(i, j)-ξ-Open Sets in Bitopological Spaces

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Abstract

The aim of this paper is to introduce a new type of sets in bitopological spaces which is conditional ξ-open set in bitopological spaces called (i, j)-ξ-open set and we study its basic properties, and also we introduce some characterizations of this set.

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1 Introduction

In 1963 Kelley J. C. [7] was first introduced the concept of bitopological spaces, where X is a nonempty set and τ_1, τ_2 are topologies on X. In 1963 Levine [8] introduced the concept of semi-open sets in topological spaces. By using this concept, several authors defined and studied stronger or weaker types of topological concept.

In this paper, we introduce the concept of a conditional ξ-open set in a bitopological space, and we study their basic properties and relationships with other concepts of sets. Throughout this paper, (X, τ_1, τ_2) is a bitopological space, and if A ⊆ Y ⊆ X, then i-Int(A) and i-Cl(A) denote respectively the
interior and closure of $A$ with respect to the topology $\tau_i$ on $X$ and $i-\text{Int}_Y(A)$, $i-\text{Cl}_Y(A)$ denote respectively the interior and the closure of $A$ with respect to the induced topology on $Y$.

2 Preliminaries

We shall give the following definitions and results.

**Definition 2.1** A subset $A$ of a space $(X, \tau)$ is called:

1. preopen [9], if $A \subseteq \text{Int}(\text{Cl}(A))$
2. semi-open [8], if $A \subseteq \text{Cl}(\text{Int}(A))$
3. $\alpha$-open [11], if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$
4. regular open [5], if $A = \text{Int}(\text{Cl}(A))$
5. regular semi-open [1], if $A = \text{sInt}(\text{sCl}(A))$

The complement of a preopen (resp., semi-open, $\alpha$-open, regular open, regular semi-open) set is said to be preclosed (resp., semi-closed, $\alpha$-closed, regular closed, regular semi-closed). The intersection of all preclosed (resp., semi-closed, $\alpha$-closed) sets of $X$ containing $A$ is called preclosure (resp., semi-closure, $\alpha$-closure) of $A$. The union of all preopen (resp., semi-open, $\alpha$-open) sets of $X$ contained in $A$ called preinterior (resp., semi-interior, $\alpha$-interior) of $A$.

A subset $A$ of a space $X$ is called $\delta$-open [15], if for each $x \in A$, there exists an open set $G$ such that $x \in G \subseteq \text{Int}(\text{Cl}(G)) \subseteq A$. A subset $A$ of a space $X$ is called $\theta$-semi-open [6] (resp., semi-$\theta$-open [2]) if for each $x \in A$, there exists a semi-open set $G$ such that $x \in G \subseteq \text{Cl}(G) \subseteq A$ (resp., $x \in G \subseteq \text{sCl}(G) \subseteq A$. A subset $A$ of a topological space $(X, \tau)$ is called $\eta$-open [13], if $A$ is a union of $\delta$-closed sets. The complement of $\eta$-open sets is called $\eta$-closed.

**Definition 2.2** A topological space $X$ is called,

1.Externally disconnected [2], if $\text{Cl}(U) \in \tau$ for every $U \in \tau$.
2. Locally indiscrete [4], if every open subset of $X$ is closed.

From the above definition we obtain:

**Remark 2.3** If $X$ is locally indiscrete space, then every semi-open subset of $X$ is closed and hence every semi-closed subset of $X$ is open.
Theorem 2.4 [9] A space $X$ is semi-$T_1$ if and only if for any point $x \in X$ the singleton set $\{x\}$ is semi-closed.

Theorem 2.5 [10] For any space $(X, \tau)$ and $(Y, \tau)$ if $A \subseteq X$, $B \subseteq Y$ then:
1. $p\text{Int}_{X \times Y}(A \times B) = p\text{Int}_X(A) \times p\text{Int}_Y(B)$
2. $s\text{Cl}_{X \times Y}(A \times B) = s\text{Cl}_X(A) \times s\text{Cl}_Y(B)$

Theorem 2.6 [10] For any topological space the following statements are true:
1. Let $(Y, \tau_Y)$ be a subspace of a space $(X, \tau)$, if $F \in SC(X)$ and $F \subseteq Y$ then $F \in SC(Y)$.
2. Let $(Y, \tau_Y)$ be a subspace of a space $(X, \tau)$, if $F \in SC(Y)$ and $Y \in SC(X)$ then $F \in SC(X)$
3. Let $(X, \tau)$ be a topological space, if $Y$ is an open subset of a space $X$ and $F \in SC(X)$, then $F \cap Y \in SC(X)$

Definition 2.7 [12] A space $X$ is said to be semi-regular if for any open set $U$ of $X$ and each point $x \in U$, there exists a regular open set $V$ of $X$ such that $x \in V \subseteq U$.

3 Basic Properties

In this section, we introduce and define a new type of sets in bitopological spaces and find some of its properties.

Definition 3.1 A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)\xi$-open, if $A$ is a $j$-open set and for all $x$ in $A$, there exist an $i$-semi-closed set $F$ such that $x \in F \subseteq A$. A subset $B$ of $X$ is called $(i, j)\xi$-closed if $B^c$ is $(i, j)\xi$-open.

The family of $(i, j)\xi$-open (resp., $(i, j)\xi$-closed) subset of $x$ is denoted by $(i, j)\xi O(X)$ (resp., $(i, j)\xi C(X)$).

From the above definition we obtain:

Corollary 3.2 A subset $A$ of a bitopological space $X$ is $(i, j)\xi$-open, if $A$ is $j$-open set and it is a union of $i$-semi-closed sets. This means that $A = \bigcup F_\alpha$, where $A$ is a $j$-open and $F_\alpha$ is an $i$-semi-closed set for each $\alpha$.

It is clear from the definition that every $(i, j)\xi$-open set is $j$-open, but the converse is not true in general as shown in the following example.
Example 3.3 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{c\}, \{a, b\}, X\}$, then $(i, j)\xi O(X) = \{\phi, \{c\}, X\}$.

It is clear that $\{a, b\}$ is $j$-open but not $(i, j)$-$\xi$-open.

Proposition 3.4 Let $(X, \tau_1, \tau_2)$ be a bitopological space if $(X, \tau_1)$ is a semi-$T_1$-space, then $(i, j)\xi O(X) = \tau_j(X)$.

Proof. Let $A$ be any subset of a space $X$ and $A$ is $j$-open set, if $A = \phi$, then $A \in (i, j)-\xi O(X)$, if $A \neq \phi$, now let $x \in A$, since $(X, \tau_1)$ is semi-$T_1$-space, then by Theorem 2.4 every singleton is $i$-semi-closed set, and hence $x \in \{x\} \subseteq A$, therefore $A \in (i, j)-\xi O(X)$, hence $\tau_j(X) \subseteq (i, j)-\xi O(X)$ but $(i, j)-\xi O(X) \subseteq \tau_j(X)$ generally, thus $(i, j)-\xi O(X) = \tau_j(X)$.

Proposition 3.5 Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A$ be a subset the space $X$. If $A \in j-\delta O(X)$ and $A$ is an $i$-closed set, then $A \in (i, j)-\xi O(X)$

Proof. If $A = \phi$, then $A \in (i, j)-\xi O(X)$, if $A \neq \phi$, let $x \in A$ since $A \in j-\delta O(X)$ and $j-\delta O(X) \subseteq \tau_j(X)$ in general so $A \in \tau_j(X)$, and since $A$ is $i$-closed so $A$ is $i$-semi-closed and $x \in A \subseteq A$, and hence $A \subseteq (i, j)-\xi O(X)$.

From Proposition 3.5 we obtain the following:

Corollary 3.6 Let $(X, \tau_1, \tau_2)$ be a bitopological space, if a subset $A$ of $X$ is $i$-regular closed and $j$-open then $A \in (i, j)-\xi O(X)$

Theorem 3.7 In a bitopological space $(X, \tau_1, \tau_2)$ if a space $(X, \tau_i)$ is locally indiscrete then $(i, j)-\xi O(X) \subseteq \tau_i$.

Proof. Let $V \in (i, j)-\xi O(X)$, then $V \in \tau_j(X)$ and for each $x \in V$, there exist $i$-semi-closed $F$ in $X$ such that $x \in F \subseteq V$, by Remark 2.3, $F$ is $i$-open, it follows that $V \in \tau_i$, and hence $(i, j)-\xi O(X) \subseteq \tau_i$.

The converse of Theorem 3.7, is not true in general, as shown in the following example:

Example 3.8 Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b, c\}, X\}$ and $\tau_2 = \{\phi, \{b, c\}, X\}$, then $(i, j)-\xi O(X) = \{\phi, \{b, c\}, X\}$ and it is clear that $(X, \tau_1)$ is locally indiscrete but $\tau_1$ is not a subset of $(i, j)-\xi O(X)$

Theorem 3.9 Let $X_1$, $X_2$ be two bitopological space and $X_1 \times X_2$ be the bitopological product, let $A_1 \in (i, j)-\xi O(X_1)$ and $A_2 \in (i, j)-\xi O(X_2)$ then $A_1 \times A_2 \in (i, j)-\xi O(X_1 \times X_2)$
Proof. Let \((x_1, x_2) \in A_1 \times A_2\) then \(x_1 \in A_1\) and \(x_2 \in A_2\), and since \(A_1 \in (i, j)\-\xi O(X_1)\) and \(A_2 \in (i, j)\-\xi O(X_2)\), then \(A_1 \in j\-\xi O(X_1)\) and \(A_2 \in j\-\xi O(X_2)\), there exist \(F_1 \in i\-SC(X_1)\) and \(F_2 \in i\-SC(X_2)\) such that \(x_1 \in F_1 \subseteq A_1\) and \(x_2 \in F_2 \subseteq A_2\). Therefore \((x_1, x_2) \in F_1 \times F_2 \subseteq A_1 \times A_2\), and since \(A_1 \in j\-\xi O(X_1)\) and \(A_2 \in j\-\xi O(X_2)\), then by Theorem 2.5 part (1) \(A_1 \times A_2 = j\-\xi Int_x \eta \in (A_1 \times A_2)\) so by Corollary 3.2 we get \(F_1 \times F_2 = i\-sCl_{x_1}(F_1) \times i\-sCl_{x_2}(F_2) = i\-sCl_{x_1 \times x_2}(F_1 \times F_2)\), hence \(F_1 \times F_2 \in i\-SC(X_1 \times X_2)\), therefore \(A_1 \times A_2 \in (i, j)\-\xi O(X)\).

Theorem 3.10 For any bitopological space \((X, \tau_1, \tau_2)\), if \(A \in \tau_1(X)\) and either \(A \in i\-\eta O(X)\) or \(A \in i\-S\theta O(X)\), then \(A \in (i, j)\-\xi O(X)\)

Proof. Let \(A \in i\-\eta O(X)\) and \(A \in \tau_1(X)\), if \(A = \emptyset\), then \(A \in (i, j)\-\xi O(X)\), if \(A \neq \emptyset\), since \(A \in i\-\eta O(X)\), then \(A = \bigcup F_\alpha\), where \(F_\alpha \in i\-\delta C(X)\) for each \(\alpha\), and since \(i\-\delta C(X) \subseteq i\-SC(X)\), so \(F_\alpha \in i\-SC(X)\) for each \(\alpha\), and \(A \in \tau_1(X)\) so by Corollary 3.2 \(A \in (i, j)\-\xi O(X)\).

On the other hand, suppose that \(A \in i\-S\theta O(X)\) and \(A \in \tau_1(X)\), if \(A = \emptyset\), then \(A \in (i, j)\-\xi O(X)\), if \(A \neq \emptyset\), since \(A \in i\-S\theta O(X)\), then for each \(x \in A\), there exist \(i\-\text{semi-open set } U\) such that \(x \in U \subseteq i\-sCl(U) \subseteq A\), this implies that \(x \in i\-sCl(U) \subseteq A\) and \(A \in \tau_1(X)\), therefore by Corollary 3.2 \(A \in (i, j)\-\xi O(X)\).

Theorem 3.11 Let \(Y\) be a subspace of a bitopological space \((X, \tau_1, \tau_2)\), if \(A \in (i, j)\-\xi O(X)\) and \(A \subseteq Y\), then \(A \in (i, j)\-\xi O(Y)\)

Proof. Let \(A \in (i, j)\-\xi O(X)\) and \(A \in \tau_1(X)\) and for each \(x \in A\), there exists \(i\-\text{semi-closed set } F \) in \(X\) such that \(x \in F \subseteq A\), since \(A \in \tau_1(X)\) and \(A \subseteq Y\), then by Theorem 2.6 \(F \in \tau_1(Y)\), and since \(F \in i\-SC(X)\) and \(F \subseteq Y\), then by Theorem 2.6 \(F \in i\-SC(Y)\), hence \(A \in (i, j)\-\xi O(Y)\).

From the above theorem we obtain:

Corollary 3.12 Let \(X\) be a bitopological space, \(A\) and \(Y\) be two subsets of \(X\) such that \(A \subseteq Y \subseteq X\), \(Y \in RO(X, \tau_1)\), \(Y \in RO(X, \tau_1)\), then \(A \in (i, j)\-\xi O(Y)\) if and only if \(A \in (i, j)\-\xi O(X)\)

Proposition 3.13 Let \(Y\) be a subspace of a bitopological space \((X, \tau_1, \tau_2)\), if \(A \in (i, j)\-\xi O(Y)\) and \(Y \in i\-SC(X)\), then for each \(x \in A\), there exists an \(i\-\text{semi-closed set } F \) in \(X\) such that \(x \in F \subseteq A\).

Proof. Let \(A \in (i, j)\-\xi O(Y)\), then \(A \in \tau_1(Y)\) and for each \(x \in A\) there exist an \(i\-\text{semi-closed set } F \) in \(Y\) such that \(x \in F \subseteq A\), and since \(Y \in i\-SC(X)\) so by Theorem 2.6 \(F \in i\-SC(X)\), which completes the proof.

Proposition 3.14 Let \(A\) and \(Y\) be any subsets of a bitopological space \(X\), if \(A \in (i, j)\-\xi O(X)\) and \(Y \in RO(X, \tau_j)\) and \(Y \in RO(X, \tau_i)\) then \(A \cap Y \in (i, j)\-\xi O(X)\).
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Proof. Let \( A \in (i, j)-\xi O(X) \), then \( A \in \tau_j(X) \) and \( A = \cup F_\alpha \), where \( F_\alpha \in i-SC(X) \) for each \( \alpha \), then \( A \cap Y = \cup F_\alpha \cap Y = \cup (F_\alpha \cap Y) \), since \( Y \in RO(X, \tau_j) \), then \( Y \) is \( j \)-open, by Theorem 2.6 \( A \cap Y \in \tau_j(X) \) and since \( Y \in RO(X, \tau_i) \) then \( Y \in i-SC(X) \) and hence \( F_\alpha \cap Y \in i-SC(X) \), for each \( \alpha \), therefore by Corollary 3.2 , \( A \cap Y \in (i, j)-\xi O(X) \).

Proposition 3.15 Let \( A \) and \( Y \) be any subsets of a bitopological space \( X \), if \( A \in (i, j)-\xi O(X) \) and \( Y \) is regular semi-open in \( \tau_i \) and \( \tau_j \), then \( A \cap Y \in (i, j)-\xi O(Y) \)

Proof. Let \( A \in (i, j)-\xi O(X) \), then \( A \in \tau_j(X) \) and \( A = \cup F_\alpha \) where \( F_\alpha \in i-SC(X) \) for each \( \alpha \), then \( A \cap Y = \cup F_\alpha \cap Y = \cup (F_\alpha \cap Y) \), since \( Y \in RSO(X, \tau_j) \), then \( Y \in j-SO(X) \) and by Theorem 2.6, \( A \cap Y \in \tau_j(Y) \) and since \( Y \in RSO(X, \tau_i) \) then \( Y \in i-SC(X) \) and hence \( F_\alpha \cap Y \in i-SC(X) \) for each \( \alpha \), since \( F_\alpha \cap Y \subseteq Y \) and \( F_\alpha \cap Y \in i-SC(X) \) for each \( \alpha \), then by Theorem 2.6, \( F_\alpha \cap Y \in i-SC(Y) \) therefore by Corollary 3.2 \( A \cap Y \in (i, j)-\xi O(Y) \).

Proposition 3.16 If \( Y \) is an \( i \)-open and \( j \)-open subspace of a bitopological space \( X \) and \( A \in (i, j)-\xi O(X) \), then \( A \cap Y \in (i, j)-\xi O(Y) \)

Proof. Let \( A \in (i, j)-\xi O(X) \), then \( A \in \tau_j(X) \) and \( A = \cup F_\alpha \) where \( F_\alpha \in i-SC(X) \) for each \( \alpha \), then \( A \cap Y = \cup F_\alpha \cap Y = \cup (F_\alpha \cap Y) \), since \( Y \) is \( j \)-open subspace of \( X \) then \( Y \in j-SO(X) \) and hence by Theorem 2.6 \( A \cap Y \in \tau_j(Y) \), and since \( Y \) is an \( i \)-open subspace of \( X \) then by Theorem 2.6 \( F_\alpha \cap Y \in i-SC(Y) \) for each \( \alpha \) then by Corollary 3.2 \( A \cap Y \in (i, j)-\xi O(Y) \).

From the above proposition we obtain the following corollary:

Corollary 3.17 If either \( Y \in RSO(X, \tau_j) \) and \( Y \in RSO(X, \tau_i) \) or \( Y \) is an \( i \)-open and \( j \)-open subspace of a bitopological space \( X \), and \( A \in (i, j)-\xi O(X) \), then \( A \cap Y \in (i, j)-\xi O(Y) \)

The following result shows that any union of \( (i, j)-\xi O(X) \) sets in bitopological space \( (X, \tau_1, \tau_2) \) is \( (i, j)-\xi O(X) \).

Proposition 3.18 Let \( \{A_\lambda : \lambda \in \Delta\} \) be family of \( (i, j)-\xi \)-open sets in bitopological space \( (X, \tau_1, \tau_2) \), then \( \cup \{A_\lambda : \lambda \in \Delta\} \) is an \( (i, j)-\xi \)-open set.

Proof. Let \( \{A_\lambda : \lambda \in \Delta\} \) be family of \( (i, j)-\xi \)-open sets in bitopological space \( (X, \tau_1, \tau_2) \). Since \( A_\lambda \) is \( j \)-open for each \( \lambda \in \Delta \) then \( \cup \{A_\lambda : \lambda \in \Delta\} \) is \( j \)-open set in a space \( X \). Suppose that \( x \in \cup A_\lambda \), this implies that there exist \( \lambda_0 \in \Delta \) such that \( x \in A_{\lambda_0} \) and since \( A_{\lambda_0} \) is \( (i, j)-\xi \)-open set, so there exists \( i \)-semi-closed set \( F \) in \( X \) such that \( x \in F \subseteq A_{\lambda_0} \subseteq \cup A_\lambda \) for all \( \lambda \in \Delta \). Therefore, \( \cup \{A_\lambda : \lambda \in \Delta\} \) is an \( (i, j)-\xi \)-open set.

The following result shows that finite intersection of \( (i, j)-\xi O(X) \) sets in bitopological space \( (X, \tau_1, \tau_2) \) is \( (i, j)-\xi O(X) \).
Proposition 3.19  Any finite intersection of \((i, j)\)-\(\xi\)-open sets in bitopological space \((X, \tau_1, \tau_2)\), is an \((i, j)\)-\(\xi\)-open set.

Proof. Let \(A_i\) be \((i, j)\)-\(\xi\)-open for \(i = 1, 2, \ldots, n\), in bitopological space \((X, \tau_1, \tau_2)\). Then \(\cap A_i\) is \(j\)-open in a space \(X\). Let \(x \in \cap A_i\), then \(x \in A_i\) for \(i = 1, 2, \ldots, n\), but \(A_i\) is \((i, j)\)-\(\xi\)-open, so there exists semi-closed \(F_i\) for each \(i = 1, 2, \ldots, n\), such that \(x \in F_i \subseteq A_i\). This implies that \(x \in \cap F_i \subseteq \cap A_i\). Therefore, \(\cap A_i\) is an \((i, j)\)-\(\xi\)-open set. Hence, the family \((i, j)\)-\(\xi\)-open subset of \((X, \tau_1, \tau_2)\) forms a bitopology on \(X\).

4  On \((i, j)\)-\(\xi\)- operators

Definition 4.1  A subset \(N\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-\(\xi\)-neighbourhood of a subset \(A\) of \(X\) if there exists an \((i, j)\)-\(\xi\)-open set \(U\) such that \(A \subseteq U \subseteq N\). When \(A = \{x\}\), we say that \(N\) is \((i, j)\)-\(\xi\)- neighbourhood of \(x\).

Definition 4.2  A point \(x \in X\) is said to be an \((i, j)\)-\(\xi\)-interior point of \(A\) if there exists an \((i, j)\)-\(\xi\)- open set \(U\) containing \(x\) such that \(U \subseteq A\). The set of all \((i, j)\)-\(\xi\)- interior points of \(A\) is said to be \((i, j)\)-\(\xi\)-interior of \(A\) and it is denoted by \((i, j)\)-\(\xi\)-Int\((A)\)

Proposition 4.3  Let \(X\) be a bitopological space and \(A \subseteq X\), \(x \in X\), then \(x\) is \((i, j)\)-\(\xi\)-interior of \(A\) if and only if \(A\) is an \((i, j)\)-\(\xi\)-neighbourhood of \(x\).

Proposition 4.4  A subset \(G\) of a bitopological space \(X\) is \((i, j)\)-\(\xi\)-open if and only if it is an \((i, j)\)-\(\xi\)-neighbourhood of each of its points .

Proposition 4.5  Let \(A\) be any subset of a bitopological space \(X\). If a point \(x\) in the \((i, j)\)-\(\xi\)-Int\((A)\), then there exists an \(i\)-semi-closed set \(F\) of \(X\) containing \(x\) and \(F \subseteq A\).

Proof. Suppose that \(x \in (i, j)\)-\(\xi\)-Int\((A)\), then there exists an \((i, j)\)-\(\xi\)-open set \(U\) of \(X\) containing \(x\) such that \(x \in U \subseteq A\). Since \(U\) is an \((i, j)\)-\(\xi\)-open set, so there exists an \(i\)-semi-closed set \(F\) such that \(x \in F \subseteq U \subseteq A\). Hence, \(x \in F \subseteq A\).

Some properties of \((i, j)\)-\(\xi\)-interior operators on a set are given in the following:

Theorem 4.6  For any subsets \(A\) and \(B\) of a bitopological space \(X\), the following statements are true:

1. The \((i, j)\)-\(\xi\)-interior of \(A\) is the union of all \((i, j)\)-\(\xi\)-open sets contained in \(A\).
2. \((i,j)\)-\(\xi\)-Int\((A)\) is an \((i,j)\)-\(\xi\)-open set in \(X\) contained in \(A\).

3. \((i,j)\)-\(\xi\)-Int\((A)\) is the largest \((i,j)\)-\(\xi\)-open set in \(X\) contained in \(A\).

4. \(A\) is an \((i,j)\)-\(\xi\)-open set if and only if \(A = (i,j)\)-\(\xi\)-Int\((A)\)

5. \((i,j)\)-\(\xi\)-Int\((\phi)\) = \(\phi\).

6. \((i,j)\)-\(\xi\)-Int\((X)\) = \(X\)

7. \((i,j)\)-\(\xi\)-Int\((A)\) \(\subseteq\) \(A\).

8. If \(A \subseteq B\), the \((i,j)\)-\(\xi\)-Int\((A)\) \(\subseteq\) \((i,j)\)-\(\xi\)-Int\((B)\).

9. \((i,j)\)-\(\xi\)-Int\((A)\) \(\cap\) \((i,j)\)-\(\xi\)-Int\((B)\) = \((i,j)\)-\(\xi\)-Int\((A \cap B)\).

10. \((i,j)\)-\(\xi\)-Int\((A)\) \(\cup\) \((i,j)\)-\(\xi\)-Int\((B)\) \(\subseteq\) \((i,j)\)-\(\xi\)-Int\((A \cup B)\).

**Proof.** Straightforward.

In general \((i,j)\)-\(\xi\)Int\((A)\) \(\cup\) \((i,j)\)-\(\xi\)Int\((B)\) \(\neq\) \((i,j)\)-\(\xi\)Int\((A \cup B)\) as it shown in the following example:

**Example 4.7** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}\) and \(\tau_2 = \{\phi, \{b, c\}, X\}\), then \((i,j)\)-\(\xi\)O\((X)\) = \(\{\phi, \{b, c\}, X\}\) if we take \(A = \{a, b\}\) and \(B = \{b, c\}\), then \((i,j)\)-\(\xi\)Int\((A)\) = \(\phi\), and \((i,j)\)-\(\xi\)Int\((B)\) = \(\{b, c\}\), \((i,j)\)-\(\xi\)Int\((A \cup B)\) = \((i,j)\)-\(\xi\)Int\((X)\) = \(X\).

In general \((i,j)\)-\(\xi\)Int\((A)\) \(\subseteq\) \(j\)-Int\((A)\), but \((i,j)\)-\(\xi\)Int\((A)\) \(\neq\) \(j\)-Int\((A)\), which is shown in the following example:

**Example 4.8** Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}\) and \(\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}\), then \((i,j)\)-\(\xi\)O\((X)\) = \(\{\phi, \{b, c\}, X\}\), if we take \(A = \{a\}\), then \((i,j)\)-\(\xi\)Int\((A)\) = \(\phi\), but \(j\)-Int\((A)\) = \(A\). Hence \((i,j)\)-\(\xi\)Int\((A)\) \(\neq\) \(j\)-Int\((A)\).

**Definition 4.9** The intersection of all \((i,j)\)-\(\xi\)-closed set containing \(F\) is called the \((i,j)\)-\(\xi\)-closure of \(F\) and we denoted it by \((i,j)\)-\(\xi\)Cl\((F)\)

**Corollary 4.10** Let \(F\) be any subset of a space \(X\). A point \(x\) in \(X\) is in the \((i,j)\)-\(\xi\)-closed of \(F\) if and only if \(F \cap U \neq \phi\) for every \((i,j)\)-\(\xi\)-open set \(U\) containing \(x\).

**Proposition 4.11** Let \(A\) be any subset of a bitopological space \(X\). If a point \(x\) in the \((i,j)\)-\(\xi\)-closure of \(A\), then \(F \cap A \neq \phi\) for every \(i\)-semi-closed set \(F\) of \(X\) containing \(x\).
Proof. Suppose that $x \in (i,j)\text{-}\xi cl(A)$, then by Corollary 4.10, $A \cap U \neq \emptyset$ for every $(i,j)\text{-}\xi$-open set $U$ of $X$ containing $x$. Since $U$ is an $(i,j)\text{-}\xi$-open set, so there exists an $i$-semi-closed set $F$ containing $x$, such that $F \subseteq U$. Hence, $F \cap A \neq \emptyset$.

Some properties of $(i,j)\text{-}\xi$-closure operators on a set are given.

**Theorem 4.12** For any subsets $A$ and $B$ of a bitopological space $X$, the following statements are true:

1. The $(i,j)\text{-}\xi$-closure of $A$ is the intersection of all $(i,j)\text{-}\xi$-closed sets containing $A$.
2. $(i,j)\text{-}\xi cl(A)$ is an $(i,j)\text{-}\xi$-closed set in $X$ containing $A$.
3. $(i,j)\text{-}\xi cl(A)$ is the smallest $(i,j)\text{-}\xi$-closed set in $X$ containing $A$.
4. $A$ is an $(i,j)\text{-}\xi$-closed set if and only if $A = (i,j)\text{-}\xi cl(A)$
5. $(i,j)\text{-}\xi cl(\emptyset) = \emptyset$.
6. $(i,j)\text{-}\xi cl(X) = X$
7. $A \subseteq (i,j)\text{-}\xi cl(A)$.
8. If $A \subseteq B$, then $(i,j)\text{-}\xi cl(A) \subseteq (i,j)\text{-}\xi cl(B)$.
9. $(i,j)\text{-}\xi cl(A) \cap (i,j)\text{-}\xi cl(B) \subseteq (i,j)\text{-}\xi cl(A \cap B)$.
10. $(i,j)\text{-}\xi cl(A) \cup (i,j)\text{-}\xi cl(B) = (i,j)\text{-}\xi Int(A \cup B)$.

**Proof.** Directly from Definition 4.9.

**Corollary 4.13** For any subset $A$ of a bitopological space $X$, then the following statements are true:

1. $X \setminus ((i,j)\text{-}\xi Cl(A)) = (i,j)\text{-}\xi Int(X \setminus A)$
2. $X \setminus ((i,j)\text{-}\xi Int(A)) = (i,j)\text{-}\xi Cl(X \setminus A)$
3. $(i,j)\text{-}\xi Int(A) = X \setminus ((i,j)\text{-}\xi Cl(X \setminus A))$

It is clear that $j\text{-}Cl(F) \subseteq (i,j)\text{-}\xi Cl(F)$, the converse may be false as shown in the following example:

**Example 4.14** Considering a space $X$ as defined in Example 3.3, if we take $F = \{a,b\}$, then $j\text{-}Cl(F) = \{a,b\}$, and $(i,j)\text{-}\xi Cl(F) = X$, this shows that $(i,j)\text{-}\xi Cl(F)$ is not a subset of $j\text{-}Cl(F)$. 

Corollary 4.15 If $A$ is any subset of a bitopological space $X$, then $(i,j)$-$\xi\text{Int}(A) \subseteq j\text{-Int}(A) \subseteq A \subseteq j\text{-Cl}(A) \subseteq (i,j)$-$\xi\text{Cl}(A)$.

**Definition 4.16** Let $A$ be a subset of a bitopological space $X$, A point $x \in X$ is said to be $(i,j)$-$\xi$-limit point of $A$ if for each $(i,j)$-$\xi$-open set $U$ containing $x$, $U \cap (A \setminus \{x\}) \neq \emptyset$. The set of all $(i,j)$-$\xi$-limit point of $A$ is called $(i,j)$-$\xi$-derived set of $A$ and is denoted by $(i,j)$-$\xi D(A)$.

In general It is clear that $(i,j)$-$\xi D(A) \subseteq j\text{-}D(A)$, but the converse may not be true as shown in the following example:

**Example 4.17** Considering the space $X$ as defined in Example 3.3 if we take $A = \{a,c\}$, So $(i,j)$-$\xi D(A) = \{a\}$ and $j\text{-}D(A) = \{b\}$, hence $(i,j)$-$\xi D(A)$ is not a subset of $j\text{-}D(A)$.

**Theorem 4.18** Let $X$ be a bitopological space and $A$ be a subset of $X$, then $A \cup (i,j)$-$\xi D(A)$ is $(i,j)$-$\xi$-closed.

**Proof.** Let $x \notin A \cup (i,j)$-$\xi D(A)$. This implies that $x \notin A$ and $x \notin (i,j)$-$\xi D(A)$. Since $x \notin (i,j)$-$\xi D(A)$, then there exists an $(i,j)$-$\xi$-open $U$ of $X$ which contains no point of $A$ other than $x$, but $x \notin A$, so $U$ contains no point of $A$, which implies that $U \subseteq X \setminus A$. Again, $U$ is an $(i,j)$-$\xi$-open set for each of its points. But as $U$ does not contain any point of $A$, no point of $U$ can be $(i,j)$-$\xi$-limit point of $A$. Therefore, no point of $U$ can belong to $(i,j)$-$\xi D(A)$. This implies that $U \subseteq X \setminus (i,j)$-$\xi DA$. Hence, it follows that $x \in X \setminus A \cap (X \setminus (i,j)$-$\xi D(A)) = X \setminus (A \cup (i,j)$-$\xi D(A))$, Therefore $A \cup (i,j)$-$\xi D(A)$ is an $(i,j)$-$\xi$-closed. Hence $(i,j)$-$\xi d(A) \subseteq A \cup (i,j)$-$\xi D(A)$.

**Corollary 4.19** If a subset $A$ of a bitopological space $X$ is $(i,j)$-$\xi$-closed, then $A$ contains the set of all its $(i,j)$-$\xi$-limit points.

**Theorem 4.20** Let $A$ be any subset of a bitopological space $X$, then the following statements are true:

1. $((i,j)$-$\xi D((i,j)$-$\xi D(A))) \setminus A \subseteq (i,j)$-$\xi D(A)$
2. $(i,j)$-$\xi D(A \cup (i,j)$-$\xi D(A)) \subseteq A \cup (i,j)$-$\xi D(A)$

**Proof.** Obvious.

**Theorem 4.21** Let $X$ be a bitopological space and $A$ be a subset of $X$, then: $(i,j)$-$\xi \text{Int}(A) = A \setminus ((i,j)$-$\xi D(X \setminus A))$

**Proof.** Obvious.
Definition 4.22 If $A$ is a subset of a bitopological space $X$, then $(i,j)$-\(\xi\)-boundary of $A$ is $(i,j)$-\(\xi\)\(Cl(A) \cap ((i,j)\cdot\xi Int(A))^c\), and denoted by $(i,j)$-\(\xi\)\(Bd(A)\)

Theorem 4.23 For any subset $A$ of a bitopological space $X$, the following statements are true:

1. $(i,j)$-\(\xi\)\(Bd(A) = (i,j)$-\(\xi\)\(Bd(X \setminus A)\)

2. $A \in (i,j)$-\(\xi\)\(O(X)\) if and only if $(i,j)$-\(\xi\)\(Bd(A) \subseteq X \setminus A\), that is $A \cap (i,j)$-\(\xi\)\(Bd(A)\) = φ.

3. $A \in (i,j)$-\(\xi\)\(C(X)\) if and only if $(i,j)$-\(\xi\)\(Bd(A) \subseteq A\).

4. $(i,j)$-\(\xi\)\(Bd((i,j)$-\(\xi\)\(Bd(A)) \subseteq (i,j)$-\(\xi\)\(Bd(A)\)

5. $(i,j)$-\(\xi\)\(Bd((i,j)$-\(\xi\)\(Int(A)) \subseteq (i,j)$-\(\xi\)\(Bd(A)\)

6. $(i,j)$-\(\xi\)\(Bd((i,j)$-\(\xi\)\(Cl(A)) \subseteq (i,j)$-\(\xi\)\(Bd(A)\)

7. $(i,j)$-\(\xi\)\(Int(A) = A \setminus ((i,j)$-\(\xi\)\(Bd(A))\)

Proof. Directly from Definition 4.22.

Theorem 4.24 Let $A$ be a subset of a bitopological space $X$, then $(i,j)$-\(\xi\)\(Bd(A) = \phi\) if and only if $A$ is both $(i,j)$-\(\xi\)-open and $(i,j)$-\(\xi\)-closed set.

Proof. Let $A$ be $(i,j)$-\(\xi\)-open and $(i,j)$-\(\xi\)-closed, then $A = (i,j)$-\(\xi\)\(Int(A) = (i,j)$-\(\xi\)\(cl(A)\), hence by Definition 4.22 $A = (i,j)$-\(\xi\)\(Cl(A) - (i,j)$-\(\xi\)\(Int(A))\) = φ.

References


(i,j)-\xi-Open sets in bitopological spaces


