On Almost $b$-Continuous Functions
in Bitopological Spaces

Z. Duszyński$^1$, N. Rajesh$^2$ and N. Balambigai$^3$

$^1$Casimirus the Great University, Department of Mathematics
pl. Weyssenhoffa 11
85072 Bydgoszcz, Poland
E-mail: imath@ukw.edu.pl
$^2$Department of Mathematics, Rajah Serfoji Govt. College
Thanjavur-613005, Tamilnadu, India
E-mail: nrajesh_topology@yahoo.co.in
$^3$Department of Mathematics, Prist University
Thanjavur, Tamilnadu, India
E-mail: balatopology@gmail.com

(Received: 1-10-13 / Accepted: 11-11-13)

Abstract

In this paper we introduce and study the concept of almost $b$-continuous functions in bitopological spaces.

Keywords: Bitopological spaces, $(i, j)$-$b$-open sets, almost $(i, j)$-$b$-continuous functions.

1 Introduction

The concept of bitopological spaces was first introduced by Kelly [4]. After the introduction of the Definition of a bitopological space by Kelly, a large number of topologists have turned their attention to the generalization of different concepts of a single topological space in this space. In this paper, we introduce and study the concept of almost $b$-continuous functions in bitopological spaces. Throughout this paper, the triple $(X, \tau_1, \tau_2)$ where $X$ is a set and $\tau_1$ and $\tau_2$
are topologies on $X$, will always denote a bitopological space. For a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$, the closure of $A$ and the interior of $A$ with respect to $\tau_i$ are denoted by $i\text{Cl}(A)$ and $i\text{Int}(A)$, respectively, for $i = 1, 2$.

2 Preliminaries

Definition 2.1 A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be:

1. $(i, j)$-semiopen [3] if $A \subset j\text{Cl}(i\text{Int}(A))$,
2. $(i, j)$-semi-preopen [6] if $A \subset j\text{Cl}(i\text{Int}(j\text{Cl}(A)))$,
3. $(i, j)$-b-open [1] if $A \subset j\text{Cl}(i\text{Int}(A)) \cup i\text{Int}(j\text{Cl}(A))$,
4. $(i, j)$-regular open [2] if $A = i\text{Int}(j\text{Cl}(A))$,

On each definition above, $i, j = 1, 2$ and $i \neq j$. The complement of an $(i, j)$-semiopen (resp. $(i, j)$-semi-preopen, $(i, j)$-b-open, $(i, j)$-regular open) set is called an $(i, j)$-semiclosed (resp. $(i, j)$-semi-preclosed, $(i, j)$-b-closed, $(i, j)$-regular closed) set.

Definition 2.2 [1] Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. Then the intersection of all $(i, j)$-b-closed sets of $X$ containing $A$ is called the $(i, j)$-b-closure of $A$ and is denoted by $(i, j)$-bCl$(A)$. The union of all $(i, j)$-b-open sets of $X$ contained in $A$ is called the $(i, j)$-b-interior of $A$ and is denoted by $(i, j)$-bInt$(A)$.

Definition 2.3 A point $x$ of $X$ is said to be the $(i, j)$-c-cluster point [5] of $A$ if $A \cap U \neq \emptyset$ for every $(i, j)$-regular open set $U$ containing $x$, the set of all $(i, j)$-c-cluster points of $A$ is called the $(i, j)$-c-closure of $A$, a subset $A$ of $X$ is said to be $(i, j)$-c-closed if the set of all $(i, j)$-c-cluster points of $A$ is a subset of $A$, the complement of an $(i, j)$-c-closed set is an called an $(i, j)$-c-open set or a subset $A$ of $X$ is called $(i, j)$-c-open if and only if there exist $(i, j)$-regular open sets $A_k, k \in I$ such that $A = \bigcup_{k \in I} A_k$.

Lemma 2.4 [1] Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ a subset of $X$. Then

1. $(i, j)$-bInt$(A)$ is an $(i, j)$-b-open set;
2. $(i, j)$-bCl$(A)$ is an $(i, j)$-b-closed set;
3. $A$ is $(i, j)$-b-open if and only if $A = (i, j)$-bInt$(A)$;
4. A is \((i,j)\)-b-closed if and only if \(A = (i,j)\)-bCl\((A)\);

5. \((i,j)\)-bInt\((X \setminus A) = X \setminus (i,j)\)-bCl\((A)\);

6. \((i,j)\)-bCl\((X \setminus A) = X \setminus (i,j)\)-bInt\((A)\).

Lemma 2.5 \([1]\) Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subset X\). A point \(x \in (i,j)\)-bCl\((A)\) if and only if \(U \cap A \neq \emptyset\) for every \((i,j)\)-b-open set \(U\) containing \(x\).

Definition 2.6 A function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is said to be \((i,j)\)-b-continuous if \(f^{-1}(B)\) is \((i,j)\)-b-open in \(X\) for each \(\sigma_i\)-open set \(B\) of \(Y\).

3 \quad \text{Almost } (i,j)\text{-b-Continuous Functions}

Definition 3.1 A function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is called an almost \((i,j)\)-b-continuous at a point \(x \in X\) if for each \(\sigma_i\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists an \((i,j)\)-b-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subset iInt(jCl(V))\). If \(f\) is almost \((i,j)\)-b-continuous at every point \(x\) of \(X\), then it is called almost \((i,j)\)-b-continuous.

It is obvious from the definition that \((i,j)\)-b-continuity implies almost \((i,j)\)-b-continuity. However, the converse is not true in general as it is shown in the following example.

Example 3.2 Let \(X = \{a, b, c, d\}, \tau_1 = \{\emptyset, \{a\}, X\}, \tau_2 = \{\emptyset, \{b, c\}, X\}, \sigma_1 = \{\emptyset, \{a\}, X\}\) and \(\sigma_2 = \{\emptyset, \{a, b\}, X\}\). Then the function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) defined by \(f(a) = d, f(b) = b, f(c) = c\) and \(f(d) = a\) is almost \((i,j)\)-b-continuous but not \((i,j)\)-b-continuous.

Theorem 3.3 For a function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\), the following statements are equivalent:

1. \(f\) is almost \((i,j)\)-b-continuous.

2. For each \(x \in X\) and each \((i,j)\)-regular open set \(V\) of \(Y\) containing \(f(x)\), there exists an \((i,j)\)-b-open \(U\) in \(X\) containing \(x\) such that \(f(U) \subset V\).

3. For each \(x \in X\) and each \((i,j)\)-\(\delta\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists an \((i,j)\)-b-open \(U\) in \(X\) containing \(x\) such that \(f(U) \subset V\).

Proof: (1) \(\Rightarrow\) (2): Let \(x \in X\) and let \(V\) be any \((i,j)\)-regular open subset of \(Y\) containing \(f(x)\). By (1), there exists an \((i,j)\)-b-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subset iInt(jCl(V))\). Since \(V\) is \((i,j)\)-regular open, \(iInt(jCl(V)) = V\). Therefore, \(f(U) \subset V\).
(2) $\Rightarrow$ (3): Let $x \in X$ and let $V$ be any $(i,j)$-$\delta$-open set of $Y$ containing $f(x)$. Then for each $f(x) \in V$, there exists a $\sigma_i$-open set $G$ containing $f(x)$ such that $G \subset i\text{Int}(j\text{Cl}(G)) \subset V$. Since $i\text{Int}(j\text{Cl}(G))$ is $(i,j)$-regular open set of $Y$ containing $f(x)$. By (2), there exists an $(i,j)$-$b$-open set $U$ in $X$ containing $x$ such that $f(U) \subset i\text{Int}(j\text{Cl}(G)) \subset V$.

(3) $\Rightarrow$ (1): Let $x \in X$ and let $V$ be any $\sigma_i$-open set of $Y$ containing $f(x)$. Then $i\text{Int}(j\text{Cl}(V))$ is $(i,j)$-$\delta$-open set of $Y$ containing $f(x)$. By (3), there exists an $(i,j)$-$b$-open set $U$ in $X$ containing $x$ such that $f(U) \subset i\text{Int}(j\text{Cl}(V))$.

Therefore, $f$ is almost $(i,j)$-$b$-continuous.

**Theorem 3.4** For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

1. $f$ is almost $(i,j)$-$b$-continuous.
2. $f^{-1}(i\text{Int}(j\text{Cl}(V)))$ is $(i,j)$-$b$-open set in $X$ for each $\sigma_i$-open set $V$ in $Y$.
3. $f^{-1}(i\text{Cl}(j\text{Int}(F)))$ is $(i,j)$-$b$-closed set in $X$ for each $\sigma_i$-closed set $F$ in $Y$.
4. $f^{-1}(F)$ is $(i,j)$-$b$-closed set in $X$ for each $(i,j)$-regular closed set $F$ of $Y$.
5. $f^{-1}(V)$ is $(i,j)$-$b$-open set in $X$ for each $(i,j)$-regular open set $V$ of $Y$.

**Proof:**

(1) $\Rightarrow$ (2): Let $V$ be any $\sigma_i$-open set in $Y$. We have to show that $f^{-1}(i\text{Int}(j\text{Cl}(V)))$ is $(i,j)$-$b$-open set in $X$. Let $x \in f^{-1}(i\text{Int}(j\text{Cl}(V)))$. Then $f(x) \in i\text{Int}(j\text{Cl}(V))$ and $i\text{Int}(j\text{Cl}(V))$ is an $(i,j)$-regular open set in $Y$. Since $f$ is almost $(i,j)$-$b$-continuous, by Theorem 3.3, there exists an $(i,j)$-$b$-open set $U$ of $X$ containing $x$ such that $f(U) \subset i\text{Int}(j\text{Cl}(V))$. Which implies that $x \in U \subset f^{-1}(i\text{Int}(j\text{Cl}(V)))$. Therefore, $f^{-1}(i\text{Int}(j\text{Cl}(V)))$ is an $(i,j)$-$b$-open set in $X$.

(2) $\Rightarrow$ (3): Let $F$ be any $\sigma_i$-closed set of $Y$. Then $Y \setminus F$ is a $\sigma_i$-open set of $Y$. By (2), $f^{-1}(i\text{Int}(j\text{Cl}(Y \setminus F)))$ is an $(i,j)$-$b$-open set in $X$ and $f^{-1}(i\text{Int}(j\text{Cl}(Y \setminus F))) = X \setminus f^{-1}(i\text{Cl}(j\text{Int}(F)))$ is an $(i,j)$-$b$-closed set in $X$.

(3) $\Rightarrow$ (4): Let $F$ be any $(i,j)$-regular closed set of $Y$. Then $F$ is a $\sigma_i$-closed set of $Y$. By (3), $f^{-1}(i\text{Cl}(j\text{Int}(F)))$ is an $(i,j)$-$b$-closed set in $X$. Since $F$ is $(i,j)$-regular closed, $f^{-1}(i\text{Cl}(j\text{Int}(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is an $(i,j)$-$b$-closed set in $X$.

(4) $\Rightarrow$ (5): Let $V$ be any $(i,j)$-regular open set of $Y$. Then $Y \setminus V$ is an $(i,j)$-regular closed set of $Y$ and by (4), we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $(i,j)$-$b$-closed set in $X$ and hence $f^{-1}(V)$ is $(i,j)$-$b$-open in $X$.

(5) $\Rightarrow$ (1): Let $x \in X$ and let $V$ be any $(i,j)$-regular open set of $Y$ containing...
\( f(x) \). Then \( x \in f^{-1}(V) \). By (5), we have \( f^{-1}(V) \) is \((i,j)\)-b-open set in \( X \). Therefore, we obtain \( f(f^{-1}(V)) \subset V \). Hence by Theorem 3.3, \( f \) is almost \((i,j)\)-b-continuous.

**Theorem 3.5** For a function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \), the following statements are equivalent:

1. \( f \) is almost \((i,j)\)-b-continuous.
2. \((i,j)\)-\(bCl(f^{-1}(V)) \subset f^{-1}(iCl(V)) \) for each \((j,i)\)-semi-preopen set \( V \) of \( Y \).
3. \( f^{-1}(iInt(F)) \subset (i,j)\)-\(bInt(f^{-1}(F)) \) for each \((j,i)\)-semi-preclosed set \( F \) of \( Y \).
4. \( f^{-1}(iInt(F)) \subset (i,j)\)-\(bInt(f^{-1}(F)) \) for each \((j,i)\)-semiclosed set \( F \) of \( Y \).
5. \((i,j)\)-\(bCl(f^{-1}(V)) \subset f^{-1}(iCl(V)) \) for each \((j,i)\)-semiopen set \( V \) of \( Y \).

Proof: (1) \(\Rightarrow\) (2): Let \( V \) be any \((j,i)\)-semi-preopen set of \( Y \). Since \( iCl(V) \) is an \((i,j)\)-regular closed set in \( Y \) and \( f \) is almost \((i,j)\)-b-continuous, by Theorem 3.4, \( f^{-1}(V) \) is \((i,j)\)-b-closed set in \( X \).

Therefore, \((i,j)\)-\(bCl(f^{-1}(V)) \subset f^{-1}(iCl(V)) \). (2) \(\Rightarrow\) (3) and (3) \(\Rightarrow\) (4) are clear.

(4) \(\Rightarrow\) (5): Let \( V \) be any \((j,i)\)-semiopen set of \( Y \). Then \( Y \setminus V \) is \((j,i)\)-semiclosed set and by (4), we have \( f^{-1}(iInt(Y \setminus V)) \subset (i,j)\)-\(bInt(f^{-1}(Y \setminus V)) \subset X \setminus (i,j)\)-\(bCl(f^{-1}(V)) \). Therefore, \((i,j)\)-\(bCl(f^{-1}(V)) \subset f^{-1}(iCl(V)) \). (5) \(\Rightarrow\) (1): Let \( F \) be any \((j,i)\)-regular closed set of \( Y \). Then \( F \) is an \((j,i)\)-semiopen set of \( Y \). By (5), we have \((i,j)\)-\(bCl(f^{-1}(F)) \subset f^{-1}(jCl(F)) = f^{-1}(F) \). This shows that \( f^{-1}(F) \) is \((i,j)\)-b-closed set in \( X \). Therefore, by Theorem 3.4, \( f \) is almost \((i,j)\)-b-continuous.

**Theorem 3.6** A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is almost \((i,j)\)-b-continuous if and only if \( f^{-1}(V) \subset (i,j)\)-\(bInt(f^{-1}(iInt(jCl(V)))) \) for each \( \sigma_1\)-open set \( V \) of \( Y \).

Proof: Let \( V \) be any \( \sigma_1\)-open set of \( Y \). Then \( V \subset iInt(jCl(V)) \) and \( iInt(jCl(V)) \) is \((i,j)\)-regular open set in \( Y \). Since \( f \) is almost \((i,j)\)-b-continuous, by Theorem 3.4, \( f^{-1}(iInt(jCl(V))) \) is \((i,j)\)-b-open set in \( X \) and hence we obtain that \( f^{-1}(V) \subset f^{-1}(iInt(jCl(V))) = (i,j)\)-\(bInt(f^{-1}(iInt(jCl(V)))) \). Conversely, let \( V \) be any \((i,j)\)-regular open set of \( Y \). Then \( V \) is \( \sigma_1\)-open set of \( Y \). By hypothesis, we have \( f^{-1}(V) \subset (i,j)\)-\(bInt(f^{-1}(iInt(jCl(V)))) \). Therefore, \( f^{-1}(V) \subset (i,j)\)-b-open set in \( X \) and hence by Theorem 3.4, \( f \) is almost \((i,j)\)-b-continuous.
Corollary 3.7 A function \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is almost \((i, j)\)-b-continuous if and only if \((i, j)\)-b\-Cl\((f^{-1}(j\text{Cl}(i\text{Int}(F)))) \subset f^{-1}(F)\) for each \(\sigma_i\)-closed set \(F\) of \(Y\).

Theorem 3.8 Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be an almost \((i, j)\)-b-continuous function and \(V \in \sigma_i \cap \sigma_j\). If \(x \in (i, j)\)-b\-Cl\((f^{-1}(V)) \setminus f^{-1}(V)\), then \(f(x) \in (i, j)\)-b\-Cl\((V)\).

Proof: Let \(x \in X\) be such that \(x \in (i, j)\)-b\-Cl\((f^{-1}(V)) \setminus f^{-1}(V)\) and suppose \(f(x) \notin (i, j)\)-b\-Cl\((V)\). Then there exists an \((i, j)\)-b-open set \(H\) containing \(f(x)\) such that \(H \cap V = \emptyset\). Then \(i\text{Int}(j\text{Cl}(H)) \cap V = \emptyset\) and \(i\text{Int}(j\text{Cl}(H))\) is an \((i, j)\)-regular open set. Since \(f\) is almost \((i, j)\)-b-continuous, by Theorem 3.4, there exists an \((i, j)\)-b-open set \(U\) in \(X\) containing \(x\) such that \(f(U) \subset i\text{Int}(j\text{Cl}(H))\). Therefore, \(f(U) \cap V = \emptyset\). However, since \(x \in (i, j)\)-b\-Cl\((f^{-1}(V))\), \(U \cap f^{-1}(V) \neq \emptyset\) for every \((i, j)\)-b-open set \(U\) in \(X\) containing \(x\), so that \(f(U) \cap V \neq \emptyset\). We have a contradiction. It follows that \(f(x) \in (i, j)\)-b\-Cl\((V)\).

Definition 3.9 Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A\) a subset of \(X\). The \((i, j)\)-b-frontier of \(A\), \((i, j)\)-b\-Fr\((A)\), is defined by \((i, j)\)-b\-Cl\((A) \cap (i, j)\)-b\-Fr\((A) = (i, j)\)-b\-Cl\((A) \setminus (i, j)\)-b\-Int\((A)\). Theorem 3.10 The set of all points \(x\) of \(X\) at which \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is not almost \((i, j)\)-b-continuous is identical with the union of the \((i, j)\)-b-frontiers of the inverse images of \((i, j)\)-regular open subsets of \(Y\) containing \(f(x)\).

Proof: If \(f\) is not almost \((i, j)\)-b-continuous at \(x \in X\), then there exists an \((i, j)\)-regular open set \(V\) containing \(f(x)\) such that for every \((i, j)\)-b-open set \(U\) of \(X\) containing \(x\), \(f(U) \cap (Y\setminus V) \neq \emptyset\). This means that for every \((i, j)\)-b-open set \(U\) of \(X\) containing \(x\), we must have \(U \cap (X\setminus f^{-1}(V)) \neq \emptyset\). Hence it follows that \(x \in (i, j)\)-b\-Cl\((X\setminus f^{-1}(V))\). But \(x \in f^{-1}(V)\) and hence \(x \in (i, j)\)-b\-Cl\((f^{-1}(V))\). This means that \(x\) belongs to the \((i, j)\)-b-frontier of \(f^{-1}(V)\). Conversely, suppose that \(x\) belongs to the \((i, j)\)-b-frontier of \(f^{-1}(V_i)\) for some \((i, j)\)-regular open subset \(V_i\) of \(Y\) such that \(f(x) \in V_i\). Suppose that \(f\) is almost \((i, j)\)-b-continuous at \(x\). Then by Theorem 3.3, there exists an \((i, j)\)-b-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subset V_i\). Then we have \(U \subset f^{-1}(V_i)\). This shows that \(x \in (i, j)\)-b\-Int\((f^{-1}(V_i))\). Therefore, we have \(x \notin (i, j)\)-b\-Cl\((X\setminus f^{-1}(V_i))\) and \(x \notin (i, j)\)-b\-Fr\((f^{-1}(V_i))\), which is a contradiction. This means that \(f\) is not almost \((i, j)\)-b-continuous.

References


