On Totally sg-Continuity, Strongly sg-Continuity and Contra sg-Continuity

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Abstract

In this paper, sg-closed sets and sg-open sets are used to define and investigate a new class of functions. Relationships between this new class and other classes of functions are established.

Keywords: Topological spaces, sg-closed set, sg-open set, totally sg-continuity, strongly sg-continuity, contra sg-continuity.

1 Introduction

Jain [9], Levine [12] and Dontchev [5] introduced totally continuous functions, strongly continuous functions and contra continuous functions, respectively.
Levine [10] also introduced and studied the concepts of generalized closed sets. The notion has been studied extensively in recent years by many topologists. As generalization of closed sets, sg-closed sets were introduced and studied by Bhattacharya and Lahiri [2]. This notion was further studied by Navalagi [14, 15]. In this paper, we will continue the study of some related functions by using sg-open sets and sg-closed sets. We introduce and characterize the concepts of totally sg-continuous, strongly sg-continuous and contra sg-continuous functions.

2 Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) (or X, Y and Z) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. We set C(X, x) = {V \in C(X) | x \in V} for x \in X, where C(X) denotes the collection of all closed subsets of (X, τ). The set of all clopen subsets of (X, τ) is denoted by CO(X, τ).

We recall the following definitions, which are useful in the sequel.

**Definition 2.1** A subset A of a space (X, τ) is called:

(i) semi-open [11] if A \subseteq cl(int(A)).
(ii) α-open [16] if A \subseteq int(cl(int(A))).

The complements of the above mentioned open sets are called their respective closed sets.
The intersection of all semi-closed sets of X containing a subset A is called the semi-closure of A and is denoted by scl(A).

**Definition 2.2** A subset A of a space (X, τ) is called:

(i) a ĝ-closed set [23] if cl(A) \subseteq U whenever A \subseteq U and U is semi-open in (X, τ). The complement of ĝ-closed set is called ĝ-open.
(ii) a *g-closed set [22] if cl(A) \subseteq U whenever A \subseteq U and U is ĝ-open in (X, τ). The complement of *g-closed set is called *g-open.
(iii) a #g-semi-closed (briefly #gs-closed) set [24] if scl(A) \subseteq U whenever A \subseteq U and U is *g-open in (X, τ). The complement of #gs-closed set is called #gs-open.
(iv) a #g-semi-closed (briefly #gs-closed) set [20] if scl(A) \subseteq U whenever A \subseteq U and U is #gs-open in (X, τ). The complement of #gs-closed set is called #gs-open.
(v) a generalized semi-closed (briefly gs-closed) set [1] if \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\). The complement of gs-closed set is called gs-open.

(vi) a semi-generalized closed (briefly sg-closed) set [2] if \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\). The complement of sg-closed set is called sg-open. The class of all sg-open sets of \((X, \tau)\) is denoted by \( \text{SG}(X, \tau) \).

(vii) a sg-clopen if it is both sg-open and sg-closed.

We set \( \text{SG}(X, x) = \{V \in \text{SG}(X, \tau) \mid x \in V\} \) for \( x \in X \).

**Remark 2.1**

From the Definitions 2.1 and 2.2, we have the following implications.

\[
\begin{array}{ccc}
\text{closed} & \xrightarrow{\alpha}\text{-closed} & \xrightarrow{\text{semi-closed}} \\
\text{# gs-closed} & \xleftarrow{\#} \text{g ~s-closed} & \xleftarrow{\#} \text{sg-closed} & \xrightarrow{\#} \text{gs-closed}
\end{array}
\]

None of the above implications is reversible as the following example shows.

**Example 2.1**

(i) Let \( X = \{a, b, c\}, \tau = \{\phi, \{a\}, X\} \). The set \( \{b\} \) is \( \alpha \)-closed, \#gs- closed and \( \# \)g s-closed but not closed.

(ii) Let \( X = \{a, b, c\}, \tau = \{\phi, \{a, b\}, X\} \). The set \( \{a, c\} \) is \( \# \)g s-closed but not \( \alpha \)-closed.

(iii) Let \( X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, c\}, X\} \). The set \( \{a, b\} \) is sg-closed, \#gs-closed but not \( \# \)g s-closed.

(iv) Let \( X = \{a, b, c\}, \tau = \{\phi, \{a, b\}, X\} \). The set \( \{b, c\} \) is sg-closed but not \( \alpha \)-closed.

(v) Let \( X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b, a\}, \{a, b\}, X\} \). The set \( \{a\} \) is semi-closed but not \( \alpha \)-closed.

(vi) Let \( X = \{a, b, c\}, \tau = \{\phi, \{a, b\}, X\} \). The set \( \{b, c\} \) is sg-closed, gs-closed but not semi-closed.

(vii) Let \( X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{a, c\}, X\} \). The set \( \{a, b\} \) is gs-closed but not sg-closed.

**Definition 2.3** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called:

(i) totally continuous [9] if the inverse image of every open subset of \((Y, \sigma)\) is a clopen subset of \((X, \tau)\).
(ii) strongly continuous [12] if the inverse image of every subset of \((Y, \sigma)\) is a clopen subset of \((X, \tau)\).

(iii) contra-continuous [5] (resp. contra-semi-continuous [6], contra-\(\alpha\)-continuous [7]) if the inverse image of every open subset of \((Y, \sigma)\) is a closed (resp. semi-closed, \(\alpha\)-closed) subset of \((X, \tau)\).

(iv) sg-continuous [21] if the inverse image of every open subset of \((Y, \sigma)\) is a sg-open subset of \((X, \tau)\).

**Definition 2.4** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is called:

(i) sg-irresolute [21] if the inverse image of every sg-closed set of \((Y, \sigma)\) is a sg-closed set of \((X, \tau)\).

(ii) sg-open [4] if for each open set \(U\) of \((X, \tau)\), \(f(U)\) is sg-open set of \((Y, \sigma)\).

**Definition 2.5** [14] Let \((X, \tau)\) be a topological space and \(A \subseteq X\). We define the sg-closure of \(A\) (briefly sg-cl\((A)\)) to be the intersection of all sg-closed sets containing \(A\).

## 3 Two Classes of Functions via sg-Clopen Sets

We introduce the following definitions:

**Definition 3.1** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be totally semi-generalized-continuous (briefly totally sg-continuous) if the inverse image of every open subset of \((Y, \sigma)\) is a sg-clopen (i.e. sg-open and sg-closed) subset of \((X, \tau)\).

It is evident that every totally continuous function is totally sg-continuous. But the converse need not be true as shown in the following example.

**Example 3.1** Let \(X = \{a, b, c\}\), \(Y = \{p, q\}\), \(\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}\) and \(\sigma = \{\phi, \{p\}, Y\}\). Define a function \(f : (X, \tau) \rightarrow (Y, \sigma)\) such that \(f(a) = p\), \(f(b) = f(c) = q\). Then clearly \(f\) is totally sg-continuous, but not totally continuous.

**Definition 3.2** A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be strongly semi-generalized-continuous (briefly strongly sg-continuous) if the inverse image of every subset of \((Y, \sigma)\) is a sg-clopen subset of \((X, \tau)\).

It is clear that strongly sg-continuous function is totally sg-continuous. But the reverse implication is not always true as shown in the following example.

**Example 3.2** Let \(X = \{a, b, c\} = Y\), \(\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}\) and \(\sigma = \{\phi, \{a\}, Y\}\). Then the identity function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is totally sg-continuous, but not strongly sg-continuous.

**Theorem 3.1** Every totally sg-continuous function into \(T_1\)-space is strongly sg-continuous.
Proof. In a $T_1$-space, singletons are closed. Hence $f^{-1}(A)$ is sg-clopen in $(X, \tau)$ for every subset $A$ of $Y$.

Remark 3.1 It is clear from the Theorem 3.1 that the classes of strongly sg-continuous functions and totally sg-continuous functions coincide when the range is a $T_1$-space.
Recall that a space $(X, \tau)$ is said to be sg-connected [3] if $X$ cannot be expressed as the union of two non-empty disjoint sg-open sets.

Theorem 3.2 If $f$ is a totally sg-continuous function from a sg-connected space $X$ onto any space $Y$, then $Y$ is an indiscrete space.

Proof. Suppose that $Y$ is not indiscrete. Let $A$ be a proper non-empty open subset of $Y$. Then $f^{-1}(A)$ is a proper non-empty sg-clopen subset of $(X, \tau)$, which is a contradiction to the fact that $X$ is sg-connected.

Definition 3.3 A space $X$ is said to be sg-$T_2$ [21] if for any pair of distinct points $x, y$ of $X$, there exist disjoint sg-open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

Lemma 3.1 The sg-closure of every sg-open set is sg-open.

Proof. Every regular open set is open and every open set is sg-open. Thus, every regular closed set is sg-closed. Now let $A$ be any sg-open set. There exists an open set $U$ such that $U \subset A \subset \text{cl}(U)$. Hence, we have $U \subset \text{sg-cl}(U) \subset \text{sg-cl}(A) \subset \text{sg-cl}(\text{cl}(U)) = \text{cl}(U)$ since $\text{cl}(U)$ is regular closed. Therefore, $\text{sg-cl}(A)$ is sg-open.

Theorem 3.3 A space $X$ is sg-$T_2$ if and only if for any pair of distinct points $x, y$ of $X$ there exist sg-open sets $U$ and $V$ such that $x \in U$, and $y \in V$ and $\text{sg-cl}(U) \cap \text{sg-cl}(V) = \phi$.

Proof. Necessity. Suppose that $X$ is sg-$T_2$. Let $x$ and $y$ be distinct points of $x$. There exist sg-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Hence $\text{sg-cl}(U) \cap \text{sg-cl}(V) = \phi$ and by Lemma 3.1, $\text{sg-cl}(U)$ is sg-open. Therefore, we obtain $\text{sg-cl}(U) \cap \text{sg-cl}(V) = \phi$.

Sufficiency. This is obvious.

Theorem 3.4 If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a totally sg-continuous injection and $Y$ is $T_0$ then $X$ is sg-$T_2$.

Proof. Let $x$ and $y$ be any pair of distinct points of $X$. Then $f(x) \neq f(y)$. Since $Y$ is $T_0$, there exists an open set $U$ containing $y$ but not $f(x)$. Then $x \not\in f^{-1}(U)$ and $y \not\in f^{-1}(U)$. Since $f$ is totally sg-continuous, $f^{-1}(U)$ is a sg-clopen subset of $X$. Also, $x \in f^{-1}(U)$ and $y \in X - f^{-1}(U)$. By Theorem 3.3, it follows that $X$ is sg-$T_2$. 
Theorem 3.5 A topological space \((X, \tau)\) is sg-connected if and only if every totally sg-continuous function from a space \((X, \tau)\) into any \(T_0\)-space \((Y, \sigma)\) is constant.

Proof. Suppose that \(X\) is not sg-connected and every totally sg-continuous function from \((X, \tau)\) to \((Y, \sigma)\) is constant. Since \((X, \tau)\) is not sg-connected, there exists a proper non-empty sg-clopen subset \(A\) of \(X\). Let \(Y = \{a, b\}\) and \(\sigma = \{\emptyset, \{a\}, \{b\}, Y\}\) be a topology for \(Y\). Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a function such that \(f(A) = \{a\}\) and \(f(Y - A) = \{b\}\). Then \(f\) is non-constant and totally sg-continuous such that \(Y\) is \(T_0\) which is a contradiction. Hence \(X\) must be sg-connected.

Converse is similar.

Theorem 3.6 Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a totally sg-continuous function and \(Y\) is a \(T_1\)-space. If \(A\) is a non-empty sg-connected subset of \(X\), then \(f(A)\) is a single point.

Definition 3.4 Let \((X, \tau)\) be a topological space. Then the set of all points \(y\) in \(X\) such that \(x\) and \(y\) cannot be separated by a sg-separation of \(X\) is said to be the quasi sg-component of \(X\).

Theorem 3.7 Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a totally sg-continuous function from a topological space \((X, \tau)\) into a \(T_1\)-space \(Y\). Then \(f\) is constant on each quasi sg-component of \(X\).

Proof. Let \(x\) and \(y\) be two points of \(X\) that lie in the same quasi-sg-component of \(X\). Assume that \(f(x) = \alpha \neq \beta = f(y)\). Since \(Y\) is \(T_1\), \(\{\alpha\}\) is closed in \(Y\) and so \(Y - \{\alpha\}\) is an open set. Since \(f\) is totally sg-continuous, therefore \(f^{-1}(\{\alpha\})\) and \(f^{-1}(Y - \{\alpha\})\) are disjoint sg-clopen subsets of \(X\). Further, \(x \in f^{-1}(\{\alpha\})\) and \(y \in f^{-1}(Y - \{\alpha\})\), which is a contradiction in view of the fact that \(y\) belongs to the quasi sg-component of \(x\) and hence \(y\) must belong to every sg-open set containing \(x\).

4 Contra-sg-Continuous Functions

Definition 4.1[17] A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is called contra-sg-continuous (briefly csg-continuous) if \(f^{-1}(V)\) is sg-open in \((X, \tau)\) for every closed set \(V\) in \((Y, \sigma)\).

It is clear that every strongly sg-continuous function is csg-continuous. But the reverse implication is not always true as shown in the following example.

Example 4.1 Let \(X = Y = \{a, b, c\}\), \(\tau = \{\emptyset, \{a, b\}, X\}\) and \(\sigma = \{\emptyset, \{b, c\}, Y\}\). Then the identity function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is csg-continuous but it is not strongly sg-continuous.
**Definition 4.2** Let A be a subset of a topological space \((X, \tau)\). The set \(\bigcap \{U \in \tau \mid A \subseteq U\}\) is called the Kernal of A [13] and is denoted by \(\ker(A)\).

**Lemma 4.1** [8] The following properties hold for subsets \(A, B\) of a space \(X\):

(i) \(x \in \ker(A)\) if and only if \(A \cap F \neq \emptyset\) for any \(F \in C(X, x)\);
(ii) \(A \subseteq \ker(A)\) and \(A = \ker(A)\) if \(A\) is open in \(X\);
(iii) If \(A \subseteq B\), then \(\ker(A) \subseteq \ker(B)\).

**Theorem 4.1** Assume that arbitrary union of sg-open sets is sg-open. The following are equivalent for a function \(f : (X, \tau) \to (Y, \sigma)\):

(i) \(f\) is csg-continuous;
(ii) for every closed subset \(F\) of \(Y\), \(f^{-1}(F) \in SG(X, \tau)\);
(iii) for each \(x \in X\) and each \(F \in C(Y, f(x))\), there exists \(U \in SG(X, \tau)\) such that \(f(U) \subseteq F\);
(iv) \(f(\text{sgcl}(A)) \subseteq \ker(f(A))\) for every subset \(A\) of \(X\);
(v) \(\text{sgcl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B))\) for every subset \(B\) of \(Y\).

**Proof.** The implications (i) \(\to\) (ii) and (ii) \(\to\) (iii) are obvious.

(iii) \(\to\) (ii). Let \(F\) be any closed set of \(Y\) and \(x \in f^{-1}(F)\). Then \(f(x) \in F\) and there exists \(U_x \in SG(X, x)\) such that \(f(U_x) \subseteq F\). Therefore, we obtain \(f^{-1}(F) = \bigcup \{U_x \mid x \in f^{-1}(F)\}\) \(\in SG(X, \tau)\).

(ii) \(\to\) (iv). Let \(A\) be any subset of \(X\). Suppose that \(y \notin \ker(f(A))\). Then by Lemma 4.1 there exists \(F \in C(X, y)\) such that \(f(A) \cap F = \emptyset\). Thus, we have \(A \cap f^{-1}(F) = \emptyset\) and \(\text{sgcl}(A) \cap f^{-1}(F) = \emptyset\). Therefore, we obtain \(f(\text{sgcl}(A)) \cap F = \emptyset\) and \(y \notin f(\text{sgcl}(A))\). This implies that \(f(\text{sgcl}(A)) \subseteq \ker(f(A))\).

(iv) \(\to\) (v). Let \(B\) be any subset of \(Y\). By (iv) and Lemma 4.1, we have \(f(\text{sgcl}(f^{-1}(B))) \subseteq \ker(f(f^{-1}(B))) \subseteq \ker(B)\) and \(\text{sgcl}(f^{-1}(B)) \subseteq f^{-1}(\ker(B))\).

(v) \(\to\) (i). Let \(V\) be any open set of \(Y\). Then by Lemma 4.1 we have \(\text{sgcl}(f^{-1}(V)) \subseteq f^{-1}(\ker(V)) = f^{-1}(V)\) and \(\text{sgcl}(f^{-1}(V)) = f^{-1}(V)\). This show that \(f^{-1}(V)\) is sg-closed in \((X, \tau)\).

**Theorem 4.2** Every contra semi-continuous function is csg-continuous.

**Proof.** The proof follows from the definitions.

**Remark 4.1** Contra sg-continuous need not be contra semi-continuous in general as shown in the following example.
Example 4.2 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{b, c\}, Y\}$. Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is csg-continuous. However, $f$ is not contra-semi-continuous, since for the closed set $F = \{a\}$, $f^{-1}(F)$ is sg-open but not semi-open in $(X, \tau)$.

Corollary 4.1 Every contra $\alpha$-continuous (resp. contra-continuous) function is csg-continuous.

Theorem 4.3 Assume that arbitrary union of sg-open sets is sg-open. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following are equivalent.

(i) $f : (X, \tau) \rightarrow (Y, \sigma)$ is sg-continuous.
(ii) For each $x$ in $X$ and each open set $V$ in $Y$ with $f(x) \in V$, there is a sg-open set $U$ in $X$ such that $x \in U$, $f(U) \subset V$.

Proof. (i) $\Rightarrow$ (ii). Let $f(x) \in V$. Since $f$ is sg-continuous we have $x \in f^{-1}(V) \in SG(X, \tau)$. Let $U = f^{-1}(V)$. Then $x \in V$ and $f(U) \subset V$.

(ii) $\Rightarrow$ (i). Let $V$ be an open set in $(Y, \sigma)$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there exists a sg-open set $U_x$ such that $x \in U_x$ and $f(U_x) \subset V$. Now $x \in U_x \subset f^{-1}(V)$ and $f^{-1}(V) = \bigcup x \in f^{-1}(V) U_x$. Therefore $f^{-1}(V)$ is sg-open in $(X, \tau)$ and consequently, $f$ is sg-continuous.

Theorem 4.4 Assume that arbitrary union of sg-open sets is sg-open. If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is csg-continuous and $Y$ is regular, then $f$ is sg-continuous.

Proof. Let $x$ be an arbitrary point of $X$ and $V$ an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $W$ in $Y$ containing $f(x)$ such that $\text{cl}(W) \subset V$. Since $f$ is csg-continuous, so by Theorem 4.1 there exists $U \in SG(X, x)$ such that $f(U) \subset \text{cl}(W)$. Then $f(U) \subset \text{cl}(W) \subset V$. Hence, by Theorem 4.3 $f$ is sg-continuous.

Theorem 4.5 Assume that arbitrary union of sg-open sets is sg-open. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $g : X \rightarrow X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then $f$ is csg-continuous if and only if $g$ is csg-continuous.

Proof. Let $x \in X$ and let $W$ be a closed subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is closed in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to $Y$. Hence $\{y \in Y | (x, y) \in W\} \subset Y$ is a closed subset of $Y$. Since $f$ is csg-continuous, $\bigcup \{f^{-1}(y) | (x, y) \in W\}$ is a sg-open subset of $X$. Further, $x \in \bigcup \{f^{-1}(y) | (x, y) \in W\} \subset g^{-1}(W)$. Hence $g^{-1}(W)$ is sg-open. Then $g$ is csg-continuous. Conversely, let $F$ be a closed subset of $Y$. Then $X \times F$ is a closed subset of $X \times Y$. Since $g$ is csg-continuous, $g^{-1}(X \times F)$ is a sg-open subset of $X$. Also, $g^{-1}(X \times F) = f^{-1}(F)$. Hence $f$ is csg-continuous.
Theorem 4.6 Assume that arbitrary union of sg-open sets is sg-open. If $X$ is a topological space and for each pair of distinct points $x_1$ and $x_2$ in $X$ there exists a map $f$ into a Urysohn topological space $Y$ such that $f(x_1) \neq f(x_2)$ and $f$ is csg-continuous at $x_1$ and $x_2$, then $X$ is sg-$T_2$.

Proof. Let $x_1$ and $x_2$ be any distinct points in $X$. Then by hypothesis there is a Urysohn space $Y$ and a function $f : (X, \tau) \to (Y, \sigma)$, which satisfies the conditions of the theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since $Y$ is Urysohn, there exist open neighbourhoods $U_{y_1}$ and $U_{y_2}$ of $y_1$ and $y_2$ respectively in $Y$ such that $\text{cl}(U_{y_1}) \cap \text{cl}(U_{y_2}) = \emptyset$. Since $f$ is csg-continuous at $x_i$, there exists a sg-open neighbourhood $W_{x_i}$ of $x_i$ in $X$ such that $f(W_{x_i}) \subset \text{cl}(U_{y_i})$ for $i = 1, 2$. Hence we get $W_{x_1} \cap W_{x_2} = \emptyset$ because $\text{cl}(U_{y_1}) \cap \text{cl}(U_{y_2}) = \emptyset$. Then $X$ is sg-$T_2$.

Corollary 4.2 Assume that arbitrary union of sg-open sets is sg-open. If $f$ is a csg-continuous injection of a topological space $X$ into a Urysohn space $Y$, then $X$ is sg-$T_2$.

Proof. For each pair of distinct points $x_1$ and $x_2$ in $X$, $f$ is csg-continuous function of $X$ into Urysohn space $Y$ such that $f(x_1) \neq f(x_2)$ because $f$ is injective. Hence by Theorem 4.6, $X$ is sg-$T_2$.

Corollary 4.3 If $f$ is a csg-continuous injection of a topological space $X$ into Ultra Hausdorff space $Y$, then $X$ is sg-$T_2$.

Proof. Let $x_1$ and $x_2$ be any distinct points in $X$. Then since $f$ is injective and $Y$ is Ultra Hausdorff $f(x_1) \neq f(x_2)$ and there exist $V_1, V_2 \in CO(Y, \sigma)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then $x_i \in f^{-1}(V) \in SG(X, \tau)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus, $X$ is sg-$T_2$.

Theorem 4.7 If $f : (X, \tau) \to (Y, \sigma)$ is a contra sg-continuous function and $g : (Y, \sigma) \to (Z, \eta)$ is a continuous function, then $(g \circ f) : (X, \tau) \to (Z, \eta)$ is csg-continuous.

Theorem 4.8 Let $f : (X, \tau) \to (Y, \sigma)$ be surjective sg-irresolute and sg-open and $g : (Y, \sigma) \to (Z, \eta)$ be any function. Then $(g \circ f) : (X, \tau) \to (Z, \eta)$ is csg-continuous if and only if $g$ is csg-continuous.

Proof. The “If” part is easy to prove. To prove the “only if” part, let $(g \circ f) : (X, \tau) \to (Z, \eta)$ be csg-continuous. Let $F$ be a closed subset of $Z$. Then $(g \circ f)^{-1}(F)$ is a sg-open subset of $X$. That is $f^{-1}(g^{-1}(F))$ is sg-open. Since $f$ is sg-open, $f(f^{-1}(g^{-1}(F)))$ is a sg-open subset of $Y$. So $g^{-1}(F)$ is sg-open in $Y$. Hence $g$ is csg-continuous.

Theorem 4.9 Let $\{X_i \mid i \in \Lambda\}$ be any family of topological spaces. If $f : X \to \Pi X_i$ is a csg-continuous function. Then $\pi_i \circ f : X \to X_i$ is csg-continuous for each $i \in \Lambda$, where $\pi_i$ is the projection of $\Pi X_i$ onto $X_i$. 
Definition 4.3 The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be csg-closed in $X \times Y$ if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in SG(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.2 The graph $f : (X, \tau) \to (Y, \sigma)$ is contra sg-closed (briefly csg-closed) in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in SG(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

Proof. The proof follows from the definition.

Theorem 4.10 Assume that arbitrary union of sg-open sets is sg-open. If $f : (X, \tau) \to (Y, \sigma)$ is csg-continuous and $Y$ is Urysohn, then $G(f)$ is contra-sg-closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist open sets $V, W$ such that $f(x) \in V$, $y \in W$ and $cl(U) \cap cl(W) = \emptyset$. Since $f$ is csg-continuous, there exists $U \in SG(X, x)$ such that $f(U) \subset cl(V)$. Therefore, we obtain $f(U) \cap cl(W) = \emptyset$. This shows that $G(f)$ is contra-sg-closed.

Theorem 4.11 A csg-continuous image of a sg-connected space is connected.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a contra-sg-continuous function of a sg-connected space $X$ onto a topological space $Y$. Let $Y$ be disconnected. Let $A$ and $B$ form a disconnected of $Y$. Then $A$ and $B$ are clopen and $Y = A \cup B$ where $A \cap B = \emptyset$. Since $f$ is a contra-sg-continuous function $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty sg-open sets in $X$. Also $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Hence $X$ is non sg-connected which is a contradiction. Therefore $Y$ is connected.

Theorem 4.12 Let $X$ be sg-connected and $Y$ be a $T_1$ space. If $f$ is csg-continuous, then $f$ is constant.

Proof. Since $Y$ is $T_1$ space, $\wedge = \{f^{-1}(\{y\}) : y \in Y\}$ is a disjoint sg-open partition of $X$. If $|\wedge| \geq 2$, then $X$ is the union of two non-empty sg-open sets. Since $X$ is sg-connected, $|\wedge| = 1$. Hence, $f$ is constant.

Definition 4.4 A topological space $(X, \tau)$ is said to be sg-normal if each pair of non-empty disjoint closed sets can be separated by disjoint sg-open sets.

Definition 4.5 [19] A topological space $(X, \tau)$ is said to be ultra normal if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 4.13 If $f : (X, \tau) \to (Y, \sigma)$ is a csg-continuous, closed injection and $Y$ is ultra normal, then $X$ is sg-normal.
Proof. Let $F_1$ and $F_2$ be a disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since $Y$ is ultra normal $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets $V_1$ and $V_2$ respectively. Hence $F_i \subseteq f^{-1}(V_i)$, $f^{-1}(V_i) \in \text{SG}(X, \tau)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus, $X$ is sg-normal.

References


