λ-Almost Summable and Statistically 
(V, λ)-Summable Sequences

Fatih Nuray¹ and Bunyamin Aydin²

¹Afyon Kocatepe University, Afyonkarahisar, Turkey
E-mail: fnuray@aku.edu.tr
²Necmettin Erbakan University, Konya, Turkey
E-mail: bunyaminaydin63@hotmail.com

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Abstract

King [3] introduced and examined the concepts of almost A-summable sequence, almost conservative matrix and almost regular matrix. In this paper, we introduce and examine the concepts of λ-almost A-summable sequence, λ-almost conservative matrix and λ-almost regular matrix. Also we introduce statistically (V, λ)-summable sequence.

Keywords: λ-sequence, λ-almost convergence, λ-almost conservative matrix, λ-almost regular matrix, λ-statistical convergence

1 Introduction and Background

Let $A = (a_{lk})$ be an infinite matrix of complex numbers and $x = (x_k)$ be a sequence of complex numbers. The sequence $\{A_l(x)\}$ defined by

$$A_l(x) = \sum_{k=1}^{\infty} a_{lk}x_k$$

is called A-transform of x whenever the series converges for $l = 1, 2, 3, \ldots$. The sequence $x$ is said to be A-summable to L if $\{A_l(x)\}$ converges to L.

Let $\ell_\infty$ denote the linear space of bounded sequences. A sequence $x \in \ell_\infty$
is said to be almost convergent to $L$ if
\[ \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{k+i} = L \]
uniformly in $i$.

The matrix $A$ is said to be conservative if $x \in c$ implies that the $A$-transform of $x$ is convergent. The matrix $A$ is said to be regular if the $A$-transform of $x$ is convergent to the limit of $x$ for each $x \in c$, where $c$ is the linear space of convergent sequences.

In the theory of summability and its applications one is usually interested in conservative or regular matrices. In [3], King introduced almost conservative and almost regular matrices.

A sequence $x \in \ell_\infty$ is said to be almost $A$-summable to $L$ if the $A$-transform of $x$ is almost convergent to $L$. The matrix $A$ is said to be almost conservative if $x \in c$ implies that the $A$-transform of $x$ is almost convergent. The matrix $A$ is said to be almost regular if the $A$-transform of $x$ almost convergent to the limit of $x$ for each $x \in c$.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to $\infty$, and $\lambda_{n+1} - \lambda_n \leq 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by
\[ v_n = v_n(x) = \frac{1}{\lambda_n} \sum_{k=n}^{n+\lambda_n} x_k := \frac{1}{\lambda_n} \sum_{k \in I_n} x_k. \]

L. Leindler in [4] defined a sequence $x = (x_k)$ to be $(V, \lambda)$-summable to number $L$ if $v_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then $(V, \lambda)$-summability is reduced to $(C, 1)$-summability. We write
\[ [V, \lambda] = \{ x = (x_k) : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0, \text{ for some } L \} \]
for set of sequences $x = (x_k)$ which are strongly $(V, \lambda)$-summable to $L$, that is, $x_k \to L[V, \lambda]$.

## 2 $\lambda$-Almost Conservative and $\lambda$-Almost Regular Matrices

In this section we introduce $\lambda$-almost conservative and $\lambda$-almost regular matrices.

**Definition 2.1** A sequence $x = (x_k)$ is said to be $\lambda$-almost convergent to number $L$ if
\[ \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} x_{k+i} = L \]
uniformly in $i$.

**Definition 2.2** The matrix $A$ is said to be $\lambda$-almost conservative if $x \in c$ implies that $A$-transform of $x$ is $\lambda$-almost convergent.

**Definition 2.3** The matrix $A$ is said to be $\lambda$-almost regular if $x \in c$ implies that $A$-transform of $x$ is $\lambda$-almost convergent to the limit of $x$ for each $x \in c$.

**Theorem 2.4** Let $A = (a_{lk})$ be an infinite matrix. Then the matrix $A$ is $\lambda$-almost conservative if and only if

(i) $\sup_n \sum_{k=0}^{\infty} \frac{1}{\lambda_n} | \sum_{j \in I_n} a_{l+j,k} | < \infty, \quad l = 0, 1, 2, \ldots,$

(ii) there exists an $\xi \in \mathbb{C}$, the set of complex numbers, such that

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{j \in I_n} \sum_{k=0}^{\infty} a_{l+j,k} = \xi$$

uniformly in $l$, and

(iii) there exists an $\xi_k \in \mathbb{C}, k=0,1,2,\ldots$ such that

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{j \in I_n} a_{l+j,k} = \xi_k$$

uniformly in $l$.

**Proof.** Suppose that $A$ is $\lambda$-almost conservative. Fix $l \in \mathbb{N}$, the set of natural numbers. Let

$$t_{nl} = \frac{1}{\lambda_n} \sum_{j \in I_n} s_{j+l}(x)$$

where $s_{j+l}(x) = \sum_{k=0}^{\infty} a_{j+l,k} x_k$. It is clear that $s_{j+l} \in c^\ast, j, n = 1, 2, \ldots$. Hence $t_{nl} \in c^\ast$, where $c^\ast$ is the continuous dual of $c$. Since $A$ is $\lambda$-almost conservative \lim_{n \to \infty} t_{nl} = t(x)$ uniformly in $l$. It follows that $\{t_{nl}(x)\}$ is bounded for $x \in c$ and fixed $l$. Hence $\{\|t_{nl}\|\}$ is bounded by the uniform bounded principle. For each $q \in \mathcal{N}$, define the sequence $y = y(l, n)$ by

$$y_k = \begin{cases} sgn \sum_{j \in I_n} a_{j+l,k}, & 0 \leq k \leq q \\ 0, & q < k. \end{cases}$$

Then $y \in c$, $\|y\| = 1$, and

$$|t_{nl}(y)| = \frac{1}{\lambda_n} \sum_{k=0}^{q} | \sum_{j \in I_n} a_{j+l,k} |.$$
Hence $|t_{nl}(y)| \leq \|t_{nl}\| |y| = \|t_{nl}\|$. Therefore $\frac{1}{\lambda_n} \sum_{k=0}^{\infty} |\sum_{j \in I_n} a_{j+k}| \leq \|t_{nl}\|$, so that (i) follows.

Since $e = (1, 1, \ldots)$ and $e_k = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ (with 1 in rank $k$) are convergent sequences, $\lim_n t_{nl}(e)$ and $\lim_n t_{nl}(e_k)$ must exist uniformly in $l$. Hence (ii) and (iii) hold.

Now assume that (i), (ii) and (iii) hold. Fix $l$ and $x \in c$. Then

$$t_{nl}(x) = \frac{1}{\lambda_n} \sum_{j \in I_n} \sum_{k=0}^{\infty} a_{j+k}x_k = \frac{1}{\lambda_n} \sum_{k=0}^{\infty} \sum_{j \in I_n} a_{j+k}x_k$$

so that

$$t_{nl}(x) \leq \frac{1}{\lambda_n} \sum_{k=0}^{\infty} |\sum_{j \in I_n} a_{j+k}x_k||x||, n = 1, 2, \ldots.$$ 

Therefore $t_{nl}(x) \leq K_1||x||$ by (i), where $K_1$ is a constant independent of $n$. Hence $t_{nl} \in c^*$, and the sequence $\{||t_{nl}||\}$ is bounded for each $l$. (ii) and (iii) imply that $\lim_n t_{nl}(e)$ and $\lim_n t_{nl}(e_k)$ exist for $l, k = 0, 1, 2, \ldots$. Since \{e, e_0, e_1, e_2, \ldots\} is a fundamental set in $c$ it follows from an elementary result of functional analysis that $\lim_n t_{nl}(x) = t_l(x)$ exists and $t_l \in c$. Therefore

$$t_l(x) = \lim_k x_k[t_l(e) - \sum_{k=0}^{\infty} t_l(e_k)] + \sum_{k=0}^{\infty} x_k t_l(e_k),$$

But $t_l(e) = \xi$ and $t_l(e_k) = \xi_k, k = 0, 1, 2, \ldots$, by (ii) and (iii), respectively. Hence $\lim_n t_{nl}(x) = t_l(x)$ exists for each $x \in c, l = 0, 1, 2, \ldots$, with

$$t(x) = \lim_k x_k[\xi - \sum_{k=0}^{\infty} \xi_k] + \sum_{k=0}^{\infty} \xi_k x_k. \quad (1)$$

Since $t_{kl} \in c^*$ for each $n$ and $l$, it has the form

$$t_{nl}(x) = \lim_k x_k[t_{nl}(e) - \sum_{k=0}^{\infty} t_{nl}(e_k)] + \sum_{k=0}^{\infty} x_k t_{nl}(e_k), \quad (2)$$

It is easy to see from (1) and (2) that convergence of $\{t_{nl}(x)\}$ to $t(x)$ is uniform in $l$, since $\lim_n t_{nl}(e) = \xi$ and $\lim_n t_{nl}(e_k) = \xi_k$ uniformly in $l$. Therefore $A$ is $\lambda$-almost conservative and the theorem is proved.

**Theorem 2.5** Let $A = (a_{lk})$ be an infinite matrix. Then the matrix $A$ is $\lambda$-almost regular if and only if

(iv) \[ \sup_n \frac{1}{\lambda_n} \sum_{k=0}^{\infty} |\sum_{j \in I_n} a_{l+j,k}| < \infty, \quad l = 0, 1, 2, \ldots, \]
Proof. Suppose that $A$ is $\lambda$-almost regular. Then $A$ is $\lambda$-almost conservative so that (iv) must hold. (v) and (vi) must hold since $A$-transform of the sequences $e_k$ and $e$ must be $\lambda$-almost convergent to $0$ and $1$, respectively.

Now suppose that (iv), (v) and (vi) hold. Then $A$ is $\lambda$-almost conservative. Therefore $\lim_{n} t_{nl}(x) = t(x)$ uniformly in $l$ for each $x \in c$. The representation (1) gives $t(x) = \lim_{k} x_k$. Hence $A$ is $\lambda$-almost regular. This proves the theorem.

3 Statistically $(V, \lambda)$-Summable Sequences

The concept of statistical convergence was introduced by Fast [1]. In [8] Schoenberg gave some basic properties of statistical convergence and also studied the concept as a summability method. A sequence $x = (x_k)$ is said to be statistically convergent to the number $L$ if for every $\epsilon > 0$,

$$\lim_{p \to \infty} \frac{1}{p} |\{k \leq p : |x_k - L| \geq \epsilon\}| = 0,$$

where the vertical bars denote the number of elements in the enclosed set. In this case we write $\text{st} - \lim x_k = L$. $\lim x_k = L$ implies $\text{st} - \lim x_k = L$, so statistical convergence may be considered as a regular summability method. This was observed by Schoenberg [8] along with the fact that the statistical limit is a linear functional on some sequence space. If $x$ is a sequence such that $x_k$ satisfies property $P$ for all $k$ except a set of natural density zero, then we say that $x_k$ satisfies $P$ for almost all $k$. In [2], Fridy proved that if $x$ is a statistically convergent sequence then there is a convergent sequence $y$ such that $x_k = y_k$ almost all $k$.

The concept of statistically summable $(C, 1)$ sequence was introduced by Moricz[5]. A sequence $x = (x_k)$ is said to be statistically summable $(C, 1)$ to $L$ if $\frac{1}{n} \sum_{k=1}^{n} x_k$ is statistically convergent to $L$.

In [6], Mursaleen introduced the concept of $\lambda$-statistical convergence. A sequence $x = (x_k)$ is said to be $\lambda$-statistically convergent to the number $L$ if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$
In this case we write $\text{st}_\lambda - \lim x_k = L$. In [7], Savas introduced the concept of almost $\lambda$-statistical convergence. A sequence $x = (x_k)$ is said to be almost $\lambda$-statistically convergent to the number $L$ if for every $\epsilon > 0$,

$$
\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_{k+i} - L| \geq \epsilon\}| = 0.
$$

uniformly in $i$.

In this section, we introduce the concept of statistically $(V, \lambda)$-summable sequence.

**Definition 3.1** A sequence $x = (x_k)$ is said to be statistically $(V, \lambda)$-summable to the number $L$ if $v_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$ is statistically convergent to $L$, i.e.,

$$
\lim_{p \to \infty} \frac{1}{p} |\{n \leq p : \left|\frac{1}{\lambda_n} \sum_{k \in I_n} x_k - L\right| \geq \epsilon\}| = 0.
$$

If $\lambda_n = n$, then statistically $(V, \lambda)$-summability is reduced to the statistically summability $(C, 1)$.

**Theorem 3.2** If $x \in \ell_\infty$ and $\text{st}_\lambda - \lim x_k = L$ then $x = (x_k)$ is statistically $(V, \lambda)$-summable to the number $L$, i.e., $\text{st} - \lim v_n(x) = L$.

**Proof.** Without loss of generality we may assume that $L = 0$. This means that if $\epsilon > 0$ and if we denote by $N_{\lambda_n}$ the number of $k \in I_n$ for which $|x_k| \geq \epsilon$, then

$$
\lim_{n \to \infty} \frac{N_{\lambda_n}}{\lambda_n} = 0.
$$

(3)

Since $(x_k)$ is bounded, we say $|x_k| \leq M$ for all $k$. Now

$$
\left|\frac{1}{\lambda_n} \sum_{k \in I_n} x_k\right| \leq \frac{N_{\lambda_n} M + (\lambda_n - N_{\lambda_n})\epsilon}{\lambda_n} = \frac{(M - \epsilon)N_{\lambda_n} + \lambda_n \epsilon}{\lambda_n} = \epsilon + \frac{(M - \epsilon) N_{\lambda_n}}{\lambda_n}
$$

where, by (3), the right side less than $2\epsilon$ provided that $n$ is sufficiently large. Thus $\lim v_n(x) = 0$. Since $\lim x_k = L$ implies $\text{st} - \lim x_k = L$, we have $\text{st} - \lim v_n(x) = 0$.

**Theorem 3.3** If $x$ statistically $(V, \lambda)$-summable to the number $L$ and $\Delta v_n = O\left(\frac{1}{n}\right)$, then $x$ is $(V, \lambda)$-summable to the number $L$, where $\Delta v_n = v_n - v_{n+1}$.

**Proof.** Assume that $x$ statistically $(V, \lambda)$-summable to the number $L$. Then $\text{st} - \lim v_n = L$ and we can choose a sequence $w$ such that $\lim w_n = L$ and $v_n = w_n$ for almost all $n$. For each $n$, write $n = m(n) + p(n)$, where $m(n) = \ldots$
\[
\max\{i \leq n : v_i = w_i\}; \text{ if the set } \{i \leq n : v_i = w_i\} \text{ is empty, take } m(n) = -1. \text{ This can occur for at most a finite number of } n. \text{ We assert that }
\lim_{n \to \infty} \frac{p(n)}{m(n)} = 0. \tag{4}
\]

For, if \( \frac{p(n)}{m(n)} > \epsilon > 0 \), then
\[
\frac{1}{n}|\{i \leq n : v_i \neq w_i\}| \leq \frac{1}{m(n) + p(n)} p(n) \leq \frac{p(n)}{p(n) + p(n)} = \frac{\epsilon}{1 + \epsilon}
\]
so if \( \frac{p(n)}{m(n)} \geq \epsilon \) for infinitely many \( n \), we would contradict \( v_n = w_n \) for almost all \( n \). Thus (4) holds. Now consider that difference between \( w_{m(n)} \) and \( v_n \). Since \( \Delta v_n = O(\frac{1}{n}) \) there is a constant \( K \) such that \( |\Delta v_n| \leq \frac{K}{n} \) for all \( n \). Therefore
\[
|w_{m(n)} - v_n| = |v_{m(n)} - v_{m(n)+p(n)}| \leq \sum_{i=m(n)}^{m(n)+p(n)-1} |\Delta v_i| \leq \frac{p(n)K}{m(n)}.
\]
By (4), the last expression tends to 0 as \( n \to \infty \), and since \( \lim_{n \to \infty} w_n = L \), we conclude that
\[
\lim_{n \to \infty} v_n = \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} x_k = L.
\]

References