Boehmians and Elzaki Transform

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Abstract

A motivation of the classical Sumudu transform “ Elzaki transform ” was
presented as a closely related transform to the Laplace transform. In the present
work, we extend the cited transform to a Schwartz space of distributions of
compact support and retain its classical properties. The Elzaki transform is
extended to the context of Boehmian spaces and, further, shown to be well de-
finied and linear mapping in the banach space of Lebesgue integrable Boehmians.
Certain theorem is also proved in some detail.

Keywords: Distribution Space, Sumudu Transform, Boehmian Spaces,
Elzaki Transform.

1 Introduction

Integral transforms play an important role in many fields of science. In the
literature, integral transforms are widely used in mathematical physics, optics,
engineering mathematics and, few others. Among these transforms which were
extensively used and applied on theory and applications are : the Mellin,
Hankel, Laplace and Sumudu transforms, to name, but a few. The Sumudu
transform is defined on a set $A$ of functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{-\frac{t}{\tau}}, \text{if } t \in (-1)^j \times [0, \infty]\right\}$$  \hspace{1cm} (1.1)

by the formula $[14,15,16]$

$$F(\zeta) = S f(\zeta) = \int_0^\infty f(\zeta t) e^{-t} dt, t \in (-\tau_1, \tau_2).$$  \hspace{1cm} (1.2)
The Sumudu transform has a strong relationship with other integral transforms. In particular, the relationship between the Sumudu transform and Laplace transform have been established by Kilicman [2010]. Recently, a motivation of the Sumudu transform, namely, Elzaki transform, is given by Elzaki [17, 18, 19]. The Elzaki transform of a function \( f(t) \) over the set (1.1) of functions of exponential order, is given by [17, 18]

\[
Ef(\zeta) = \zeta \int_0^\infty f(t) e^{-\zeta t} dt, \zeta \in (-\tau_1, \tau_2).
\]  

(1.3)

In the above citations, the transform (1.3) is noted to facilitate the process of solving order and partial differential equations where examples are solved.

Let \( f \) be a function of exponential order. Let \( Lf \) and \( Ef \) be the Laplace and Elzaki transforms of \( f \), respectively, then

\[
Ef(\zeta) = \zeta Lf \left( \frac{1}{\zeta} \right).
\]

(1.4)

and hence

\[
Lf \left( \frac{1}{\zeta} \right) = \zeta E \left( \frac{1}{\zeta} \right).
\]

(1.5)

Following, are considered as general properties of Elzaki transform.

1. If \( a \) and \( b \) are non-negative real numbers then

\[
Ef(\zeta) = aEf(\zeta) + bEg(\zeta).
\]

2. \( \lim_{t \to 0} f(t) = \lim_{\zeta \to 0} Ef(\zeta) = f(0) \).

The convolution product between two \( L^1 \) functions \( f \) and \( g \) is defined by

\[
f * g(x) = \int_0^\infty f(t) g(x-t) dt
\]

(1.6)

then

\[
Ef(f * g)(\zeta) = \frac{f(\zeta)g(\zeta)}{\zeta}, \text{ see [19].}
\]

2 The Elzaki Transform of Distributions

Let \( \varepsilon(R_+) \) be the space of smooth functions of arbitrary support on \( R_+ \) and \( \hat{\varepsilon}(R_+) \) be its strong dual of distributions of compact support. Denote by \( D(R_+) \), the subspace of \( \varepsilon(R_+) \) of test functions of compact support then its dual space \( \hat{D}(R_+) \) consists of Schwartz distributions. Certainly, \( D \subset \varepsilon \) and hence \( \varepsilon \subset \hat{\varepsilon} \subset \hat{D} \). The kernel function \( \zeta e^{-\zeta t} \) of Elzaki transform is clearly a member of \( \varepsilon(R_+) \). Hence, we define the generalized Elzaki transform \( \hat{E} \) on \( \hat{\varepsilon}(R_+) \) by the equation

\[
\hat{E}f(\zeta) = \left< f(t), \zeta e^{-\zeta t} \right>.
\]

(2.1)
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for every distribution $f \in \mathcal{E}(\mathbb{R}^+)$. 

**Theorem 2.1.** $\hat{E}$ is a well-defined mapping in the space $\mathcal{E}(\mathbb{R}^+)$. 

**Proof.** of this theorem is immediate, since $\zeta e^{-\tau} \in \mathcal{E}(\mathbb{R}^+)$. 

**Theorem 2.2.** $\hat{E}$ is infinitely smooth and 

$$
\frac{d^k}{d\zeta^k} \hat{E} f (\zeta) = \left\langle f (t), \frac{d^k}{d\zeta^k} \left( \zeta e^{-\tau} \right) \right\rangle.
$$

for every $f \in \mathcal{E}(\mathbb{R}^+)$. 

This theorem can be proved by an argument similar to that used in [9, Theorem 2.9.1.], detailed proof thus avoided. Next, is a theorem describing linearity of the distributional Elzaki transform. 

**Theorem 2.3.** $\hat{E}$ is linear.

Let $f, g \in \mathcal{E}(\mathbb{R}^+)$. We define the generalized convolution between $f$ and $g$ by 

$$
\langle f * g (x), \psi (x) \rangle = \langle f (x), \langle g (t), \psi (x + t) \rangle \rangle. \tag{2.2}
$$

for every $\psi \in \mathcal{E}(\mathbb{R}^+)$. Hence using (2.1) and (2.2) together with simple calculations yields 

$$
\hat{E} (f * g) (\zeta) = \frac{\hat{E} f (\zeta) \hat{E} g (\zeta)}{\zeta}.
$$

**Theorem 2.4.** Let $f \in \mathcal{E}(\mathbb{R}^+)$ and $g (t) = \begin{cases} f (t - \tau), & t \geq \tau \\ 0, & t < \tau \end{cases}$ then 

$$
\hat{E} g (\zeta) = e^{-\tau} \hat{E} f (\zeta).
$$

**Proof.** It is clear that $g \in \mathcal{E}(\mathbb{R}^+)$. The translation property of distributions through $\tau$ [9, p.26], implies 

$$
\hat{E} g (\zeta) = \langle f (t - \tau), \zeta e^{-\tau(1/\zeta)} \rangle = e^{-\tau} \hat{E} f (\zeta).
$$

Hence, the theorem.

**Theorem 2.5.** Let $f \in \mathcal{E}(\mathbb{R}^+)$ then the following holds 

\begin{enumerate}
  
  \item $\hat{E} (tf (t)) (\zeta) = \zeta^2 \frac{d}{d\zeta} \hat{E} f (\zeta) - \zeta \hat{E} f (\zeta)$. 
  
  \item $\hat{E} (t^2 f (t)) (\zeta) = \zeta^4 \frac{d^2}{d\zeta^2} \hat{E} f (\zeta)$.
\end{enumerate}
Proof. Considering properties of Elzaki transform (2.1) and Theorem 2.2, we get
\[
\frac{d}{d\zeta} \hat{E}(\zeta) = \frac{d}{d\zeta} \left\langle f(t), \zeta e^{-t/\zeta} \right\rangle = \int f(t) \frac{d}{d\zeta} \left( \zeta e^{-t/\zeta} \right)
\]
Differentiating inside the inner product yields
\[
\frac{d}{d\zeta} \hat{E}(\zeta) = \left\langle tf(t), \frac{1}{\zeta} e^{-t/\zeta} + e^{-t/\zeta} \right\rangle
\]
Properties of distributions imply
\[
\frac{d}{d\zeta} \hat{E}(\zeta) = \left\langle tf(t), 1 \frac{1}{\zeta} e^{-t/\zeta} \right\rangle + \left\langle f(t), e^{-t/\zeta} \right\rangle
\]
Multiplying both sides by $\zeta^2$ and rearranging complete the proof of the first part of the theorem. Proof of the second part is similar. Hence we avoid the same. This proof is therefore completed.

Theorem 2.6. (Shifting Theorem). Let $f \in \hat{\mathcal{E}}(R_+)$ then
\[
\hat{E}\left(e^{at} f(t)\right)(\zeta) = \frac{1}{1-a\zeta} \hat{E}\left(\frac{\zeta}{1-a\zeta}\right).
\]
The proof is straightforward.

3 Bohemians

Let $G$ be a linear space and $S$ be a subspace of $G$. We assume that to each pair of elements $f \in G$ and $\phi \in S$, is assigned the product $f \ast \phi$ such that the following conditions are satisfied:

1. $\phi, \psi \in S \Rightarrow \phi \ast \psi \in S$ and $\phi \ast \psi = \psi \ast \phi$.
2. $f \in G, \phi, \psi \in S \Rightarrow (f \ast \phi) \ast \psi = f \ast (\phi \ast \psi)$.
3. If $f, g \in G, \phi \in S$ and $\lambda \in R$, then $(f + g) \ast \phi = f \ast \phi + g \ast \phi$ and $\lambda (f \ast \phi) = (\lambda f) \ast \phi$. Let $\Delta$ be a family of sequences from $S$, such that

1. If $f, g \in G, (\gamma_n) \in \Delta$ and $f \ast \gamma_n = g \ast \gamma_n \ (n = 1, 2, ...)$, then $f = g$.
2. $(\gamma_n), (\tau_n) \in \Delta \Rightarrow (\gamma_n \ast \tau_n) \in \Delta$.

then each elements of $\Delta$ will be called delta sequence.

Consider the class $A$ of pairs of sequences defined by
\[
A = \left\{((f_n), (\gamma_n)) : (f_n) \subseteq G^N, (\gamma_n) \in \Delta \right\}.
\]
for each $n \in \mathbb{N}$. Then, an element $((f_n), (\gamma_n)) \in A$ is called a quotient of sequences, denoted by $\frac{f_n}{\gamma_n}$ if
\[
f_i \ast \gamma_j = f_j \ast \gamma_i, \forall i, j \in \mathbb{N}.
\]
Two quotients of sequences $\frac{f_n}{\gamma_n}$ and $\frac{g_n}{\tau_n}$ are said to be equivalent, $\frac{f_n}{\gamma_n} \sim \frac{g_n}{\tau_n}$, if 

$$f_i * \gamma_j = g_j * \tau_i, \forall i, j \in \mathbb{N}.$$ 

The relation $\sim$ is an equivalent relation on $A$ and hence, splits $A$ into equivalence classes. The equivalence class containing $\frac{f_n}{\gamma_n}$ is denoted by $[\frac{f_n}{\gamma_n}]$. These equivalence classes are called Boehmians and the space of all Boehmians is denoted by $H$.

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way 

$$\left[\frac{f_n}{\gamma_n}\right] + \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n*\tau_n + g_n*\gamma_n}{\gamma_n*\tau_n}\right]$$

and

$$a \left[\frac{f_n}{\gamma_n}\right] = \left[\frac{af_n}{\gamma_n}\right], a \in \mathbb{C}.$$ 

The operation $*$ and the differentiation are defined by

$$\left[\frac{f_n}{\gamma_n}\right] * \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n*\gamma_n}{\gamma_n*\tau_n}\right]$$

and

$$D^\alpha \left[\frac{f_n}{\gamma_n}\right] = \left[\frac{D^\alpha f_n}{\gamma_n}\right].$$

Many a time, $G$ is equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $*$ are given by:

1. If $f_n \to f$ as $n \to \infty$ in $G$ and, $\phi \in S$ is any fixed element, then

$$f_n * \phi \to f * \phi \text{ in } G \ (\text{as } n \to \infty).$$

2. If $f_n \to f$ as $n \to \infty$ in $G$ and $(\gamma_n) \in \Delta$, then

$$f_n * \gamma_n \to f \text{ in } G \ (\text{as } n \to \infty).$$

The operation $*$ can be extended to $H \times S$ by the following definition.

**Definition 3.1.** If $\left[\frac{f_n}{\gamma_n}\right] \in H$ and $\phi \in S$, then $\left[\frac{f_n}{\gamma_n}\right] * \phi = \left[\frac{f_n*\phi}{\gamma_n}\right]$.

In $H$, two types of convergence, $\delta-$convergence and $\Delta-$convergence, are defined as follows:

**Definition 3.2.** A sequence of Boehmians $(\beta_n)$ in $H$ is said to be $\delta-$convergent to a Boehmian $\beta$ in $H$, denoted by $\beta_n \to \delta \beta$, if there exists a delta sequence $(\gamma_n)$ such that

$$(\beta_n * \gamma_n), (\beta * \gamma_n) \in G, \forall k, n \in \mathbb{N},$$

$$(\beta_n * \gamma_n)$$
and

\[(\beta_n \ast \gamma_k) \to (\beta \ast \gamma_k) \text{ as } n \to \infty, \text{ in } G, \text{ for every } k \in \mathbb{N}.\]

The following lemma is equivalent for the statement of \(\delta\)-convergence

**Lemma 3.3.** \(\beta_n \overset{\delta}{\to} \beta (n \to \infty) \text{ in } H \text{ if and only if there is } f_{n,k}, f_k \in G \text{ and } \gamma_k \in \Delta \text{ such that } \beta_n = \left[\frac{f_{n,k}}{\gamma_k}\right], \beta = \left[\frac{f_k}{\gamma_n}\right] \text{ and for each } k \in \mathbb{N}, \)

\[f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } G.\]

**Definition 3.4.** A sequence of Boehmians \((\beta_n)\) in \(H\) is said to be \(\Delta\)-convergent to a Boehmian \(\beta\) in \(H\), denoted by \(\beta_n \overset{\Delta}{\to} \beta\), if there exists a \((\gamma_n)\) \(\in \Delta\) such that \((\beta_n - \beta) \ast \gamma_n \in G, \forall n \in \mathbb{N}\), and \((\beta_n - \beta) \ast \gamma_n \to 0 \text{ as } n \to \infty \text{ in } G.\) see [1-5,78,10,11,13]

4 The Elzaki Transform of Boehmians

Let \(G = L^1(R_+)\) and \(S = D(R_+)\). Let \(\Delta\) be the collection of sequences \((\gamma_n)\) from \(D(R_+)\) such that

1. \(\int_{R_+} \gamma_n(t) \, dt = 1.\)
2. \(\|\gamma_n\|_{L^1} < B \text{ for all } (\gamma_n) \in \Delta \text{ where } B \text{ is certain positive constant.}\)
3. \(|x| > \epsilon |\gamma_n(t)| \, dt \to 0 \text{ as } n \to \infty, \epsilon > 0.\)

The corresponding space of Boehmians \(H(L^1, D, *, \Delta)\) is a convolution algebra with the operations

\[\left[\frac{f_n}{\gamma_n}\right] + \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n + g_n \ast \gamma_n}{\gamma_n \ast \gamma_n}\right].\]  

(4.1)

and

\[a \left[\frac{f_n}{\gamma_n}\right] = \left[\frac{af_n}{\gamma_n}\right], a \in R.\]  

(4.2)

The operation \(*\) in \(H(L^1, D, *, \Delta)\) can be defined by

\[\left[\frac{f_n}{\gamma_n}\right] \ast \left[\frac{g_n}{\gamma_n}\right] = \left[\frac{f_n \ast g_n}{\gamma_n \ast \gamma_n}\right].\]  

(4.3)

Differentiation in \(H(L^1, D, *, \Delta)\) is defined by

\[D^k \left[\frac{f_n}{\gamma_n}\right] = \left[\frac{D^k f_n}{\gamma_n}\right], k \in \mathbb{N}.\]  

(4.4)

If \((\gamma_n) \in \Delta\) then, certainly, \(E\gamma_n(\zeta) \to \zeta \text{ as uniformly } n \to \infty, \text{ on compact subsets of } R_+.\)
Lemma 4.1. If $f_n \in L^1$ such that $\left[ f_n \right] \in H (L^1, D, \ast, \Delta)$ then
\[ Ef_n (\zeta) = R_+ \zeta e^{\frac{1}{\zeta}} f_n (t) \, dt. \]
converges uniformly on each compact set of $R_+$.

Proof. Since $E \gamma_n \to \zeta$ as $n \to \infty$ on compact subsets of $R_+$, $E \gamma_n > 0$ for almost all $k \in \mathbb{N}$ and hence
\[ Ef_n (\zeta) = Ef_n (\zeta) \frac{E \gamma_n (\zeta)}{E \gamma_n (\zeta)} = \frac{\zeta E (f_n \ast \gamma_n) (\zeta)}{E \gamma_n (\zeta)} = \frac{\zeta E (f_k \ast \gamma_n) (\zeta)}{E \gamma_n (\zeta)} = \frac{E f_k (\zeta)}{E \gamma_n (\zeta)} E \gamma_n (\zeta) \quad \text{on } K. \]
where $K$ is certain compact subset of $R_+$. Considering limit as $n \to \infty$ we get
\[ Ef_n (\zeta) \to \frac{\zeta E f_k (\zeta)}{E \gamma_n (\zeta)}. \]
From Theorem 4.1 we define the Elzaki transform of $\beta \in H (L^1, D, \ast, \Delta)$,
where $\beta = \left[ f_n \right]$, by the formula
\[ \tilde{E} \beta = \lim_{n \to \infty} E f_n, \]
on compact subsets of $R_+$. Now, we show the above definition is well defined.

For, if $\beta_1 = \beta_2$ where, $\beta_1 = \left[ f_n \right]$, $\beta_2 = \left[ g_m \right]$ then $f_n \ast \gamma_m = g_m \ast \gamma_n = g_n \ast \gamma_m$.
Employing Elzaki transform on both sides yields
\[ \frac{Ef_n (\zeta)}{\zeta} E \gamma_m (\zeta) = \frac{Eg_n (\zeta)}{\zeta} E \gamma_m (\zeta). \]
Hence, allowing $m \to \infty$ yields
\[ \lim_{n \to \infty} Ef_n (\zeta) = \lim_{n \to \infty} Eg_n (\zeta) \]
Hence
\[ \tilde{E} \beta_1 = \tilde{E} \beta_2. \]
This completes the proof.

Theorem 4.2. Let $x_1, x_2 \in H (L^1, D, \ast, \Delta)$ and $a \in \mathbb{C}$ then
\begin{enumerate}
  \item $\tilde{E} (x_1 + x_2) = \tilde{E} x_1 + \tilde{E} x_2$.
  \item $\tilde{E} (ax_1) = a \tilde{E} x_1$.
  \item $\tilde{E} (x_1 \ast \gamma_n) = \tilde{E} (\gamma_n \ast x_1) = \frac{1}{\zeta} \tilde{E} x_1$.
\end{enumerate}
\( (4) \tilde{E}x_1 = 0 \Rightarrow x_1 = 0. \)
\( (5) x_n \xrightarrow{\Delta} x \in H(L^1, D, *, \Delta) \Rightarrow \)
\[ \tilde{E}x_n \xrightarrow{\Delta} \tilde{E}x \in H(L^1, D, *, \Delta) \text{ as } n \to \infty \]
on compact subsets.

**Proof.** of Parts (1 – 2), and (4) follows from the corresponding properties of the classical Elzaki transform. Proof of Part(3): Let \( x_1 \in H(L^1, D, *, \Delta) \) such that \( x_1 = \left[ \frac{f_{\gamma}}{\gamma_n} \right] \) then \( x_1 \ast \gamma_n = \left[ \frac{f_{\gamma} \ast \gamma_n}{\gamma_n} \right] \). Hence,

\[ \tilde{E}(x_1 \ast \gamma_n) = \frac{1}{\zeta} \lim_{n \to \infty} Ef_n(\zeta) = \frac{1}{\zeta} \tilde{E}x_1. \]

Finally, the proof of Part (5) have analysis similar to that employed in the proof of Part (f) of from [7, Theorem 2]. This completes the proof of the theorem.

**References**


