Coefficient Estimates for $\lambda$–Bazilevič Functions of Bi-univalent Functions

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Abstract

In this paper, we introduce two new subclasses of the function class $\Sigma$ of $\lambda$–Bazilevič functions of bi-univalent functions defined in the open unit disc. We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. The results presented in this paper would generalize some recent works of Xu et al. and Ali et al.

Keywords: Analytic functions, Univalent functions, Bazilevič functions, Bi-univalent functions, Coefficient estimates.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]

which are analytic in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$. We also denote by $\mathcal{S}$ the subclass of the normalized analytic function class $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $U$ (see [1-4]). Familiar subclasses of starlike functions of order $\xi(0 \leq \xi < 1)$ and convex functions of order $\xi$ for
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which either of the quantity

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \xi$$

and

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \xi.$$ 

The class consisting these two functions are given by $S^*(\xi)$ and $K(\xi)$, respectively. For a constant $\beta \in (-\pi/2, \pi/2)$, a function $f$ is univalent on $U$ and satisfies the condition that $\Re\{e^{i\theta}zf'(z)/f(z)\} > 0$ in $U$. We denote this class by $TS^*$ (see [2]).

It is well known that every function $f(z) \in S$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega, \quad (|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}).$$

In fact, the inverse function is given by

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \quad (2)$$

A function $f \in S$ is bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. We denote by $\Sigma$ the class of all bi-univalent functions in $U$ given by the Taylor-Maclaurin series expansion (1). Lewin [5] investigated the class $\Sigma$ of bi-univalent functions and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [6]). Srivastava et al. [7], and Frasin and Aouf [8] introduced subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Recently, Xu et al. [9], Goyal and Goswami [10] and Ali et al. [11] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients.

Let $f$ and $g$ be analytic functions in $U$, we say that $f$ is subordinate to $g$, written as $f(z) \prec g(z)$ if there exists a Schwarz function $\omega(z)$ in $U$, with $\omega(0) = 0$ and $|\omega(z)| < 1(z \in U)$, such that $f(z) = g(\omega(z))$. In particular, when $g$ is univalent, then the above subordination is equivalent to $f(0) = 0$ and $f(U) \subseteq g(U)$.

Let

$$H(U) = \{h : U \to \mathbb{C}, \Re\{h(z)\} > 0 \text{ and } h(0) = 1, h(\overline{z}) = \overline{h(z)}(z \in U)\}.$$ 

Assume that $\varphi$ is an analytic univalent function with positive part in $U$, $\varphi(U)$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$, and $\varphi'(0) > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1z + B_2z^2 + \cdots, \quad (B_1 > 0).$$ \quad (3)
Obviously, $\varphi(U) \subseteq H(U)$.

Wang et al.[13] (also see Li [14]) introduced and investigated the class of $\lambda$–Bazilevič functions consists of functions $f \in A$ satisfying the subordination:

$$
\left(1 - \lambda\right)\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{g(z)}\right)^{\alpha+i\mu} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) \left(\frac{f'(z)}{g'(z)}\right)^{\alpha+i\mu} \prec \frac{1 + Az}{1 + Bz}
$$

$(\alpha \geq 0, \lambda \geq 0, \mu, A, B \in R$ and $A \neq B, -1 \leq B \leq 1; g \in S^*(\xi))$.

In this paper, using the subordination, we introduce the following two classes of $\lambda$–Bazilevič functions of bi-univalent functions.

**Definition 1.1** Let the function $f(z)$, defined by (1), be in the analytic function class $A$. We say that $f(z) \in U_{p,q}^{\alpha}(\beta, b, \lambda)$ if the following conditions are satisfied:

$$
f(z) \in \Sigma
$$

and

$$
\left\{ \frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left(1 - \lambda\right)\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) \left(\frac{f'(z)}{z}\right)^{\alpha} \right] - i \tan \beta \right\} \in p(U) \quad (4)
$$

and

$$
\left\{ \frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left(1 - \lambda\right)\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) \left(\frac{f'(z)}{z}\right)^{\alpha} \right] - i \tan \beta \right\} \in q(U) \quad (5)
$$

where $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2}); b \in C\{0\}; \alpha \geq 0, \lambda \geq 0; n \in N_0$, the function $g(\omega) = f^{-1}(\omega)$ is given by (2).

**Definition 1.2** Let the function $f(z)$ of the form (1), be in the analytic function class $A$. We say that $f(z) \in L_{p,q}^{\beta}(\beta, b, \lambda)$ if the following conditions are satisfied:

$$
f(z) \in \Sigma
$$

and

$$
\frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left(1 - \lambda\right)\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) \left(\frac{f'(z)}{z}\right)^{\alpha} \right] - i \tan \beta \prec \varphi(z)
$$
and

\[ \frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left( 1 - \lambda \right) \frac{\omega g'(\omega)}{g(\omega)} \left( \frac{g(\omega)}{\omega} \right)^\alpha + \lambda \left( 1 + \frac{\omega g''(\omega)}{g'(\omega)} \right) (g'(\omega))^\alpha - 1 \right] \]

\[ -i \tan \beta \prec \varphi(\omega) \]

\[ (g(\omega) = f^{-1}(\omega); z, \omega \in U). \]

For \( f(z) \in L_\alpha^\varphi(\beta, b, \lambda) \) and \( \varphi = \left( \frac{1 + Az}{1 + Bz} \right) \eta (0 < \eta \leq 1) \), Definition 1.1 readily yields the following class \( B_\alpha^\varphi(\beta, b, \lambda) \) satisfying:

\[ f(z) \in \Sigma \]

and

\[ \left| \arg \frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left( 1 - \lambda \right) \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\alpha + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) (f'(z))^\alpha - 1 \right] \right| < \frac{\eta \pi}{2} \]

and

\[ \left| \arg \frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left( 1 - \lambda \right) \frac{\omega g'(\omega)}{g(\omega)} \left( \frac{g(\omega)}{\omega} \right)^\alpha + \lambda \left( 1 + \frac{\omega g''(\omega)}{g'(\omega)} \right) (g'(\omega))^\alpha - 1 \right] \right| < \frac{\eta \pi}{2} \]

\[ (g(\omega) = f^{-1}(\omega); 0 < \eta \leq 1; z, \omega \in U) \]

For \( f(z) \in L_\alpha^\varphi(\beta, b, \lambda) \) and \( \varphi = \left( \frac{1 + Az}{1 + Bz} \right) (-1 \leq B < A \leq 1) \), Definition 1.2 readily yields the following class \( L_\alpha^{A,B}(\beta, b, \lambda) \) satisfying:

\[ f(z) \in \Sigma \]

and

\[ \frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left( 1 - \lambda \right) \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\alpha + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) (f'(z))^\alpha - 1 \right] \]

\[ -i \tan \beta \prec \frac{1 + Az}{1 + Bz} \]

and

\[ \frac{e^{i\beta}}{\cos \beta} \left[ 1 + \frac{1}{b} \left( 1 - \lambda \right) \frac{\omega g'(\omega)}{g(\omega)} \left( \frac{g(\omega)}{\omega} \right)^\alpha + \lambda \left( 1 + \frac{\omega g''(\omega)}{g'(\omega)} \right) (g'(\omega))^\alpha - 1 \right] \]

\[ -i \tan \beta \prec \frac{1 + A\omega}{1 + B\omega} \]

\[ (g(\omega) = f^{-1}(\omega); z, \omega \in U). \]
For suitable choices of $p, q$ and by specializing the parameters $b, \lambda, \alpha, \eta, \beta$ involved in the class $U^{p,q}_{\alpha}(\beta, b, \lambda)$, $L^{\alpha}_{\xi}(\beta, b, \lambda)$ and $B^{\eta}_{\alpha}(\beta, b, \lambda)$, we also obtain the following subclasses which were studied in many earlier works:

1. $S^{\Sigma}(\xi) = L_{1}^{-}\frac{2^{-1}}{0}(0, 1, 0)$ (Bi-Starlike function) (Brannan and Taha [6]);
2. $K^{\Sigma}(\xi) = L_{0}^{-}\frac{2^{-1}}{1}(0, 1, 1)$ (Bi-Starlike function) (Brannan and Taha [6]);
3. $U^{(p,q)} = U^{p,q}_{1}(0, 1, 0)$ (Xu et al. [9]);
4. $M^{\Sigma}(\alpha, \phi) = U^{p,p}_{0}(0, 1, \lambda)(\text{General Bi-Mocanu-convex function of Ma-Minda})$ (Ali et al.[11]);
5. $B^{\Sigma}(\alpha, \phi) = L^{\phi}_{\alpha}(0, 1, 0)$ (Bi-Bazilevič functions of Ma-Minda type [16]).

In this paper, estimates on the initial coefficients for class $U^{p,q}_{\alpha}(\beta, b, \lambda)$, $L^{\alpha}_{\xi}(\beta, b, \lambda)$ and $B^{\eta}_{\alpha}(\beta, b, \lambda)$ are obtained. Several related classes are also considered, and a connection to earlier known results is made.

2 Coefficient Bounds for the Function Class $U^{p,q}_{\alpha}(\beta, b, \lambda)$

Theorem 2.1 Suppose that $f(z) \in A$ of the form (1), be in the class $U^{p,q}_{\alpha}(\beta, b, \lambda)$. Then

$$|a_{2}| \leq \min \left\{ \frac{|b| \cos \beta \sqrt{|p'(0)|^{2} + |q'(0)|^{2}}}{(\alpha + 1)(\lambda + 1)}, \frac{(|p''(0)| + |q''(0)|)|b| \cos \beta}{2(\alpha + 2)[\lambda + \alpha(1 + 3\lambda) + 1]} \right\}$$

and

$$|a_{3}| \leq \min \left\{ \frac{(|p''(0)| + |q''(0)|)|b| \cos \beta}{4(\alpha + 2)(2\lambda + 1)}, \frac{|b| \cos \beta}{2(\alpha + 2)}, \frac{|p''(0)|5\lambda + \alpha(3\lambda + 1) + 3 + |q''(0)|(3\lambda + 1)|1 - \alpha|}{(2\lambda + 1)[\lambda + \alpha(3\lambda + 1) + 1]} \right\}.$$ 

Proof. It follows from the conditions (4) and (5) that

$$e^{i\beta} \left\{ 1 + \frac{1}{b} \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\alpha} + \lambda(1 + \frac{zf''(z)}{f'(z)})f'(z) \right] - 1 \right\}$$

$$= p(z) \cos \beta + i \sin \beta$$

and

$$e^{i\beta} \left\{ 1 + \frac{1}{b} \left[ (1 - \lambda) \frac{\omega g'(\omega)}{g(\omega)} \left( \frac{g(\omega)}{\omega} \right)^{\alpha} + \lambda(1 + \frac{\omega g''(\omega)}{g'(\omega)})g'(\omega) \right] - 1 \right\}$$
where $p(z) \in H(U), q(\omega) \in H(U)$. Furthermore, the functions $p(z)$ and $q(\omega)$ have the following series expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, p_m = \frac{p^{(m)}(0)}{m!} \quad (m \in \mathbb{N})$$

and

$$q(\omega) = 1 + q_1 \omega + q_2 \omega^2 + \cdots, q_m = \frac{q^{(m)}(0)}{m!} \quad (m \in \mathbb{N})$$

respectively. Now, in view of the series expansions (10) and (11), by equating the coefficients in (8) and (9), we get

$$\frac{e^{i\beta}}{b} [(\alpha + 1)(\lambda + 1)] a_2 = p_1 \cos \beta,$$

$$\frac{e^{i\beta}}{b} [(\alpha + 2)(2\lambda + 1)a_3 + \frac{(\alpha - 1)(\alpha + 2)(3\lambda + 1)}{2} a_2^2] = p_2 \cos \beta,$$

$$-\frac{e^{i\beta}}{b} [(\alpha + 1)(\lambda + 1)] a_2 = q_1 \cos \beta,$$

and

$$\frac{e^{i\beta}}{b} [(\alpha + 2)(2\lambda + 1)(2a_2^2 - a_3) + \frac{(\alpha - 1)(\alpha + 2)(3\lambda + 1)}{2} a_2^2] = q_2 \cos \beta.$$
Since $p_1 = p'(0), p_2 = \frac{p''(0)}{2}, q_1 = q'(0), q_2 = \frac{q''(0)}{2}$, it follows from (19) and (20) that
\[
|a_2| \leq \frac{|b| \cos \beta \sqrt{|p'(0)|^2 + |q'(0)|^2}}{(\alpha + 1)(\lambda + 1)}
\]
and
\[
|a_2| \leq \sqrt{\frac{(|p''(0)| + |q''(0)|)|b| \cos \beta}{2(\alpha + 2)[\lambda + \alpha(1 + 3\lambda) + 1]}},
\]
which gives us the desired estimate on $|a_2|$ as asserted in (6).

Next, in order to find the bound on $|a_3|$, by subtracting (13) from (15), we get
\[
ed^{i\beta} \frac{b}{2(\alpha + 2)(2\lambda + 1)} (a_3 - a_2^2) = (p_2 - q_2) \cos \beta.
\]
(21)
Thus, upon substituting the value of $a_2^2$ from (16) and (19) into (21), it follows that
\[
a_3 = \frac{(p_2 - q_2) b \cos \beta}{2e^{i\beta}(\alpha + 2)(2\lambda + 1)} + \frac{b^2 \cos^2 \beta(p_1^2 + q_1^2)}{2e^{2i\beta}(\alpha + 1)^2(\lambda + 1)^2},
\]
which yields
\[
|a_3| \leq \frac{(|p''(0)| + |q''(0)|)|b| \cos \beta}{4(\alpha + 2)(2\lambda + 1)} + \frac{|b|^2 \cos^2 \beta(|p'(0)|^2 + |q'(0)|^2)}{(\alpha + 1)^2(\lambda + 1)^2}.
\]
On the other hand, by using (16) and (10) in (21), we obtain
\[
a_3 = \frac{b \cos \beta}{2e^{i\beta}(\alpha + 2)} \cdot \frac{[5\lambda + \alpha(1 + 3\lambda) + 3]p_2 + (3\lambda + 1)(1 - \alpha)q_2}{(2\lambda + 1)[\lambda + \alpha(1 + 3\lambda) + 1]},
\]
it follows that
\[
|a_3| \leq \frac{|b| \cos \beta}{4(\alpha + 2)} \cdot \frac{[5\lambda + \alpha(1 + 3\lambda) + 3]|p''(0)| + (3\lambda + 1)|1 - \alpha||q''(0)|}{(2\lambda + 1)[\lambda + \alpha(1 + 3\lambda) + 1]}.
\]
This completes the proof of Theorem 2.1.

For $b = 1, \beta = 0$, Theorem 2.1 readily yields the following coefficient estimates for $U_{\alpha,\beta}^p(0, 1, \lambda)$.

**Corollary 2.2** Suppose that $f(z) \in \mathcal{A}$ of the form (1), be in the class $U_{\alpha,\beta}^p(0, 1, \lambda)$. Then
\[
|a_2| \leq \min \left\{ \sqrt{|p'(0)|^2 + |q'(0)|^2}, \sqrt{\frac{|p''(0)| + |q''(0)|}{2(\alpha + 2)[\lambda + \alpha(3\lambda + 1) + 1]}} \right\}
\]
and
\[
|a_3| \leq \min \left\{ \frac{|p''(0)| + |q''(0)|}{4(\alpha + 2)(2\lambda + 1)} + \frac{(|p'(0)|^2 + |q'(0)|^2)}{2(\alpha + 1)^2(\lambda + 1)^2}, \frac{|p''(0)||5\lambda + \alpha(3\lambda + 1) + 3| + |q''(0)|(3\lambda + 1)|1 - \alpha|}{4(\alpha + 2)(2\lambda + 1)[\lambda + \alpha(3\lambda + 1) + 1]} \right\}.
\]
For \( b = 1, \beta = 0, \alpha = 0, \lambda = 0 \), we obtain the results in [15] by S. Bulut.

For \( b = 1, \beta = 0, \alpha = 1, \lambda = 0 \), Theorem 2.1 readily improve coefficient estimates for \( U(p, q) \) in [9] as follows.

**Corollary 2.3** Suppose that \( f(z) \in \mathcal{A} \) of the form (1), be in the class \( U(p, q) \). Then

\[
|a_2| \leq \min \left\{ \frac{|p'(0)|}{2}, \sqrt{\frac{|p''(0)| + |q''(0)|}{12}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{|p''(0)| + |q''(0)|}{12}, \frac{(p'(0))^2}{4}, \frac{|p''(0)|}{6} \right\}.
\]

3 Coefficient Bounds for the Function Class \( L_\alpha^c(\beta, b, \lambda) \) and \( B_\alpha^q(\beta, b, \lambda) \)

In order to prove our main results, we first recall the following lemmas.

**Lemma 3.1** (see [12]) If \( p(z) \in \mathcal{P} \), then \( |p_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p(z) \) analytic in \( U \) for which \( \Re\{p(z)\} > 0 \), \( p(z) = 1 + p_1z + p_2z^2 + \cdots \) for \( z \in U \).

**Theorem 3.2** Suppose that \( f(z) \in \mathcal{A} \) of the form (1), be in the class \( L_\alpha^c(\beta, b, \lambda) \). Then

\[
|a_2| \leq \min \left\{ \frac{|B_1||b| \cos \beta}{(\alpha + 1)(\lambda + 1)}, \frac{2|b| \cos \beta(|B_1| + |B_2 - B_1|)}{(\alpha + 2)[\lambda + \alpha(3\lambda + 1) + 1]}, \frac{|B_1|\sqrt{2|B_1||b| \cos \beta}}{\sqrt{|B_1|^2b \cos \beta(\alpha + 2)[\lambda + \alpha(3\lambda + 1) + 1] - 2(B_2 - B_1)e^{i\beta}(\alpha + 1)^2(\lambda + 1)^2}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{|B_1||b| \cos \beta}{(\alpha + 2)(2\lambda + 1)}, \frac{|B_1|^2|b|^2 \cos^2 \beta}{(\alpha + 1)^2(\lambda + 1)^2}, Q_1, Q_2 \right\},
\]

where

\[
Q_1 = \frac{|b| \cos \beta\{5\lambda + (\alpha + |1 - \alpha|)(1 + 3\lambda) + 3|B_1| + 4(2\lambda + 1)|B_2 - B_1|\}}{2(\alpha + 2)(2\lambda + 1)[\lambda + \alpha(3\lambda + 1) + 1]},
\]

\[
Q_2 = \frac{|B_1||b|\{B_1^2|b| \cos(\alpha + 2)[5\lambda + (\alpha + |1 - \alpha|)(3\lambda + 1) + 3] + Q_3\}}{2(\alpha + 2)(2\lambda + 1)[B_1^2b(\alpha + 2)[\lambda + \alpha(3\lambda + 1) + 1] - Q_4]},
\]

\[
Q_3 = 4|B_2 - B_1|(\alpha + 1)^2(\lambda + 1)^2
\]

and

\[
Q_4 = 2(B_2 - B_1)(1 + i \tan \beta)(1 + \alpha)^2(1 + \lambda)^2.
\]
Proof. Let \( f \in L_\phi^\alpha(\beta, b, \lambda) \), consider the analytic functions \( u, v : U \to U \), with \( u(0) = v(0) = 0 \), such that

\[
e^{i\beta}\left\{ 1 + \frac{1}{b} \left[ (1 - \lambda)\frac{zf'(z)}{f(z)} + \lambda(1 + zf''(z))(f'(z))\alpha - 1 \right] \right\}
\]

\[
= \varphi(u(z)) \cos \beta + i \sin \beta
\] (24)

and

\[
e^{i\beta}\left\{ 1 + \frac{1}{b} \left[ (1 - \lambda)\frac{\omega g'(\omega)}{g(\omega)} + \lambda(1 + \omega g''(\omega))(g'(\omega))\alpha - 1 \right] \right\}
\]

\[
= \varphi(v(\omega)) \cos \beta + i \sin \beta,
\] (25)

where \( g := f^{-1} \).

Define the function \( m \) and \( n \) by

\[
m(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + m_1 z + m_2 z^2 + \cdots,
\]

and

\[
n(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + n_1 z + n_2 z^2 + \cdots,
\]

it follows that

\[
u(z) = \frac{m(z) - 1}{m(z) + 1} = \frac{m_1}{2} z + \frac{1}{2}(m_2 - \frac{m_1^2}{2})z^2 + \cdots
\] (26)

and

\[
v(z) = \frac{n(z) - 1}{n(z) + 1} = \frac{n_1}{2} z + \frac{1}{2}(n_2 - \frac{n_1^2}{2})z^2 + \cdots
\] (27)

It is clear that \( m \) and \( n \) are analytic in \( U \) and \( m(0) = n(0) = 1 \). Since \( u, v : U \to U \), the function \( m \) and \( n \) have positive real part in \( U \), by virtue of Lemma 3.1, we have \( |m_i| \leq 2 \) and \( |n_i| \leq 2 \quad (i = 1, 2, \cdots) \).

From (24), (25), (26) and (27), it follows that

\[
e^{i\beta}\left\{ 1 + \frac{1}{b} \left[ (1 - \lambda)\frac{zf'(z)}{f(z)} + \lambda(1 + zf''(z))(f'(z))\alpha - 1 \right] \right\}
\]

\[
= \varphi\left(\frac{m(z) - 1}{m(z) + 1}\right) \cos \beta + i \sin \beta
\] (28)

and

\[
e^{i\beta}\left\{ 1 + \frac{1}{b} \left[ (1 - \lambda)\frac{\omega g'(\omega)}{g(\omega)} + \lambda(1 + \omega g''(\omega))(g'(\omega))\alpha - 1 \right] \right\}
\]
\[ \varphi(n(\omega) - 1) \cos \beta + i \sin \beta. \]  

(29)

According to (3), it is evident that

\[ \varphi(u(z)) = 1 + \frac{m_1B_1}{2} z + \left( \frac{m_2B_1}{2} + \frac{m_1^2(B_2 - B_1)}{4} \right) z^2 + \ldots \]

and

\[ \varphi(v(\omega)) = 1 + \frac{n_1B_1}{2} \omega + \left( \frac{n_2B_1}{2} + \frac{n_1^2(B_2 - B_1)}{4} \right) \omega^2 + \ldots \]

Since

\[ e^{i\beta} \left\{ 1 + \frac{1}{b} \left[ (1 - \lambda) \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\alpha} + \lambda(1 + zf''(z))^{\alpha - 1} \right] \right\} \]

\[ = e^{i\beta} + \frac{e^{i\beta}(\alpha + 1)(\lambda + 1)}{b} a_2 z + \left[ (\alpha + 2)(2\lambda + 1)a_3 \right] z^2 + \ldots \]

and

\[ e^{i\beta} \left\{ 1 + \frac{1}{b} \left[ (1 - \lambda) \frac{\omega g'(\omega)}{g(\omega)} \left( \frac{g(\omega)}{\omega} \right)^{\alpha} + \lambda(1 + \omega g''(\omega))^{\alpha - 1} \right] \right\} \]

\[ = e^{i\beta} - \frac{e^{i\beta}(\alpha + 1)(\lambda + 1)}{b} a_2 \omega + \left[ (\alpha + 2)(2\lambda + 1)(2a_2^2 - a_3) \right] \omega^2 + \ldots \]

By equating the coefficients in (28) and (29), we get

\[ \frac{e^{i\beta}(\alpha + 1)(\lambda + 1)}{b \cos \beta} a_2 = \frac{m_1B_1}{2}, \]

(30)

\[ \frac{e^{i\beta}}{b \cos \beta} \left[ (\alpha + 2)(1 + 2\lambda) a_3 + \frac{(\alpha - 1)(\alpha + 2)(1 + 3\lambda)}{2} a_2^2 \right] \]

\[ = \frac{m_2B_1}{2} + \frac{m_1^2(B_2 - B_1)}{4} \]

(31)

\[ - \frac{e^{i\beta}(\alpha + 1)(\lambda + 1)}{b \cos \beta} a_2 = \frac{n_1B_1}{2}, \]

(32)

and

\[ \frac{e^{i\beta}}{b \cos \beta} \left[ (\alpha + 2)(2\lambda + 1)(a_2^2 - a_3) + \frac{(\alpha - 1)(\alpha + 2)(3\lambda + 1)}{2} a_2^2 \right] \]

...
\[
= \frac{\frac{n_2 B_1}{2} + \frac{n_1^2 (B_2 - B_1)}{4}}{2}
\]

We find from (30) and (32) that

\[
m_1 = -n_1
\]

and

\[
\frac{2 e^{2i\beta} (\alpha + 1)^2 (\lambda + 1)^2}{b^2 \cos^2 \beta} a_2^2 = \frac{B_1^2}{4} (m_1^2 + n_1^2).
\]

Also, from (31) and (33), we obtain

\[
e^{i\beta} \frac{b \cos \beta}{2} [\alpha + 2(\lambda + \alpha(1 + 3\lambda) + 1)] a_2^2 = \frac{(m_2 + n_2) B_1}{2} + \frac{(B_2 - B_1) (m_1^2 + n_1^2)}{4}
\]

From (35) and (36), we have

\[
a_2^2 = \frac{(m_2 + n_2) B_1^3 b^2 \cos^2 \beta}{2e^{i\beta} B_1^2 \cos \beta (\alpha + 2)[\lambda + \alpha(3\lambda + 1) + 1] - 4(B_2 - B_1) e^{2i\beta} (\alpha + 1)^2 (\lambda + 1)^2}
\]

Substituting value of \(a_2^2\) from (35), (36) and (37) in (38), we get

\[
a_3 = \frac{(m_2 - n_2) B_1 \cos \beta}{4e^{i\beta}(\alpha + 2)(2\lambda + 1)} + \frac{B_1^2 b^2 \cos^2 \beta (m_1^2 + n_1^2)}{8e^{2i\beta}(\alpha + 1)^2(\lambda + 1)^2},
\]

\[
a_3 = \frac{b \cos \beta \{ (5\lambda + \alpha(3\lambda + 1) + 3|m_2 + (3\lambda + 1)(1 - \alpha)n_2)B_1 + W_3 \}}{4e^{i\beta}(\alpha + 2)(2\lambda + 1)[\lambda + \alpha(1 + 3\lambda) + 1]}
\]
and
\[ a_3 = \frac{B_1 b \cos \beta (W_1 m_2 + W_2 n_2)}{4 e^{i\beta}(\alpha + 2) (2\lambda + 1) \{ B_1^2 b \cos \beta (\alpha + 2) [\lambda + \alpha (1 + 3\lambda) + 1] - W_1 \}}, \quad (41) \]

where
\[ W_1 = (\alpha + 2) [5\lambda + \alpha (1 + 3\lambda) + 3] B_1^2 b \cos \beta - 2 (B_2 - B_1) e^{i\beta} (\alpha + 1)^2 (\lambda + 1)^2, \]
\[ W_2 = (\alpha + 2)(3\lambda + 1)(1 - \alpha) B_1^2 b \cos \beta + 2 (B_2 - B_1) e^{i\beta} (\alpha + 1)^2 (\lambda + 1)^2, \]
\[ W_3 = 2 (2\lambda + 1) (B_2 - B_1) m_1^2 \]

and
\[ W_4 = 2 (B_2 - B_1) e^{i\beta} (\alpha + 1)^2 (\lambda + 1)^2. \]

Using (39), (40) and (41), we have
\[ |a_3| \leq |B_1| |b| \cos \beta \frac{[5\lambda + (\alpha + 1 - \alpha)] (3\lambda + 1) + 3] B_1 | + 4(2\lambda + 1) |B_2 - B_1|}{2(\alpha + 2)(2\lambda + 1)[\lambda + \alpha (1 + 3\lambda) + 1] \}, \quad (42) \]

and
\[ |a_3| \leq \frac{|b| \cos \beta \{ [5\lambda + (\alpha + 1 - \alpha)] (3\lambda + 1) + 3] B_1 | + 4(2\lambda + 1) |B_2 - B_1|}{2(\alpha + 2)(2\lambda + 1)[B_1^2 b (\alpha + 2) [\lambda + \alpha (1 + 3\lambda) + 1] - W_5]}, \quad (43) \]

where
\[ W_5 = 4 |B_2 - B_1| (\alpha + 1)^2 (\lambda + 1)^2 \]

and
\[ W_6 = 2 (B_2 - B_1) (1 + \tan \beta) (1 + \alpha)^2 (1 + \lambda)^2. \]

From (42), (43) and (44), we obtain the desired estimate on \(|a_3|\) given in (23). This is the end of Theorem 3.2.

Let \( b = 1, \beta = 0, \lambda = 0 \), Theorem 3.2 improves Theorem 2.8 in [6] by E. Deniz as follows.

**Corollary 3.3** Suppose that \( f \in \mathcal{A} \) of the form (1), be in the class \( B_{\Sigma}(\alpha, \varphi) \). Then
\[ |a_2| \leq \min \left\{ \frac{|B_1|}{\alpha + 1}, \sqrt{2 |B_1| + |B_2 - B_1|} \frac{2 |B_1| + |B_2 - B_1|}{(\alpha + 2)(\alpha + 1)}, \frac{|B_1| \sqrt{2 |B_1|}}{\sqrt{|B_1^2 (\alpha + 2) (\alpha + 1) - 2 (B_2 - B_1) (\alpha + 1)^2|}} \right\} \]
and

$$|a_3| \leq \min \left\{ \frac{|B_1|}{\alpha + 2} + \frac{|B_1|^2}{(1 + \alpha)^2}, \frac{2|B_2 - B_1|}{2(\alpha + 2)(\alpha + 1)} \right\},$$

$$\frac{|B_1|((\alpha + 2)(\alpha + |1 - \alpha| + 3)|B_1| + 4|B_2 - B_1|((\alpha + 1)^2)}{2(\alpha + 2)|B_1^2(\alpha + 2)(\alpha + 1) - 2(B_2 - B_1)(\alpha + 1)^2|}.$$

Also, let $b = 1, \beta = 0, \alpha = 0$ in Theorem 3.2, we obtain the following Corollary, which improves Theorem 2.3 in [11].

**Corollary 3.4** Suppose that $f(z) \in \mathcal{A}$ of the form (1), be in the class $M_\Sigma(\lambda, \varphi)$. Then

$$|a_2| \leq \min \left\{ \frac{|B_1|}{(\lambda + 1)}, \sqrt{\frac{|B_1| + |B_2 - B_1|}{(\lambda + 1)}} \right\},$$

and

$$|a_3| \leq \min \left\{ \frac{|B_1|}{2(\lambda + 1)} + \frac{|B_1|^2}{(\lambda + 1)^2}, \frac{|B_1| + |B_2 - B_1|}{\lambda + 1}, \frac{|B_1|2(\lambda + 1)|B_1|^2 + (\lambda + 1)^2|B_2 - B_1|}{2(\lambda + 1)(\lambda + 1)|B_1^2 - (B_2 - B_1)(\lambda + 1)|} \right\}.$$
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\[ Q_\lambda(\alpha, \beta, \eta, b) = \eta |b| \sqrt{\eta |b| \cos \beta (\alpha + 2) [5\lambda + (\alpha + |1 - \alpha|)(3\lambda + 1) + 3] + W_7}, \]

\[ W_7 = 2(1 - \eta)(\alpha + 1)^2(\lambda + 1)^2 \]

and

\[ W_8 = (1 - \eta)(1 + i \tan \beta)(\alpha + 1)^2(\lambda + 1)^2. \]

Especially, for $b = 1, \beta = 0, \alpha = 0$, Corollary 3.5 readily yields the following coefficient estimates for $B_0^\eta(0, 1, \lambda)$,

**Corollary 3.6** Suppose that $f(z) \in A$ of the form (1), be in the class $B_0^\eta(0, 1, \lambda)$. Then

\[ |a_2| \leq \min \left\{ \frac{2\eta}{(\lambda + 1)}, \sqrt{\frac{2\eta(2 - \eta)}{(\lambda + 1)^3}}, \frac{2\eta}{\sqrt{(\lambda + 1)|2\eta + (1 - \eta)(\lambda + 1)|}} \right\} \]

and

\[ |a_3| \leq \min \left\{ \frac{\eta}{(2\lambda + 1)} + \frac{4\eta^2}{(\lambda + 1)^2}, \frac{2\eta(2 - \eta)}{(\lambda + 1)} + \frac{\eta(4\eta(2\lambda + 1) + (1 - \eta)(\lambda + 1)^2)}{(2\lambda + 1)(\lambda + 1)|2\eta + (1 - \eta)(\lambda + 1)|} \right\}. \]

By setting $\varphi(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} (0 < \gamma \leq 1)$ in Theorem 3.2, we get the following Corollary:

**Corollary 3.7** Suppose that $f(z) \in A$ of the form (1), be in the class $L_{\alpha - 2\gamma - 1}^\eta(\beta, b, \lambda)$. Then

\[ |a_2| \leq \min \left\{ \frac{2(1 - \gamma)|b| \cos \beta}{(\alpha + 1)(\lambda + 1)} \cdot \sqrt{\frac{4|b| \cos \beta(1 - \gamma)}{(\alpha + 2)[\lambda + \alpha(3\lambda + 1) + 1]}}, \right. \]

\[ \frac{2\sqrt{(1 - \gamma)}}{\sqrt{|(\alpha + 2)[\lambda + \alpha(1 + 3\lambda) + 1]|}} \]

and

\[ |a_3| \leq \min \left\{ \frac{2(1 - \gamma)|b| \cos \beta}{(\alpha + 2)(2\lambda + 1)} + \frac{4(1 - \gamma)^2|b|^2 \cos^2 \beta}{(\alpha + 1)^2(\lambda + 1)^2}, M_1, M_2 \right\}, \]

where

\[ M_1 = \frac{(1 - \gamma)|b| \cos \beta[5\lambda + (\alpha + |1 - \alpha|)(3\lambda + 1) + 3]}{(\alpha + 2)(2\lambda + 1)[\lambda + \alpha(1 + 3\lambda) + 1]} \]

and

\[ M_2 = \frac{(1 - \gamma)|b| \cos \beta[5\lambda + (\alpha + |1 - \alpha|)(3\lambda + 1) + 3]}{(\alpha + 2)(2\lambda + 1)[\lambda + \alpha(3\lambda + 1) + 1]}. \]
Especially, for $b = 1, \beta = 0, \alpha = 1, \lambda = 0$, Corollary 3.7 readily improves the result in [9].

**Corollary 3.8** Suppose that $f(z) \in \mathcal{A}$ of the form (1), be in the class $L_{1-2\gamma, -1}^{-1}(0, 1, 0)$. Then

$$|a_2| \leq \min \left\{ (1 - \gamma), \sqrt{\frac{2(1 - \gamma)}{3}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{2(1 - \gamma)}{3} + (1 - \gamma)^2, \frac{2(1 - \gamma)}{3} \right\} = \frac{2(1 - \gamma)}{3}.$$ 

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**References**


