An Extension in the Domain of an n-Norm
Defined on the Space of p-Summable Sequences

J.K. Srivastava\(^1\) and Pradeep Kumar Singh\(^2\)

\(^1,\(^2\)Department of Mathematics and Statistics
D.D.U Gorakhpur University, Gorakhpur
Uttar Pradesh, India – 273009
E-mail: jks_ddugu@yahoo.com
E-mail: pradeep3789@gmail.com

(Received: 2-8-15 / Accepted: 17-2-16)

Abstract
In [6], we have already studied the space of \(p\)-summable sequences (i.e. \(l^p\)) as an n-normed space by defining a new n-norm \(\|\cdot,\cdots,\cdot\|_p\) on it. In this paper, we shall extend the domain of the definition of n-norm \(\|\cdot,\cdots,\cdot\|_p\) from the space \(l^p\) to different vector subspaces of the vector space \(l^\infty\) containing \(l^p\). Further, we shall discuss on their derived norms also.

Keywords: \(l^p\) space, \(l^\infty\) space, norms, n-norms, derived norms.

1 Introduction

In [9], Gähler initialy introduced the theory of 2-norm, defined on a linear space, while that of n-norm can be found in [3] and has been studied in many papers such as [1, 2, 5]. Research works on sequence spaces regarded as n-normed space can be found in [1, 4, 6, 7, 8].

Definition 1.1. Let \(X\) be a vector space over \(\mathbb{K}(=\mathbb{R} \text{ or } \mathbb{C})\) of dimension \(d \geq n(n \geq 2)\). A non-negative real valued function \(\|.,\cdots,\|\) defined on \(X^n\) satisfying the four conditions:

\[(N1) \|x^1, x^2, \cdots, x^n\| = 0 \text{ if and only if } x^1, x^2, \cdots, x^n \text{ are linearly dependent;}\]
(N2) \(\|x_1, x_2, \ldots, x^n\|\) is invariant under the permutation of \(x_1, x_2, \ldots, x^n\);

(N3) \(\|\alpha \cdot x_1, x_2, \ldots, x^n\| = |\alpha| \cdot \|x_1, x_2, \ldots, x^n\|;\)

(N4) \(\|x^n + y, x^2, \ldots, x^n\| \leq \|x_1, x^2, \ldots, x^n\| + \|y, x^2, \ldots, x^n\|;\)

for all \(x_1, x_2, \ldots, x^n, y \in X\) and for all \(\alpha \in \mathbb{K}\), is called an \textit{n-norm on X}, and the pair \((X, \|\cdot\|_n)\) is called an \textit{n-normed space}.

**Definition 1.2.** Two \(n\)-norms \(\|\cdot\|_1\) and \(\|\cdot\|_2\) defined on a linear space \(X\) are said to be 
\(\textit{equivalent}\) if and only if \(\exists K_1, K_2 > 0\) such that:

\[K_1 \cdot \|x^1, x_2, \ldots, x^n\|_1 \leq \|x^1, x_2, \ldots, x^n\|_2 \leq K_2 \cdot \|x^1, x_2, \ldots, x^n\|_1\]

for all \(x_1, x_2, \ldots, x^n \in X\).

**Definition 1.3.** Let \((X, \|\cdot\|)\) is an \(n\)-normed space and \(\{e^1, \ldots, e^n\}\) is a set of linearly independent vectors in \(X\) then both of the functions \(\|\cdot\|_\infty\) and \(\|\cdot\|_d\) define a norm on \(X\) (known as \textit{derived norm} with respect to the set \(\{e^1, \ldots, e^n\}\)) and they are equivalent, where

1. \(\|x\|_\infty = \max\{\|x, e^{t_1}, \ldots, e^{t_{n-1}}\| : \{t_1, \ldots, t_{n-1}\} \subseteq \{1, \ldots, n\}\}\)

2. \(\|x\|_d = \left(\sum_{\{t_1, \ldots, t_{n-1}\} \subseteq \{1, \ldots, n\}} \|x, e^{t_1}, \ldots, e^{t_{n-1}}\|^q\right)^{1/q}; 1 \leq q < \infty.\)

In this paper, we shall focus on \(l^p\) and \(l^\infty\), where

\[l^p = \left\{x = (x_i)_{i=0}^\infty : \sum_{i=0}^\infty |x_i|^p < \infty \text{ where } x_i \in \mathbb{K}, \text{ for all } i = 0, 1, 2, \ldots\right\}\]

and

\[l^\infty = \left\{x = (x_i)_{i=0}^\infty : \sup_{0 \leq i < \infty} |x_i| < \infty\right\}.\]

As we know that \((l^p, \|\cdot\|_p)\) is a Banach space where \(\|x\|_p = \{\sum_{i=0}^\infty |x_i|^p\}^{1/p}\) as well as \((l^\infty, \|\cdot\|_\infty)\) also becomes a Banach space with norm \(\|x\|_\infty = \sup_{0 \leq i < \infty} |x_i|\); while \((l^p, \|\cdot\|_\infty)\) forms simply a normed space.

In [6], for our convenience and need we have denoted the set of whole numbers as \(\mathbb{N} = \{0, 1, 2, \ldots\}\), which is also considered as a sequence \(\mathbb{N} = (0, 1, 2, \ldots)\). Further, we have denoted the sequence \(\mathbb{N} = (0, 1, 2, \ldots)\) in the form of \textit{n-consecutive terms notation} as

\[\mathbb{N} = (0, 1, 2, \ldots) = (nl, nl + 1, \ldots, nl + (n - 1))_{i=0}^\infty\]

and expressed as

\[\mathbb{N} = (n \cdot 0 = 0, n \cdot 0 + 1 = 1, \ldots, n \cdot 0 + (n - 1) = n - 1,\]

\[n \cdot 1 = n, n \cdot 1 + 1 = n + 1, \ldots, n \cdot 1 + (n - 1) = 2n - 1, \ldots)\]
Let $\tilde{N} = (\tilde{m}_{nk}, \tilde{m}_{nk+1}, \ldots, \tilde{m}_{nk+(n-1)})_{k=0}^{\infty}$ be a rearrangement of the sequence $N$. Then for any $n$ vectors

$$x^t = (x_{nl}^t, x_{nl+1}^t, \ldots, x_{nl+(n-1)}^t)_{l=0}^{\infty} \in \ell^p; \quad t = 1, 2, \ldots, n$$

the $n$ vectors

$$\tilde{x}^t = (x_{nk}^t, x_{nk+1}^t, \ldots, x_{nk+(n-1)}^t)_{k=0}^{\infty}; \quad t = 1, 2, \ldots, n$$

are called parallel rearrangements of $x^1, x^2, \ldots, x^n$ respectively.

In [6], we have observed that $(\ell^p, \|\cdot\|_p)$ is an $n$-normed space, but not complete where

$$\|x^1, x^2, \ldots, x^n\|_p = \sup\{|\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n| : \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n \text{ are parallel rearrangements of } x^1, x^2, \ldots, x^n \text{ respectively}\}.$$  \hspace{1cm} (1)

and

$$|\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n| = \left(\sum_{k=0}^{\infty} \left| \begin{array}{cccc} x_{nk}^1 & x_{nk+1}^1 & \cdots & x_{nk+(n-1)}^1 \\ x_{nk}^2 & x_{nk+1}^2 & \cdots & x_{nk+(n-1)}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{nk}^n & x_{nk+1}^n & \cdots & x_{nk+(n-1)}^n \end{array} \right| \right)^{1/p}. \hspace{1cm} (2)$$

Moreover, we have

$$\|x^1, x^2, \ldots, x^n\|_p \leq n!\|x^{\pi_1}\|_p \cdot \|x^{\pi_2}\|_{p/\infty} \cdots \|x^{\pi_n}\|_{p/\infty}; \hspace{1cm} (3)$$

where $\{\pi_1, \pi_2, \ldots, \pi_n\}$ is any permutation of $\{1, 2, \ldots, n\}$ and $\|x^{\pi_i}\|_{p/\infty}$ means either $\|x^{\pi_i}\|_p$ or $\|x^{\pi_i}\|_\infty$ is taken freely.

In [1], Malčeski investigated that the function

$$\|x^1, x^2, \ldots, x^n\|_\infty := \sup_{i_1, \ldots, i_n} \left| \begin{array}{cccc} x_{i_1}^1 & x_{i_2}^1 & \cdots & x_{i_n}^1 \\ x_{i_1}^2 & x_{i_2}^2 & \cdots & x_{i_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_1}^n & x_{i_2}^n & \cdots & x_{i_n}^n \end{array} \right|$$  \hspace{1cm} (4)

defines an $n$-norm on $\ell^\infty$, where $i_1, \ldots, i_n \in \mathbb{N}$. But $\ell^p$ is a subspace of $\ell^\infty$ therefore we can show that $\|\cdot, \|_\infty$ forms an $n$-norm on $\ell^p$ also.

In [6, 7], we have already proved that these two $n$-norms $\|\cdot, \|_p$ and $\|\cdot, \|_\infty$ defined on $\ell^p$ are non-equivalent. Where as their derived norms with respect to the linearly independent set $\{e^1, \ldots, e^n\}$ are equivalent and equivalent to $\|\cdot\|_\infty$, where $e^t = (\delta^t_i)_{i=0}^{\infty}$. For details see [7].
2 Results

Here, our aim is to investigate the possibilities of extending the domain of definition of the n-norm \[\|\cdot, \ldots, \cdot\|_p\] from the domain \(l^p\) to other sequence spaces containing \(l^p\).

If we take any arbitrary \(z \in l^\infty\) and define

\[l^p + [z] = \{x + \alpha z : x \in l^p \quad \text{and} \quad \alpha \in \mathbb{K}\},\]

then obviously \(l^p + [z]\) is a subspace of \((l^\infty, \|\cdot\|_\infty)\).

**Theorem 2.1.** The function \[\|\cdot, \ldots, \cdot\|_p\] defines an n-norm on \(l^p + [z]\), for every arbitrary \(z \in l^\infty\). Moreover,

\[
\|x_1^1 + z_1^1, x_2^2 + z_2^2, \ldots, x_n^n + z_n^n\|_p \leq n! \|x_1^{\pi_1}\|_p \cdot \|x_2^{\pi_2} + z_2^{\pi_2}\|_\infty \cdots \|x_n^{\pi_n} + z_n^{\pi_n}\|_\infty
+ n! \|x_2^{\pi_2}\|_p \cdot \|x_2^{\pi_3} + z_2^{\pi_3}\|_\infty \cdots \|x_n^{\pi_n} + z_n^{\pi_n}\|_\infty \cdot \|z_1^{\pi_1}\|_\infty
\]

(5)

for every scalar multiples \(z^1, z^2, \ldots, z^n\) of \(z\) and any permutation \(\{\pi_1, \pi_2, \ldots, \pi_n\}\) of \(\{1, 2, \ldots, n\}\).

**Proof:** The proof is similar to proving that \[\|\cdot, \ldots, \cdot\|_p\] defines an n-norm on \(l^p\), as we have done in [6]. Here we are going to establish the inequality only. Let \(x_1^1 + z_1^1, x_2^2 + z_2^2, \ldots, x_n^n + z_n^n \in l^p + [z]\) and \(N = (m_{nk}, m_{nk+1}, \ldots, m_{nk+(n-1)})^\infty_{k=0}\) be a rearrangement of the sequence \(N\). By the properties of determinant we have

\[
\begin{vmatrix}
\frac{x_1^{\pi_1}}{m_{nk}} + \frac{z_1^{\pi_1}}{m_{nk}} & \frac{x_1^{\pi_2}}{m_{nk}} + \frac{z_1^{\pi_2}}{m_{nk}} & \cdots & \frac{x_1^{\pi_{n-1}}}{m_{nk}} + \frac{z_1^{\pi_{n-1}}}{m_{nk}} & \frac{x_1^{\pi_n}}{m_{nk}} + \frac{z_1^{\pi_n}}{m_{nk}} \\
\frac{x_2^{\pi_1}}{m_{nk}} + \frac{z_2^{\pi_1}}{m_{nk}} & \frac{x_2^{\pi_2}}{m_{nk}} + \frac{z_2^{\pi_2}}{m_{nk}} & \cdots & \frac{x_2^{\pi_{n-1}}}{m_{nk}} + \frac{z_2^{\pi_{n-1}}}{m_{nk}} & \frac{x_2^{\pi_n}}{m_{nk}} + \frac{z_2^{\pi_n}}{m_{nk}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{x_n^{\pi_1}}{m_{nk}} + \frac{z_n^{\pi_1}}{m_{nk}} & \frac{x_n^{\pi_2}}{m_{nk}} + \frac{z_n^{\pi_2}}{m_{nk}} & \cdots & \frac{x_n^{\pi_{n-1}}}{m_{nk}} + \frac{z_n^{\pi_{n-1}}}{m_{nk}} & \frac{x_n^{\pi_n}}{m_{nk}} + \frac{z_n^{\pi_n}}{m_{nk}} \\
\end{vmatrix} = \det
\begin{vmatrix}
\frac{x_1^{\pi_1}}{m_{nk} + 1} & \frac{x_1^{\pi_2}}{m_{nk} + 1} & \cdots & \frac{x_1^{\pi_{n-1}}}{m_{nk} + 1} & \frac{x_1^{\pi_n}}{m_{nk} + 1} \\
\frac{x_2^{\pi_1}}{m_{nk} + 1} & \frac{x_2^{\pi_2}}{m_{nk} + 1} & \cdots & \frac{x_2^{\pi_{n-1}}}{m_{nk} + 1} & \frac{x_2^{\pi_n}}{m_{nk} + 1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{x_n^{\pi_1}}{m_{nk} + 1} & \frac{x_n^{\pi_2}}{m_{nk} + 1} & \cdots & \frac{x_n^{\pi_{n-1}}}{m_{nk} + 1} & \frac{x_n^{\pi_n}}{m_{nk} + 1} \\
\end{vmatrix}
\]

As we know that the expansion of a determinant of order \(n\) consists of sum of \(n!\) terms, among which each term is again a product of \(n\) terms, therefore
breaking the last determinant along second row and using Minkowski inequality; (Keeping the fact in mind that \( z^1, z^2 \) are linearly dependent.) by definition (2) we have
\[
\|z^1 + z^2, x^2 + \bar{z}^2, \ldots, x^n + \bar{z}^n \|_p \leq n!\|x^1\|_p \cdot \|x^2 + z^2\|_\infty \cdots \|x^n + z^n\|_\infty \\
+ n!\|x^2\|_p \cdot \|x^2 + z^2\|_\infty \cdots \|x^n + z^n\|_\infty \cdot \|z^1\|_\infty .
\]

Above is true for any rearrangement \( \bar{N} \) of \( N \) and breaking determinant along any row, therefore
\[
\|x^1 + z^1, x^2 + z^2, \ldots, x^n + z^n\|_p \leq n!\|x^{p_1}\|_p \cdot \|x^{p_2} + z^{p_2}\|_\infty \cdots \|x^{p_n} + z^{p_n}\|_\infty \\
+ n!\|x^{p_2}\|_p \cdot \|x^{p_3} + z^{p_3}\|_\infty \cdots \|x^{p_n} + z^{p_n}\|_\infty \cdot \|z^{p_1}\|_\infty .
\]

In general, if we take any \( n - 1 \) arbitrary vectors \( v^1, v^2, \ldots, v^{n-1} \in l^\infty \) and define
\[
l^p + [v^1] + [v^2] + \cdots + [v^{n-1}] = \\
\{ x + \alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_{n-1} v^{n-1} : x \in l^p \text{ and } \alpha_j \in \mathbb{K}; j = 1, 2, \ldots, n - 1 \}.
\]

It is clear that, \( l^p + [v^1] + [v^2] + \cdots + [v^{n-1}] \) is a subspace of \( (l^\infty, \| \cdot \|_\infty) \).

**Corollary 2.2.** The function \( \| \cdot, \cdot, \cdot, \cdot \|_p \) defines an \( n \)-norm on \( l^p + [v^1] + [v^2] + \cdots + [v^{n-1}] \) for \( v^1, v^2, \ldots, v^{n-1} \in l^\infty \). Moreover,
\[
\|x^1 + z^1, x^2 + z^2, \ldots, x^n + z^n\|_p \leq n!\|x^{p_1}\|_p \cdot \|x^{p_2} + z^{p_2}\|_\infty \cdots \|x^{p_n} + z^{p_n}\|_\infty \\
+ n!\|x^{p_2}\|_p \cdot \|x^{p_3} + z^{p_3}\|_\infty \cdots \|x^{p_n} + z^{p_n}\|_\infty \cdot \|z^{p_1}\|_\infty \\
+ n!\|x^{p_3}\|_p \cdot \|x^{p_4} + z^{p_4}\|_\infty \cdots \|x^{p_n} + z^{p_n}\|_\infty \cdot \|z^{p_1}\|_\infty \cdot \|z^{p_2}\|_\infty \\
+ \cdots + n!\|x^{p_n}\|_p \cdot \|z^{p_1}\|_\infty \cdots \|z^{p_{n-1}}\|_\infty
\]

where each of the vectors \( z^1, z^2, \ldots, z^n \) is linear combination of \( v^1, v^2, \ldots, v^{n-1} \).

**Proof:** The proof can be done similar to the proof of theorem 2.1 by breaking the determinant successively along every row.

Obviously, the function \( \| \cdot, \cdot, \cdot, \cdot \|_\infty \) defines an \( n \)-norm on \( l^p + [v^1] + [v^2] + \cdots + [v^{n-1}] \) also. Since, the \( n \)-norms \( \| \cdot, \cdot, \cdot, \cdot \|_p \) and \( \| \cdot, \cdot, \cdot, \cdot \|_\infty \) are non-equivalent on \( l^p \) (for details see [7]), therefore they must be non-equivalent on \( l^p + [v^1] + [v^2] + \cdots + [v^{n-1}] \) also. But, it is easy to show that their derived norms with respect to the linearly independent set \( \{ e^1, \ldots, e^n : e^t = (\delta^t_i)_{i=0}^n ; \ t = 1, 2, \ldots, n \} \) are equivalent and equivalent to \( \| \cdot \|_\infty \).

Under some conditions, the function \( \| \cdot, \cdot, \cdot, \cdot \|_p \) may define an \( n \)-norm on \( l^p + [v^1] + [v^2] + \cdots + [v^n] \) for \( v^1, v^2, \ldots, v^n \in l^\infty \), before discussing such conditions, let us consider the following theorem.
Theorem 2.3. The function $\|\cdot, \cdots, \cdot\|_p$ need not be an n-norm on $l^p + [v^1] + [v^2] + \cdots + [v^n]$ for $v^1, v^2, \ldots, v^n \in l^\infty$. Moreover, the function $\|\cdot, \cdots, \cdot\|_p$ fails to be an n-norm on $l^\infty$.

Proof: Taking $n$ vectors $v^t = (v^t_i)_{i=0}^\infty \in l^\infty$; $t = 1, 2, \ldots, n$ where

$$v^t_i = \begin{cases} 1, & \text{if } i \equiv (t-1)(mod\ n) \\ 0, & \text{otherwise} \end{cases}$$

obviously, $\|v^t\|_\infty = 1$ therefore $v^t \in l^\infty$ for every $t = 1, 2, \ldots, n$. But

$$\|v^1, \cdots, v^n\|_p = \infty.$$ 

Thus the function $\|\cdot, \cdots, \cdot\|_p$ need not be an n-norm on $l^p + [v^1] + [v^2] + \cdots + [v^n]$ for $v^1, v^2, \ldots, v^n \in l^\infty$. Moreover, the function $\|\cdot, \cdots, \cdot\|_p$ fails to be an n-norm on $l^\infty$.

Now we shall investigate those circumstances under which the function $\|\cdot, \cdots, \cdot\|_p$ define an n-norm on $l^p + [v^1] + [v^2] + \cdots + [v^n]$ for $v^1, v^2, \ldots, v^n \in l^\infty$.

Lemma 2.4. If $\|x^1, x^2\|_p < \infty$ for $x^1, x^2 \in l^\infty$ then for every $x^3, \ldots, x^n \in l^\infty$

$$\|x^1, x^2, \cdots, x^n\|_p < \infty.$$ 

Moreover,

$$\|x^1, x^2, \cdots, x^n\|_p \leq \frac{n!}{2} \|x^3\|_\infty \cdots \|x^n\|_\infty \|x^1, x^2\|_p.$$ 

Proof: Let $\bar{N} = (\bar{m}_{nk}, \bar{m}_{nk+1}, \ldots, \bar{m}_{nk+(n-1)})_{k=0}^\infty$ be a rearrangement of the sequence $N$. Obviously, expanding the determinant

$$\det \begin{pmatrix} x^1_{\bar{m}_{nk}} & x^1_{\bar{m}_{nk+1}} & \cdots & x^1_{\bar{m}_{nk+(n-1)}} \\ x^2_{\bar{m}_{nk}} & x^2_{\bar{m}_{nk+1}} & \cdots & x^2_{\bar{m}_{nk+(n-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ x^n_{\bar{m}_{nk}} & x^n_{\bar{m}_{nk+1}} & \cdots & x^n_{\bar{m}_{nk+(n-1)}} \end{pmatrix}$$

in the form of the sum of the terms like

$$x^3_{\bar{m}_{nk+j_3}} \cdot x^4_{\bar{m}_{nk+j_4}} \cdots x^n_{\bar{m}_{nk+j_n}} \cdot \begin{vmatrix} x^1_{\bar{m}_{nk+j_1}} & x^1_{\bar{m}_{nk+j_2}} \\ x^2_{\bar{m}_{nk+j_1}} & x^2_{\bar{m}_{nk+j_2}} \end{vmatrix}$$

and then using the Minkowski inequality equation (2) gives

$$\|\bar{x}, \bar{x}, \ldots, \bar{x}\| = \frac{n!}{2} \|x^3\|_\infty \cdots \|x^n\|_\infty \|x^1, x^2\|_p.$$
Hence, we have the lemma.

**Example:** If we take $x^1, x^2 \in l^\infty$ as follows:

$$x^1 = (x^1_i)_{i=0}^\infty \quad \text{and} \quad x^2 = (x^2_i)_{i=0}^\infty$$

where

$$x^1_i = \begin{cases} 1 & ; i = 0, 1 \\ \frac{1}{i^{1/p}} & ; i \geq 2 \end{cases}$$

and

$$x^2_i = \begin{cases} 0 & ; i = 0 \\ 1 & ; i = 1, 2 \\ \frac{1}{(i-1)^{1/p}} & ; i \geq 3 \end{cases}$$

obviously, $x^1, x^2 \notin l^p$ but $x^1, x^2 \in l^\infty$. Whereas

$$\|x^1, x^2\|_p < \infty$$

for, let $\overline{N} = (\overline{m}_{nk}, \overline{m}_{nk+1}, \ldots, \overline{m}_{nk+(n-1)})_{k=0}^\infty$ be a rearrangement of the sequence $N$ then for every $t \in \mathbb{N}$, we have

$$\left( \sum_{k=0}^{t} \left| \text{det} \left( \begin{array}{cc} x^1_{m_{2k}} & x^1_{m_{2k+1}} \\ x^2_{m_{2k}} & x^2_{m_{2k+1}} \end{array} \right) \right|^p \right)^{1/p} \leq \left( \sum_{k=0}^{t} \left| \frac{x^1_{m_{2k}} \cdot x^2_{m_{2k+1}}}{m_{2k+1}} \right|^p \right)^{1/p} + \left( \sum_{k=0}^{t} \left| \frac{x^1_{m_{2k+1}} \cdot x^2_{m_{2k}}}{m_{2k}} \right|^p \right)^{1/p}$$

but both

$$\sum_{k=0}^{t} \left| \frac{x^1_{m_{2k}} \cdot x^2_{m_{2k+1}}}{m_{2k+1}} \right|^p \leq 2 \left( 1 + \sum_{i=1}^{\infty} \frac{1}{i^2} \right)$$

and

$$\sum_{k=0}^{t} \left| \frac{x^1_{m_{2k+1}} \cdot x^2_{m_{2k}}}{m_{2k}} \right|^p \leq 2 \left( 1 + \sum_{i=1}^{\infty} \frac{1}{i^2} \right)$$

therefore

$$\|x^1, x^2\|_p < \infty.$$

**Lemma 2.5.** If for $x^1, x^2, \ldots, x^n \in l^\infty$,

$$\|x^1, x^2, \ldots, x^n\|_p < \infty$$

then

$$\|z^1, z^2, \ldots, z^n\|_p < \infty$$

for each $z^t \; ; t = 1, 2, \ldots, n$ is linear combination of $x^1, x^2, \ldots, x^n$.

**Theorem 2.6.** If $\|v^1, \ldots, v^n\|_p < \infty$ for $v^1, v^2, \ldots, v^n \in l^\infty$, then the function $\|\cdot, \cdot, \cdot\|_p$ defines an $n$-norm on $l^p + [v^1] + [v^2] + \cdots + [v^n]$.

**Proof:** In view of lemma 2.5, the proof is similar to the proof of corollary 2.2.
References


