The Relationship between M-Weakly Compact Operator and Order Weakly Compact Operator

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(Received: 12-6-13 / Accepted: 21-7-13)

Abstract

In this note, we will show that the class of order weakly compact operators bigger than the class of M-weakly compact operators. Under a new condition, we will show that each M-weakly compact operator is an order weakly compact operator. We will show that, if Banach lattice \( E \) be an AM-space with unit and has the property (b), then the class of M-weakly compact operators from \( E \) into Banach space \( Y \) coincides with that of order weakly compact operators from \( E \) into \( Y \). Also we establish some relationship between M-weakly compact operators and weakly compact operators and \( b \)-weakly compact operators and order weakly compact operators.

Keywords: Banach lattice, order weakly compact operator, M-weakly compact operator, \( b \)-weakly compact operator, AM-space.

1 Introduction

The class of order weakly compact operators bigger than the class of M-weakly compact operators. In this note by combining Theorems 3.1 and 3.2, we will show that, if Banach lattice \( E \) is an AM-space with unit and has the property (b), then the class of M-weakly compact operators on \( E \) coincides with that of order weakly compact operators on \( E \).

A vector lattice \( E \) is an ordered vector space in which \( \text{sup}(x, y) \) exists for every \( x, y \in E \). A sequence \( \{x_n\} \) in a vector lattice \( E \) is said to be disjoint whenever
\( n \neq m \) implies \( |x_n| \land |x_m| = 0 \). A vector lattice \( E \) is called \( \sigma \)-Dedekind complete whenever every countable subset that is bounded from above has a supremum. A subset \( B \) of a vector lattice \( E \) is said to be solid if it follows from \( |y| \leq |x| \) with \( x \in B \) and \( y \in E \) that \( y \in B \). A solid vector subspace of a vector lattice \( E \) is referred to as an ideal. Let \( E \) be a vector lattice, for each \( x, y \in E \) with \( x \leq y \), the set \([x, y] = \{ z \in E : x \leq z \leq y \}\) is called an order interval. A subset of \( E \) is said to be order bounded if it is included in some order interval. If \( E \) is a vector lattice, we denote by \( E^\sim \) its order dual. Recall from \([2]\) that a subset \( A \) of a vector lattice \( E \) is called b-order bounded in \( E \) if it is order bounded in the order bidual \((E^\sim)^\sim\). A vector lattice \( E \) is said to have property (b) if \( A \subset E \) is order bounded whenever \( A \) is b-order bounded in \( E \). A Banach lattice is a Banach space \((E, \| . \|)\) such that \( E \) is a vector lattice and its norm satisfies the following property: for each \( x, y \in E \) such that \( |x| \leq |y| \), we have \( \|x\| \leq \|y\| \). If \( E \) is a Banach lattice, its topological dual \( E' \), endowed with the dual norm, is also a Banach lattice. A norm \( \| . \| \) of a Banach lattice \( E \) is order continuous if for each net \((x_\alpha)\) such that \( x_\alpha \downarrow 0 \) in \( E \), the net \((x_\alpha)\) converges to 0 for the norm \( \| . \| \). A Banach lattice \( E \) is said to be an AM-space if for each \( x, y \in E \) such that \( \inf(x, y) = 0 \) we have \( \|x + y\| = \max\{\|x\|,\|y\|\} \). The Banach lattice \( E \) is an AL-space if its topological dual \( E' \) is an AM-space. A Banach lattice \( E \) is said to be a KB-space whenever every increasing norm bounded sequence of \( E^+ \) is norm convergent.

We will use the term operator \( T : E \to F \) between two Banach lattices to mean a linear mapping.

## 2 Main Result of Relationship

**Definition 2.1** Let \( T : X \to Y \) be an operator between two Banach spaces. Then, \( T \) is said to be weakly compact whenever \( T \) carries the closed unit ball of \( X \) onto a relatively weakly compact subset of \( Y \), the collection of weakly compact operators will be denoted by \( W(X,Y) \).

**Definition 2.2** A continuous operator \( T : E \to Y \) from a Banach lattice into a Banach space is said to be \( M \)-weakly compact whenever \( \lim_n \|Tx_n\| = 0 \) holds for every norm bounded disjoint sequence \( \{x_n\} \) of \( E \), denoted by \( W_M(E,Y) \).

**Definition 2.3** A continuous operator \( T : E \to Y \) from a Banach lattice into a Banach space is said to be \( b \)-weakly compact whenever \( T \) carries each \( b \)-order bounded subset of \( E \) into relatively weakly compact subset of \( Y \), denoted by \( W_b(E,Y) \).
\begin{definition}
Finally, a continuous operator \(T : E \to Y\) from a Banach lattice into a Banach space is order weakly compact whenever \(T[0,x]\) is a relatively weakly compact subset of \(Y\) for each \(x \in E^+\), denoted by \(W_o(E,Y)\).
\end{definition}

\begin{theorem}
For a Banach lattice \(E\), the following statements are equivalent:
\begin{enumerate}
\item \(E\) has order continuous norm.
\item If \(0 \leq x_n \leq x\) holds in \(E\), then \(\\{x_n\}\) is norm couchy sequence.
\item \(E\) is \(\sigma\)-Dedekind complete, and \(x_n \downarrow 0\) in \(E\) implies \(\|x_n\| \downarrow 0\).
\item \(E\) is an ideal of \(E''\).
\item Each order interval of \(E\) is weakly compact.
\end{enumerate}
\end{theorem}

\begin{proof}
(1) \(\Rightarrow\) (2) Let \(0 \leq x_\alpha \leq x\) hold in \(E\), and let \(\varepsilon > 0\). By Lemma 12.8 of [1] there exists a net \((y_\lambda) \subseteq E\) with \(y_\lambda - x_\alpha \downarrow 0\). Thus, there exists \(\lambda_0\) and \(\alpha_0\) such that \(\|y_\lambda - x_\alpha\| < \varepsilon\) holds for all \(\lambda \geq \lambda_0\) and \(\alpha \geq \alpha_0\). From the inequality
\[
\|x_\alpha - x_\beta\| \leq \|x_\alpha - y_\lambda\| + \|x_\beta - y_\lambda\|,
\]
we see that \(\|x_\alpha - x_\beta\| < 2\varepsilon\) holds for all \(\alpha, \beta \geq \alpha_0\). Hence, \((x_\alpha)\) is a norm couchy net.

(2) \(\Rightarrow\) (3) It follows immediately from Theorem 11.2(2) of [1].

(3) \(\Rightarrow\) (1) Let \(x_\alpha \downarrow 0\). If \((x_\alpha)\) is not a norm Cauchy net, then there exist some \(\varepsilon > 0\) and a sequence \(\{\alpha_n\}\) of indices with \(\alpha_n \uparrow\), and \(\|x_{\alpha_n} - x_{\alpha_{n+1}}\| \geq \varepsilon\) for all \(n\). Since \(E\) is \(\sigma\)-Dedekind complete, there exists some \(x \in E\) with \(x_{\alpha_n} \downarrow x\).

Now from our hypothesis, we see that \((x_{\alpha_n})\) is a norm Cauchy sequence, which contradicts \(\|x_{\alpha_n} - x_{\alpha_{n+1}}\| \geq \varepsilon\). Thus, \((x_\alpha)\) is a norm Cauchy net, and so \((x_\alpha)\) is norm convergent to some \(y \in E\). By Theorem 11.2(2) of [1] we see that \(y = 0\), and so \(\|x_\alpha\| \downarrow 0\) holds.

The other equivalences follow easily from Theorems 11.13 and 11.10 of [1].
\end{proof}

\begin{theorem}
Let \(E\) be a Banach lattice. \(E\) is a KB-space if and only if \(I : E \to E\) is a b-weakly compact operator.
\end{theorem}

\begin{proof}
Let \(E\) be KB-space and \(A\) be an b-order bounded subset of \(E\). Since \(E\) by Proposition 2.1 of [2] has property (b), \(A\) is an order bounded subset of \(E\) and thus there exists some \(x \in E^+\) for which \(A \subseteq [-x,x]\). Then, by Theorem 2.5, \([-x,x]\) and hence \(A\) is a relatively weakly compact subset of \(E\).

Conversely, let \(I : E \to E\) be b-weakly compact and \(\{x_n\}\) be an increasing, norm bounded sequence in \(E^+\). We wish to show \(\{x_n\}\) is norm convergent. Let us define \(x'' : (E^+)'' \to \mathbb{R}\) by \(x''(f) = \lim_n f(x_n)\) for each \(f \in (E^+)''\). \(x''\) is additive on \((E^+)''\) and extends to an element of \((E^+)''\) which we shall also denote by \(x''\). We have \(0 \leq x_n \leq x''\) in \(E''\) for each \(n\). Therefore, \(\{x_n\}\) is an b-order bounded subset of \(E\). By b-weak compactness of \(I\), we obtain a subsequence \(\{x'_{n_k}\}\) of \(\{x_n\}\) such that \(x'_{n_k} \to x\) in \(\sigma(E,E')\) for some \(x \in E\).
Since \( \{x_n\} \) is increasing, \( x = \sup_k x_{n_k} \) and we have \( x = \sup_n x_n \). Thus \( x_n \to x \) in \( \sigma(E, E') \). \( x - x_n \downarrow 0, x - x_n \to 0 \) in \( \sigma(E, E') \) now yield \( x - x_n \to 0 \) in the norm topology.

**Theorem 2.7** M-weakly compact operators are weakly compact operators.

Proof. Assume first that \( T : E \to Y \) is an M-weakly compact operator. Denote by \( U \) and \( V \) the Closed unit balls of \( E \) and \( Y \), respectively, and let \( \varepsilon > 0 \). By Theorem 18.9(1) of [1], there exists some \( u \in E^+ \) such that \( \| T(|x| - u)^+ \| < \varepsilon \) holds for all \( x \in U \), and consequently from the identity \( |x| = |x| \wedge u + (|x| - u)^+ \) we see that

\[
T(U^+) \subseteq T[0, u] + \varepsilon V. \quad (*)
\]

On the other hand, if \( \{u_n\} \) is disjoint sequence of \([0, u]\), then it follows from our hypothesis that \( \lim \| Tu_n \| = 0 \), and thus by Theorem 18.1 of [1] the set \( T[0, u] \) is relatively weakly compact. Now (*) combined with Theorem 10.17 of [1] shows that \( T(U^+) \) (and hence \( T(U) \)) is relatively weakly compact, and so \( T \) is a weakly compact operator.

### 3 Main Result of Equality

Recall from [1] that Banach space \( X \) has the Dunford-pettis property whenever \( x_n \to 0 \) in \( \sigma(X, X') \) and \( x'_n \to 0 \) in \( \sigma(X', X'') \) imply \( \lim x'_n(x_n) = 0 \), and we say that an operator \( T : X \to Y \) between two Banach spaces is a Dunford-pettis operator whenever \( x_n \to 0 \) in \( \sigma(X, X') \) implies \( \lim \| Tx_n \| = 0 \).

**Theorem 3.1** Let \( T \) is an operator from AM-space with unit \( E \) into Banach space \( Y \). Then the following assertion are equivalent:

1. \( T \) is M-weakly compact.
2. \( T \) is weakly compact.
3. \( T \) is Dunford-pettis.
4. \( T \) is b-weakly compact.

Proof. (1) \( \Rightarrow \) (2) Follows from Theorem 2.6.

(2) \( \Rightarrow \) (3) From Theorem 19.6 of [1] \( E \) has the Duonford-pettis property. Then from Theorem 19.4 of [1] it follows that every weakly compact operators from \( E \) which has the Duonford-pettis property into an arbitrary Banach space \( Y \) is a Duonford-pettis operator.

(3) \( \Rightarrow \) (1) \( E' \) is an AL-space so it will be KB-space and then \( E' \) has the order continuous norm. Then from Theorem 3.7.10 of [5] every Duonford-pettis operator from \( E \) into \( Y \) is a M-weakly compact operator.

(2) \( \Rightarrow \) (4) Obvious.

(4) \( \Rightarrow \) (2) Since \( E \) is AM-space with unit so from Theorem 12.20 of [1] its closed unit ball is like an order interval. So we have the result.
Theorem 3.2 Let $E$ is a Banach lattice with property (b). Then every order weakly compact operator from $E$ into Banach space $Y$ is a $b$-weakly compact operator.

Proof. Let $E$ has the property (b) and $T$ from $E$ into Banach space $Y$ is order weakly compact operator and $A$ is a $b$-order bounded subset of $E$. Since $E$ has the property (b) we can choose $x \in E^+$ with $A \subseteq [-x,x]$. Therefore

$$
T(A)^w \subseteq T([-x,x])^w.
$$

Therefor by hypothesis, we will result.

4 Conclusion

In the following, we establish some relationships between some class of operators.

i) Each weakly compact operator from Banach lattice $E$ into Banach space $Y$ is $b$-weakly compact operator.

ii) Each $b$-weakly compact operator from Banach lattice $E$ into Banach space $Y$ is order weakly compact.

iii) Now by Theorem 2.7, i , ii, we will have

$$
W_M(E,Y) \subset W(E,Y) \subset W_b(E,Y) \subset W_o(E,Y) \quad (**)
$$

iv) Since the norm of $c_0$ is order continuous, by Theorem 2.5, $[0,x]$ is weakly compact in $c_0$, then $I : c_0 \to c_0$ is order weakly compact. But $c_0$ is not KB-space, then by Theorem 2.6, $I : c_0 \to c_0$ is not b-weakly compact operator. Therefore, by (**) every order weakly compact operator is not M-weakly compact and weakly compact operator.

v) Since $L_1([0,1])$ is a KB-space therefor $I : L_1([0,1]) \to L_1([0,1])$ is b-weakly compact operator. But its not weakly compact operator. By (**) every $b$-weakly compact operator is not M-weakly compact operator.

vi) By theorems 19.6 and 17.5 of [1], operator $T : l^1 \to l^\infty$ defined by

$$
T(\alpha_1, \alpha_2, ...) = \left( \sum_{n=1}^\infty \alpha_n, \sum_{n=1}^\infty \alpha_n, ... \right) = \left[ \sum_{n=1}^\infty \alpha_n \right] (1,1,1, ...)
$$

is weakly compact. The sequence $\{e_n\}$ of the standard unit vectors is a norm bounded disjoint sequence of $l^1$ satisfying $Te_n = (1,1,1, ...)$ for each $n$. This
follow that $T$ is not M-weakly compact. Then every weakly compact is not M-weakly compact.

If $E$ is an AM-space with unit and has the property (b), by Theorems 3.1 and 3.2 we will have

$$W_o(E, Y) = W_b(E, y) = W_M(E, Y) = W(E, Y).$$

References


