, p) = P_p^{(p)}(n + p + 1).$$

The **permanent** of an $n \times n$ matrix $A = (a_{ij})$ is given by

$$\text{per } (A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i \sigma(i)},$$

where $S_n$ represents the symmetric group of degree $n$.

Brualdi and Gibson [1] proposed a method to compute permanent of a matrix. Let $A = (a_{ij})$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call $A$ is **contractible** on column $k$, if column $k$ contains exactly two non zero elements. Suppose that $A$ is contractible on column $k$ with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m - 1) \times (n - 1)$ matrix $A_{i\neq k}$ obtained from $A$ replacing row $i$ with $a_{jk}r_i + a_{ik}r_j$ and deleting row $j$ and column $k$ is called the contraction of $A$ on column $k$ relative to rows $i$ and $j$. If $A$ is contractible on row $k$ with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{\neq i}$ is called the contraction of $A$ on row $k$ relative to columns $i$ and $j$. We know that if $A$ is an integer matrix and $B$ is a contraction of $A$ [1], then

$$\text{per } A = \text{per } B.$$  

(4)

A matrix $A$ is called **convertible** if there exists an $n$-square $(1, -1)$-matrix $H$ such that $\text{per } A = \det(A \circ H)$, here $\circ$ denotes Hadamard product of $A$ and $H$. The matrix $H$ is called as **converter** of $A$. Let $H$ be a $(1, -1)$-matrix such that

$$h_{i,j} = \begin{cases} -1, & i + 1 = j \\ 1, & \text{otherwise} \end{cases}$$

(5)

Klein [5] established a generalization for Fibonacci numbers for a constant integer $m \geq 2$

$$A_n^{(m)} = A_{n-1}^{(m)} + A_{n-m}^{(m)}, \quad \text{for } n > m + 1,$$

$$A_n^{(m)} = n - 1, \quad \text{for } 1 < n \leq m + 1.$$  

(6)
In particular, \( F_n = A_n^{(2)} \) are the standard Fibonacci numbers. Taking into account Klein’s generalization, let us consider the sequence \( \{u_n\} \) given below:

\[
    u_n^{(k)} = au_{n-1}^{(k)} + b^k cu_{n-k-1}^{(k)}. \tag{7}
\]

Here \( k > 1 \) and \( u_0^{(k)} = 1, u_1^{(k)} = d, u_2^{(k)} = ad \) and \( u_k^{(k)} = a^{k-1}d \). The first few terms of the sequence given in following table:

<table>
<thead>
<tr>
<th>( k \backslash n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_n^{(2)} )</td>
<td>d</td>
<td>da</td>
<td>( da^2 + b^2c )</td>
<td>( da^3 + ab^2c + cdb^2 )</td>
<td>( da^4 + a^2cb^2 + 2cb^2da )</td>
</tr>
<tr>
<td>( u_n^{(3)} )</td>
<td>d</td>
<td>da</td>
<td>( da^2 )</td>
<td>( da^3 + b^3c )</td>
<td>( da^4 + ab^2c + b^3dc )</td>
</tr>
<tr>
<td>( u_n^{(4)} )</td>
<td>d</td>
<td>da</td>
<td>( da^2 )</td>
<td>( da^3 )</td>
<td>( da^4 + cb^4 )</td>
</tr>
<tr>
<td>( u_n^{(5)} )</td>
<td>d</td>
<td>da</td>
<td>( da^2 )</td>
<td>( da^3 )</td>
<td>( da^4 )</td>
</tr>
</tbody>
</table>

2 \ Lower \( k \)-Hessenberg Matrices and the \( \{u_n\} \) Sequence

Let us define the \( n \)-square Hessenberg matrix \( H_n(k) = (h_{ij}) \) as follows:

\[
    h_{ij} = \begin{cases} 
    a, & \text{for } i = j = 1, 2, \ldots, n - 1 \\
    b, & \text{for } j = i + 1 \\
    c, & \text{for } i = j + k \\
    d, & \text{for } i = j = n \\
    0, & \text{otherwise}
    \end{cases} \tag{8}
\]

where \( 2 \leq k \leq n - 1 \) and \( a, b, c, d \in \mathbb{R} \).

**Example 2.1** For \( k = 3 \) and \( n = 7 \);

\[
    H_7(3) = \begin{pmatrix} 
    a & b & 0 & 0 & 0 & 0 & 0 \\
    0 & a & b & 0 & 0 & 0 & 0 \\
    0 & 0 & a & b & 0 & 0 & 0 \\
    c & 0 & 0 & a & b & 0 & 0 \\
    0 & c & 0 & 0 & a & b & 0 \\
    0 & 0 & c & 0 & 0 & a & b \\
    0 & 0 & 0 & c & 0 & 0 & d 
    \end{pmatrix}.
\]

**Theorem 2.2** Let \( H_n(k) \) be as in 8, then

\[
    \text{per} H_n(k) = u_n^{(k)},
\]

for \( 2 \leq k < n \), where \( u_n^{(k)} \) is the \( n \)th term of the sequence given by 7.
Proof. By the definition of $H_n(k)$, it can be contracted on column $n$. Let $H_n^{(r)}(k)$ be the $r$th contraction of the matrix $H_n(k)$. For $r = 1$,

$$H_n^{(1)}(k) = \begin{pmatrix}
a & b \\
0 & a & b \\
\vdots & 0 & a & b \\
0 & \ddots & 0 & a & b \\
0 & 0 & \cdots & 0 & a & b \\
c & \ddots & \ddots & \ddots & \cdots \\
0 & \ddots & 0 & 0 & \cdots & 0 & a & b \\
\vdots & c & 0 & 0 & \cdots & 0 & a & b \\
0 & \cdots & 0 & dc & bc & 0 & \cdots & 0 & da \\
\end{pmatrix}. $$

Using the consecutive contraction method on the last column, we get,

$$H_n^{(r)}(k) = \begin{pmatrix}
a & b \\
0 & a & b \\
\vdots & 0 & a & b \\
0 & \cdots & 0 & a & b \\
0 & 0 & \cdots & 0 & a & b \\
c & \ddots & \ddots & \ddots & \cdots \\
0 & \ddots & 0 & 0 & \cdots & 0 & a & b \\
\vdots & c & 0 & 0 & \cdots & 0 & a & b \\
0 & \cdots & 0 & cu_r^{(k)} & bcu_{r-1}^{(k)} & b^2cu_{r-2}^{(k)} & \cdots & b^{k-1}cu_{r-k+1}^{(k)} & u_{r+1}^{(k)} \\
\end{pmatrix}. $$

Here $2 \leq r \leq n - k - 1$ and

$$H_n^{(r)}(k) = \begin{pmatrix}
a & b \\
0 & a & b \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \cdots & 0 & a & b \\
0 & 0 & \cdots & 0 & a & b \\
\end{pmatrix}, $$

where $n - k - 1 < r \leq n - 3$. Then, continuing with this process, we get

$$H_n^{(n-2)}(k) = \begin{pmatrix}
a & b \\
b^{k-1}cu_{n-k-1}^{(k)} & u_{n-1}^{(k)} \\
\end{pmatrix}. $$
By applying (4), we have \( \text{per} H_n(k) = \text{per} H_n^{(n-2)}(k) = u_n^{(k)} \), as desired. ■

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>( k )-Fibonacci numbers</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Fibonacci-( p ) numbers</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>( \text{Pell}(p, i) ) numbers</td>
</tr>
</tbody>
</table>

As it can be seen from the previous table, the matrix \( H_n(k) \) is a general form of the matrices given by 2 and 3. Moreover, for \( a = 2, b = 1, c = -1 \) and \( d = 1 \), the permanent of the sequence gives \( k \)-Fibonacci numbers.

**Theorem 2.3** Let us consider the matrix \( H_n(k) = (h_{ij}) \) with \( h_{i,i+1} = 1, h_{i,i} = 2, \) and \( h_{i+k,i} = -1 \), where \( 2 \leq k \leq n \). Then

\[
\text{per} H_n(k) = \sum_{i=1}^n g_i^{(k)}.
\]

**Proof.** By the contraction method on column \( n \), one can see that

\[
H_n^{(1)}(k) = \begin{pmatrix}
2 & 1 \\
0 & 2 & 1 \\
\vdots & 0 & 2 & 1 \\
0 & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
-1 & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & 0 & 0 & \cdots & 0 & 2 & 1 \\
\vdots & -1 & 0 & 0 & \cdots & 0 & 2 & 1 \\
0 & \cdots & -2 & -1 & 0 & \cdots & 0 & 4
\end{pmatrix}.
\]

By the recursive contraction method on the last column, we get

\[
H_n^{(r)}(k) = \begin{pmatrix}
2 & 1 \\
0 & 2 & 1 \\
\vdots & 0 & 2 & 1 \\
0 & \cdots & 0 & 2 & 1 \\
0 & 0 & \cdots & 0 & 2 & 1 \\
-1 & \cdots & 0 & 0 & \cdots & 2 & 1 \\
0 & \cdots & 0 & -\sum_{i=1}^{r-2} g_i^k & -\sum_{i=1}^{r-1} g_i^k & \cdots & -\sum_{i=1}^{r-k+2} g_i^k & -\sum_{i=1}^{r-k+1} g_i^k \\
\vdots & -1 & 0 & 0 & \cdots & 0 & 2 & 1 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 2 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 4
\end{pmatrix}
\]
for $2 \leq r \leq n - k - 1$ and

$$H^{(r)}_{n}(k) = \begin{pmatrix}
2 & 1 & \cdots \\
0 & 2 & \cdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & 2 & 1 \\
0 & 0 & \cdots & 0 & 2 & \cdots & 1 \\
-\sum_{i=1}^{n-k} g_i^k & -\sum_{i=1}^{n-k-1} g_i^k & \cdots & \cdots & -\sum_{i=1}^{n-r+2} g_i^k & \sum_{i=1}^{n-r+2} g_i^k
\end{pmatrix}$$

for $n - k - 1 < r \leq n - 3$. Going with this process, one gets

$$H^{(n-2)}_{n}(k) = \begin{pmatrix}
2 & \cdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & 2 & 1 \\
0 & 0 & \cdots & 0 & 2 & \cdots & 1 \\
-\sum_{i=1}^{n-k} g_i^k & -\sum_{i=1}^{n-k-1} g_i^k & \cdots & \cdots & -\sum_{i=1}^{n-r+2} g_i^k & \sum_{i=1}^{n-r+2} g_i^k
\end{pmatrix}.$$ 

By applying 4, we have

$$\text{per} \, H_n(k) = \text{per} \, H^{(n-2)}_{n}(k) = \sum_{i=1}^{n} g_i^{(k)},$$

which is the sum of $k$-Fibonacci numbers given by 1. ■

**Theorem 2.4** Let us consider the $n$-square Hessenberg matrix $M_n(k) = (m_{ij})$ as

$$m_{ij} = \begin{cases} 
a, & \text{for } i = j = 1, 2, \ldots, n-1 \\
b, & \text{for } j = i + 1 \\
c, & \text{for } i = j + k \\
d, & \text{for } i = j = n \\
o, & \text{otherwise}
\end{cases}$$

where $2 \leq k \leq n - 1$. Then

$$\det M_n(k) = u^{(k)}_n.$$ 

**Proof.** It can be seen by using the converter matrix given with 5. ■

### 3 Appendix A

Using the following Maple 13 source code, it is possible to get the matrix and the steps of the contraction method. Here $n$ is the order of the matrix and $s$ is the shifting diagonal (i.e, $s = k$).

```maple
restart:
> a:=..:b:=..:c:=..:d:=..:s:=..:n:=..:with(LinearAlgebra):
> permanent:=proc(n)
> local i,j,k,p,C;
> p:=(i,j)->piecewise(i=j+s+1,c,j=i+1,b,j=n and i=n,d,i=j,a);
> C:=Matrix(n,n,p):
```

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for k from 1 to n-1 do
print(k,C):
for j from 1 to n+1-k do
C[n-k,j]:=C[n+1-k,n+1-k]*C[n-k,j]+C[n-k,n+1-k]*C[n+1-k,j]:
od:
C:=DeleteRow(DeleteColumn(Matrix(n+1-k,n+1-k,C),n+1-k),n+1-k):
od:
print(k,eval(C)):
end proc:

with(LinearAlgebra):
permanent(n);

References


