Quasi-Open Sets in Bispaces

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Abstract

The notions of quasi-open sets, quasi-continuity, semi-open sets and quasi-Hausdorffness were studied in [4]. Here we study the same in more general structure of a bispace and investigate how far several results as valid in a bitopological space are affected in a bispace.

Keywords: Bispaces, Quasi open sets, Quasi continuity, Semi open sets, Quasi Hausdorffness.

1 Introduction

The notion of a topological space was generalized to a σ-space (or simply space) by A.D. Alexandroff [1] weakening the union requirements. J.C. Kelly [6] introduced the idea of a bitopological space. Several works on bitopological spaces have been done in [4]. The concept of σ-space was used by Lahiri and Das [8] to generalize the notion of a bitopological space to a bispace where several ideas like pairwise Hausdorffness, pairwise bicompletness etc. were also studied. The concept of quasi-open sets and quasi-continuity were studied by M.C. Datta [4] in a bitopological space. Indeed quasi-continuity is weaker than Pervin’s continuity [10] and quasi-Hausdorffness is more general than a pairwise Hausdorffness in a bitopological space [4]. Here we have studied the ideas of quasi open sets, quasi-continuity and quasi-Hausdorffness in a bispace and investigate how far several results as valid in a bitopological space are
affected in a bispace. J. Swart [11] introduced the idea of least upper bound topology in a bitopological space whose members are called semi-open set [4]. In fact this notion is not same with the notion of semi-open set introduced by Levine[9] and [3]. In this paper we have also studied the idea of semi-open set due to M.C. Datta [4] in a bispace.

**Definition 1.1** A set \( X \) is called an Alexandroff space or simply a space if in it is chosen a system of subsets \( F \) satisfying the following axioms:

1. The intersection of a countable number of sets from \( F \) is a set in \( F \).
2. The union of a finite number of sets from \( F \) is a set in \( F \).
3. The void set \( \emptyset \) is a set in \( F \).
4. The whole set \( X \) is a set in \( F \).

Sets of \( F \) are called closed sets. Their complementary sets are called open. It is clear that instead of closed sets in the definition of the space, one may put open sets with subject to the conditions of countable summability, finite intersectibility and the condition that \( X \) and void set \( \emptyset \) should be open. The collection of all such open sets will sometimes be denoted by \( \tau \) and the space by \( (X, \tau) \). Note that, in general, \( \tau \) is not a topology as can be easily seen by taking \( X = \mathbb{R} \), the set of real numbers and \( \tau \) as the collection of all \( F_\sigma \)-sets in \( \mathbb{R} \).

**Definition 1.2** To every set \( M \) of \( (X, \tau) \) we correlate its closure \( M \), the intersection of all closed sets containing \( M \). Sometimes the closure of a set \( M \) will be denoted by \( \tau - \text{cl} \ M \) or simply \( \text{cl} \ M \) when there is no confusion about \( \tau \).

Generally the closure of a set in a space is not a closed set.

From the axioms, it easily follows that

1) \( M \cup N = \overline{M} \cup \overline{N} \);  
2) \( M \subset \overline{M} \);  
3) \( \overline{M} = \overline{\overline{M}} \);  
4) \( \overline{\emptyset} = \emptyset \).

**Definition 1.3** The interior of a set \( M \) in \( (X, \tau) \) is defined as the union of all open sets contained in \( M \) and is denoted by \( \tau - \text{int} \ M \) or \( \text{int} \ M \) when there is no confusion.

**Definition 1.4** Let \( X \) be a nonempty set. If \( \tau_1 \) and \( \tau_2 \) be two collections of subsets of \( X \) such that \( (X, \tau_1) \) and \( (X, \tau_2) \) are two spaces, then \( X \) is called a bispace and is denoted by \( (X, \tau_1, \tau_2) \).

## 2 Quasi Open Sets

Throughout our discussion, \( (X, \tau_1, \tau_2) \) or simply \( X \) stands for a bispace, \( \mathbb{R} \) stands for the set of real numbers, \( \mathbb{Q} \) stands for the set of rational numbers and sets are always subsets of \( X \) unless otherwise stated.
Definition 2.1 A subset $A$ in a bispace $(X, \tau_1, \tau_2)$ is said to be quasi-open if for every $x \in A$ there exists a $\tau_1$-open neighbourhood $U_x \subset A$ or a $\tau_2$-open neighbourhood $V_x \subset A$.

In bitopological space quasi open sets are precisely the unions of $\tau_1$-open and $\tau_2$-open sets (proposition 2.2 [4]). But in bispace quasi open sets may not be the unions of $\tau_1$-open and $\tau_2$-open sets as shown in the following example:

Example 2.2 Let $X = [0, 2] - Q$, where $Q$ is the set of rational numbers. Let $\{F_i\}$ be the collection of all countable subsets in $[0, 1] - Q$ and $\{G_i\}$ be the collection of all countable subsets in $[1, 2] - Q$.

Let $\tau_1 = \{X, \phi, F_i\}$, $\tau_2 = \{X, \phi, G_i\}$ and $A = \left[\frac{1}{2}, \frac{3}{2}\right] - Q$. Then $A$ is neither $\tau_1$ open nor $\tau_2$ open, but $A$ is quasi-open because $A = \left( \bigcup_{r \in A_1} \{r\} \right) \cup \left( \bigcup_{q \in A_2} \{q\} \right)$ where $A_1 = \left[\frac{1}{2}, 1\right] - Q$, $A_2 = \left[1, \frac{3}{2}\right] - Q$ and each $\{r\}$ is $\tau_1$-open and $\{q\}$ is $\tau_2$-open.

However any quasi open set is the union of $\tau_1$-open and $\tau_2$-open sets in the form $(\bigcup U_{x_i}) \cup (\bigcup V_{x_j})$ where $U_{x_i}$ and $V_{x_j}$ are respectively the $\tau_1$-open and $\tau_2$-open neighbourhoods of $x_i$ and $x_j$.

Note 1: Clearly every $\tau_1$-open($\tau_2$-open) set is quasi-open and arbitrary union of quasi-open sets is quasi-open. But finite intersection of quasi-open sets need not be quasi-open as shown in the following example.

Example 2.3 Let $X = [0, 3]$.

Let $\{G_i\}$ be the collection of all countable subsets in $[0, 1] - Q$ and $\{F_i\}$ be the collection of all countable subsets in $[2, 3] - Q$.

Let $\tau_1 = \{X, \phi, G_i \cup \{\sqrt{2}\}\}$ and $\tau_2 = \{X, \phi, F_i \cup \{\sqrt{2}\}\}$. Then clearly each $\tau_1$-open or $\tau_2$-open set is quasi-open but the intersection of any $\tau_1$-open and $\tau_2$-open set other than $X$ and $\phi$ is $\{\sqrt{2}\}$ which is not quasi-open.

Definition 2.4 A subset $A$ in a bispace $(X, \tau_1, \tau_2)$ is said to be quasi-closed set if its complement is quasi-open.

Note 2: Clearly every $\tau_1$-closed ($\tau_2$-closed) set is quasi-closed. Arbitrary intersection of quasi-closed sets is quasi-closed but finite union of quasi-closed sets need not be quasi-closed as shown in the above example 2.3 that union of any two $\tau_1$-closed and $\tau_2$-closed set is $X - \{\sqrt{2}\}$ which is not quasi-closed.

In bitopological space every quasi-closed set is the intersection of a $\tau_1$-closed and $\tau_2$-closed sets but in a bispace this is not true. Because in Example 2.2, $A^c = \left( \bigcap_{r \in A_1} (X - \{r\}) \right) \cap \left( \bigcap_{q \in A_2} (X - \{q\}) \right)$ which is not the intersection of $\tau_1$-closed and $\tau_2$-closed sets.
Definition 2.5 The quasi-closure of a subset \( A \) in a bispace \( (X, \tau_1, \tau_2) \) is the set \( (\tau_1 - cl(A)) \cap (\tau_2 - cl(A)) \). The quasi-closure of \( A \) is denoted by \( A^q \).

Theorem 2.6 Quasi-closure of a set is quasi-closed.

Proof: \( A^q = (\tau_1 - cl(A)) \cap (\tau_2 - cl(A)) = (\cap P_i) \cap (\cap Q_i) \), where \( \{P_i\} \) and \( \{Q_i\} \) are family of \( \tau_1 \)-closed and \( \tau_2 \)-closed set respectively satisfying the conditions \( A \subset P_i \) and \( A \subset Q_i \).

Therefore \( A^c = (\cap P_i)^c \cup (\cap Q_i)^c = (\cup P_i^c) \cup (\cup Q_i^c) \) which implies that \( A^c \) is quasi open and hence \( A \) is quasi closed.

Note 3: If \( A \subset (X, \tau_1, \tau_2) \), then \( A \) is the smallest quasi closed set containing \( A \).

Theorem 2.7 If \( x \in \overline{A} \) then every open set \( U \) containing \( x \) intersects \( A \).

Proof: If possible, let there exist an open set \( U \) containing \( x \) such that \( A \cap U = \emptyset \) which implies that \( A \subset X - U \) where \( X - U \) is closed set. Since \( x \notin X - U \) and \( X - U \) is a closed set containing \( A \), \( x \notin \overline{A} \). So if \( x \in \overline{A} \), then every open set \( U \) containing \( x \) intersects \( A \).

Corollary 2.8 \( \overline{A} \subset A \cup A' \).

Definition 2.9 Let \( (X, \tau) \) be a space. A family of open sets \( B \) is said to form a base (open) for \( \tau \) if and only if every open set can be expressed as countable union of members of \( B \).

Theorem 2.10 A collection of subsets \( B \) of a set \( X \) forms an open base of a suitable space structure \( \tau \) of \( X \) if and only if

1) the null set \( \emptyset \in B \)
2) \( X \) is the countable union of some sets belonging to \( B \).
3) intersection of any two sets belonging to \( B \) is expressible as countable union of some sets belonging to \( B \).

Definition 2.11 Let \( (X, \tau) \) be a space. A family of subsets \( S \) of \( X \) is said to form a subbase of a space structure \( \tau \) if the collection of subsets obtained as the intersection of all finite sub-collections of \( S \) constitute a base of \( \tau \).

Theorem 2.12 A collection of subsets \( S \) of a given set \( X \) forms a subbase of a suitable space structure of \( X \) if and only if

1) either \( \emptyset \in S \) or \( \emptyset \) is the intersection of a finite number of subsets belonging to \( S \)
2) \( X \) is the countable union of subsets belonging to \( S \).
Proof: Let \( S \) form a subbase of a space structure \( \tau \) of \( X \) and let \( B \) be the base generated by \( S \). As \( \phi \in B \) either \( \phi \in S \) or \( S \) must contains some subsets, finite in number whose intersection is the null set \( \phi \). As \( X \) is the countable union of some sets belonging to \( B \), so \( X = \bigcup_{i=1}^{\infty} V_i \), where \( V_i \in B \).

Again, each \( V_i \) is the intersection of finite numbers of subsets belonging to \( S \).\( i.e, V_i = S^1_i \cap S^2_i \cap \ldots \cap S^k_i \), for some finite number \( k \), where \( S^l_i \in S \).

So \( V_i \subset S^1_i \) (taking first number of the intersection) for all \( i = 1, 2, \ldots, \infty \).

Therefore \( X = \bigcup_{i=1}^{\infty} V_i \subset \bigcup_{i=1}^{\infty} S^1_i \), on the other hand \( \bigcup_{i=1}^{\infty} S^1_i \subset X \). Therefore, \( X = \bigcup_{i=1}^{\infty} S^1_i \), i.e, \( X \) is the countable union of subsets belongs to \( S \).

Conversely, let the condition hold. Let \( B \) be the set formed by the intersection of finite members of \( S \). Clearly \( \phi \in B \) by (1). Since \( S \subset B \) and \( X \) is the countable union of subsets belonging to \( S \), therefore \( X \in B \).

Let \( V_1, V_2 \in B \), and let \( V_1 = A_1 \cap A_2 \cap \ldots \cap A_k \), \( V_2 = B_1 \cap B_2 \cap \ldots \cap B_m \) where \( A_i, B_j \in S \), \( i = 1, 2, \ldots, k \) and \( j = 1, 2, \ldots, m \). So \( V_1 \cap V_2 = A_1 \cap A_2 \cap \ldots \cap A_k \cap B_1 \cap \ldots \cap B_m = V_3 \) (say) \( \in B \). So \( V_1 \cap V_2 = \bigcup_{i=1}^{\infty} V_i \), where \( V_i = V_3 \) for all \( i \).

So \( B \) form a base and hence \( S \) form a subbase of space structure \( \tau \) of \( X \).

Definition 2.13 Let \((X, \tau_1, \tau_2)\) be a bispace. The space \((X, \tau)\) is called least upper bound space generated by \( \tau_1 \) and \( \tau_2 \) if \( \tau \) is generated by subbase \( \tau_1 \cup \tau_2 \).

Definition 2.14 Let \((X, \tau_1, \tau_2)\) be a bispace. A \( \subset X \) is said to be semi-open if it is open in the least upper bound space structure \( \tau \) generated by \( \tau_1 \) and \( \tau_2 \).

Definition 2.15 Complement of semi-open set is called semi-closed. Semi-closure of any set \( A \subset X \) is intersection of all semi-closed sets containing \( A \).

In \((X, \tau_1, \tau_2)\) every \( \tau_1 \)-closed or \( \tau_2 \) closed is semi-closed. For let \( A \) be \( \tau_1 \)-closed. Then \( A^c \) is \( \tau_1 \)-open. So \( A^c \in \tau_1 \cup \tau_2 \) which implies that \( A^c \) is a subbasic open set of least upper bound space structure \( \tau \) generated by \( \tau_1 \) and \( \tau_2 \). Therefore \( A^c \) is semi-open and hence \( A \) is semi-closed. Similarly every \( \tau_2 \)-closed set is semi-closed set.

Remark 2.16 In a bitopological space every quasi-closed is semi-closed, but in bispace quasi-closed sets may not be semi-closed i.e, quasi-open set may not be semi-open as shown in the following example.
Example 2.17 Consider $X, \tau_1, \tau_2$ and $A \subset X$ as in example 2.2. Then $A$ is a quasi-open and uncountable set. Clearly $\tau_1 \cup \tau_2$ form a subbase of least upper bound space structure $\tau$. Other than $X, \phi$ every member of $\tau_1 \cup \tau_2$ is countable set. So finite intersection of member of $\tau_1 \cup \tau_2$ is a quasi-open and uncountable set. Clearly every continuous mapping is quasi open in $(X_1, \tau_1, \tau_2)$.

Definition 2.18 Let $f : (X_1, \tau_1, \tau_2) \rightarrow (X_2, \tau'_1, \tau'_2)$ be a mapping. Then $f$ is said to be quasi continuous if the inverse image of every quasi open set in $(X_2, \tau'_1, \tau'_2)$ is quasi open in $(X_1, \tau_1, \tau_2)$.

Definition 2.19 A function $f$ maps a bispace $(X, \tau_1, \tau_2)$ into a bispace $(X', \tau'_1, \tau'_2)$ is said to be $(\tau_1 \tau'_1, \tau_2 \tau'_2)$ continuous or simply continuous if and only if the induced mappings $f_1 : (X, \tau_1) \rightarrow (X', \tau'_1)$ and $f_2 : (X, \tau_2) \rightarrow (X', \tau'_2)$ are continuous.

Clearly every continuous mapping is quasi continuous. But converse may not be true as shown in the following example.

Example 2.20 Example of a function which is quasi continuous but not continuous:

Let $X = [0, 2]$, $\tau_1 = \{X, \phi, G_i \}$, where $G_i$ are countable subsets in $[0, 1] - Q$ and $\tau_2 = \{X, \phi, F_i \}$, where $F_i$ are countable subsets in $[1, 2] - Q$ and $\tau'_2 = \tau_1 = \{X, \phi, P_i \}$, where $P_i$ are countable subsets in $[0, 2] - Q$. Let $f : (X, \tau_1, \tau_2) \rightarrow (X', \tau'_1, \tau'_2)$ be the identity mapping. Clearly $\sqrt{2} \in \tau'_1$ but $f^{-1}(\sqrt{2}) = \{\sqrt{2}\} \notin \tau_1$. Also $\{1/\sqrt{2}\} \in \tau'_2$ but $f^{-1}(1/\sqrt{2}) = \{1/\sqrt{2}\} \notin \tau_2$. So the identity mapping $f$ is not continuous.

Let $G$ be any quasi open set in $(X, \tau'_1, \tau'_2)$, let $x \in f^{-1}(G) = G$, where $G$ is a countable set in $[0, 2] - Q$. So $x$ is an irrational number in $[0, 2] - Q$. Now if $x \in [0, 1] - Q$ then $\{x\} \in \tau_1$ and if $x \in [1, 2] - Q$ then $\{x\} \in \tau_2$. Thus in any case, for any $x \in f^{-1}(G)$, there is $\tau_1$-open or $\tau_2$-open set $\{x\}$ such that $x \in \{x\} \subset G$. So $f^{-1}(G)$ is quasi open in $(X, \tau_1, \tau_2)$ and hence $f$ is quasi continuous.

Definition 2.21 A space (or a set) is called bicompact if every open cover of it has a finite subcover.

Definition 2.22 A set $A \subset (X, \tau_1, \tau_2)$ is said to be semi-bicompact if it is compact in the least upper bound space structure generated by $\tau_1$ and $\tau_2$; in other words, $A$ is semi-bicompact if and only if any given covering of $A$ by semi-open subsets of $X$ there exists a finite sub-covering.
In bitopological space the quasi-continuous image of a semi-compact set is semi-compact. But in bispace it is not true.

**Proposition 2.23** Let \( f : (X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2) \) be continuous and surjective and let \((X, \tau_1, \tau_2)\) be semi-bicompact. Then \((X', \tau'_1, \tau'_2)\) is semi-bicompact.

**Proof:** Let \( \{U_{a}\}_{a \in \Lambda} \) be a covering of \( X' \), where each \( U_{a} \) is semi-open in \((X', \tau'_1, \tau'_2)\). Then \( U_{a} \) is of the form \( \bigcup_{i,j=1}^{\infty} (V_{ai} \cap W_{aj}) \) where \( V_{ai} \) is \( \tau'_1 \)-open and \( W_{aj} \) is \( \tau'_2 \)-open.

Then \( f^{-1}(U_{a}) = \bigcup_{i,j=1}^{\infty} (f^{-1}(V_{ai}) \cap f^{-1}(W_{aj})) \). Since \( V_{ai} \) is \( \tau'_1 \)-open, \( f^{-1}(V_{ai}) \) is \( \tau_1 \)-open for each \( i = i_{1}, i_{2}, \ldots, i_{k} \) and similarly \( f^{-1}(W_{aj}) \) is \( \tau_2 \)-open for each \( j = j_{1}, j_{2}, \ldots, j_{m} \). Therefore, \( \{f^{-1}(U_{a})\} \) is a covering of \( X \) where \( f^{-1}(U_{a}) \) is semi open in \((X, \tau_1, \tau_2)\). Since \( X \) is semi-bicompact, there exists finite subset \( \Lambda_{1} \subset \Lambda \) such that \( \{f^{-1}(U_{a})\}_{a \in \Lambda_{1}} \) covers \( X \). Therefore \( \{U_{a}\}_{a \in \Lambda_{1}} \) is finite subcovers of \( X' \). Therefore \( X' \) is semi-bicompact.

## 3 Quasi-Hausdorff Space

**Definition 3.1** A bispace \((X, \tau_1, \tau_2)\) is said to be quasi Hausdorff if given \( x_{1} \neq x_{2} \), there exist quasi open sets \( U_{1}, U_{2} \) such that \( x_{1} \in U_{1}, x_{2} \in U_{2} \) and \( U_{1} \cap U_{2} = \emptyset \).

**Theorem 3.2** Every quasi Hausdorff bispace \((X, \tau_1, \tau_2)\) is Hausdorff with respect to least upper bound space structure \( \tau \) generated by \( \tau_1 \) and \( \tau_2 \).

**Proof:** Since \((X, \tau_1, \tau_2)\) is quasi-Hausdorff, for every \( x_{1}, x_{2} \in X, x_{1} \neq x_{2} \), there exist quasi-open sets \( U_{1} \) and \( U_{2} \) such that \( x_{1} \in U_{1}, x_{2} \in U_{2}, U_{1} \cap U_{2} = \emptyset \).

So \( U_{1} \) and \( U_{2} \) is of the form \( U_{1} = (\bigcup_{i} U_{x_{i}}) \cup (\bigcup_{j} V_{x_{j}}) \) and \( U_{2} = (\bigcup_{i} U_{y_{i}}) \cup (\bigcup_{j} V_{y_{j}}) \)

where \( U_{x_{i}}, U_{y_{i}} \in \tau_{1} \) and \( V_{x_{j}}, V_{y_{j}} \in \tau_{2} \).

Since \( U_{1} \cap U_{2} = \emptyset \), i.e., \( (\bigcup_{i} U_{x_{i}}) \cup (\bigcup_{j} V_{x_{j}}) \cap (\bigcup_{i} U_{y_{i}}) \cup (\bigcup_{j} V_{y_{j}}) = \emptyset \), and since \( x_{1} \in U_{1} \), there exists a \( U_{x_{i}} \) (or \( V_{x_{j}} \)) such that \( x_{1} \in U_{x_{i}} \) (or \( V_{x_{j}} \)) and since \( x_{2} \in U_{2} \) implies there exists a \( U_{y_{i}} \) (or \( V_{y_{j}} \)) such that \( x_{2} \in U_{y_{i}} \) (or \( V_{y_{j}} \)).

So in any case \((U_{x_{i}}, U_{y_{i}}) \) or \((U_{x_{j}}, V_{y_{j}}) \) or \((V_{x_{j}}, U_{y_{i}}) \) or \((V_{x_{j}}, V_{y_{j}}) \) is a pair of \( \tau \)-open sets which separates strongly \( x_{1}, x_{2} \), since \( U_{x_{i}} \cap V_{y_{i}} = \emptyset \), \( U_{x_{j}} \cap V_{y_{j}} = \emptyset \), \( V_{x_{j}} \cap U_{y_{i}} = \emptyset \), \( V_{x_{j}} \cap V_{y_{j}} = \emptyset \). So \((X, \tau)\) is Hausdorff.

**Lemma 3.3** Let \((X, \tau_1, \tau_2)\) be a quasi Hausdorff bispace and let \( U_{1} \) and \( U_{2} \) be two quasi open sets such that \( U_{1} \cap U_{2} = \emptyset \) then \( \overline{U_{1}} \cap \overline{U_{2}} = \emptyset \) where \( \overline{U_{1}} \) is the quasi closure of \( U_{1} \).
Proof: Let $U_1$ and $U_2$ be quasi open sets such that $U_1 \cap U_2 = \emptyset$. Suppose $\overline{U_1} \cap U_2 \neq \emptyset$. Let $y \in \overline{U_1} \cap U_2$. Now $y \in \overline{U_1}$ implies that every $\tau_1$-neighbourhood and $\tau_2$-neighbourhood of $y$ meet $U_1$.

But $U_2$ is a quasi open set containing $y$. So there exists a $\tau_1$-open neighbourhood or a $\tau_2$-open neighbourhood $W$ of $y$ such that $W \subset U_2$. Since $U_1 \cap U_2 = \emptyset$ implies $U_1 \cap W = \emptyset$ which contradicts the above fact. So $\overline{U_1} \cap U_2 = \emptyset$.

Theorem 3.4 A semi-bicompact subset in a quasi-Hausdorff bispace is quasi-closed if the following condition 'C' is satisfied.

'C': Quasi-closure of every quasi-open set is quasi-open.

Proof: Let $A$ be a semi-bicompact subset of a quasi-Hausdorff space $(X, \tau_1, \tau_2)$ which satisfies condition 'C'. We show $X - A$ is quasi open.

Let $s \in X - A$. Then $s \neq a_i$ for each $a_i \in A$. Since $(X, \tau_1, \tau_2)$ is quasi-Hausdorff there exist quasi-open sets $U_{a_i}$ and $V_{a_i}$ containing $a$ and $s$ respectively such that $U_{a_i} \cap V_{a_i} = \emptyset$. So by lemma 3.3, we get $U_{a_i} \cap V_{a_i} = \emptyset$.

The collection $\{U_{a_i}\}_{a_i \in A}$ is a quasi-open covering of $A$. Now $U_{a_i}$ is of the form $U_{a_i} = (\bigcup_{x \in \Lambda_1} U^{a_i}_x) \cup (\bigcup_{y \in \Lambda_2} V^{a_i}_y)$ where $U^{a_i}_x$ and $V^{a_i}_y$ are $\tau_1$-open and $\tau_2$-open sets respectively. So $\bigcup \{ (\bigcup U^{a_i}_x) \cup (\bigcup V^{a_i}_y) : a_i \in A \} \supset A$. Since $U^{a_i}_x$ and $V^{a_i}_y$ are also semi-open sets, the collection $\{ U^{a_i}_x, V^{a_i}_y : a_i \in A, x \in \Lambda_1, y \in \Lambda_2 \}$ form an semi-open cover for $A$.

So there exist finite number of sets $U^{a_1}_{x_1}, U^{a_2}_{x_2}, U^{a_3}_{x_3}, ..., U^{a_n}_{x_n}, V^{a_1'}_{y_1}, V^{a_2'}_{y_2}, ..., V^{a_m'}_{y_m}$ such that $A \subset U^{a_1}_{x_1} \cup U^{a_2}_{x_2} \cup ... \cup U^{a_n}_{x_n} \cup V^{a_1'}_{y_1} \cup V^{a_2'}_{y_2} \cup ... \cup V^{a_m'}_{y_m}$. Clearly $U^{a_1}_{x_1} \cap V^{a_1}_{x_1} = \emptyset, V^{a_1'}_{y_1} \cap V^{a_1}_{x_1} = \emptyset, since U^{a_i}_{x_i} \subset U_{a_i}, V^{a_i'}_{y_i} \subset U_{a_i}$.

Let $W_s = V^{a_1}_{x_1} \cap V^{a_2}_{x_2} \cap ... \cap V^{a_n}_{x_n} \cap V^{a_1'}_{y_1} \cap V^{a_2'}_{y_2} \cap ... \cap V^{a_m'}_{y_m} \cap V^{a_2'}_{y_2} \cap ... \cap V^{a_m'}_{y_m} = W_s$.

But $V^{a_1}_{x_1} \cap V^{a_2}_{x_2} \cap ... \cap V^{a_n}_{x_n} \cap V^{a_1'}_{y_1} \cap V^{a_2'}_{y_2} \cap ... \cap V^{a_m'}_{y_m} = W_s'$ is quasi open, by condition 'C'.

So $W_s$ contain a quasi open set $W_s'$ containing $s$. Also $A \cap W_s \subset (U^{a_1}_{x_1} \cup U^{a_2}_{x_2} \cup ... \cup U^{a_n}_{x_n} \cup V^{a_1'}_{y_1} \cup V^{a_2'}_{y_2} \cup ... \cup V^{a_m'}_{y_m}) \cap W_s = \emptyset$.

Thus for each $s \in X - A$, there exists a quasi open set $W_s' \subset W_s$ such that $W_s \cap A = \emptyset$ that is $s \in W_s' \subset W_s \subset X - A$. Hence $X - A$ is quasi open implies $A$ is quasi closed.
References


