Properties of $\beta^*$-Homeomorphisms in Topological Spaces

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Abstract

The concept of $\beta^*$-homeomorphisms is introduced and investigated by Palanimani [13] earlier. In the present paper we investigate some more properties of $\beta^*$-homeomorphisms and also investigate contra $\beta^*$-homeomorphisms.

Keywords: $\beta^*$-closed, $\beta^*$-continuous, $\beta^*$-closed map, $\beta^*$-homeomorphism, contra $\beta^*$-irresolute and contra $\beta^*$-homeomorphism.

1 Introduction

Throughout the present paper, \((X, \tau), (Y, \sigma)\) and \((Z, \eta)\) (or \(X, Y\) and \(Z\)) represent nonempty topological spaces on which no separation axioms are assumed unless otherwise mentioned. The closure and interior of a subset \(A \subseteq X\) will be denoted by \(\text{Cl}(A)\) and \(\text{Int}(A)\), respectively. The present paper is a continuation of [13] due to one of the present authors, we investigate more properties of functions preserving \(\beta^*\)-closed sets. In section 2, we recall some definitions on functions and we need some properties on functions (c.f. Lemma 2.5 and Theorem 2.6). In section 3, for a topological space \((X, \tau)\), we introduce and investigate groups of functions, say \(\beta^*h(X, \tau)\) preserving \(\beta^*\)-closed sets respectively, they contain the homeomorphism group \(h(X, \tau)\) as a subgroup (cf. Theorem 3.3). Moreover, these groups have an important property that they are one of topological invariants (Theorem 3.3). Using the concept of contra \(\beta^*\)-irresoluteness, In section 4, we construct more groups of functions, say \(\beta^*h(X, \tau) \cup \text{con-}\beta^*h(X, \tau)\) for a topological space \((X, \tau)\); they contain the homeomorphism group \(h(X, \tau)\) as a subgroup (cf. Theorem 4.4).

2 Preliminaries

We need the following definition, lemma and Theorem.

Definition 2.1 A subset \(A\) of a topological space \((X, \tau)\) is called a

(i) generalized closed (briefly, \(g\)-closed) \([6]\) if \(\text{Cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

(ii) \(\beta^*\)-closed \([11]\) if \(\text{Cl} (\text{Int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\)-open in \((X, \tau)\).

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.2 \([12]\) Let \(f : X \to Y\) from a topological space \(X\) into a topological space \(Y\) is called \(\beta^*\)-continuous if the inverse image of every closed set in \(Y\) is \(\beta^*\)-closed in \(X\).

Definition 2.3 \([13]\) Let \(f : X \to Y\) from a topological space \(X\) into a topological space \(Y\) is called \(\beta^*\)-closed map if for each closed set \(F\) of \(X\), \(f(F)\) is \(\beta^*\)-closed in \(Y\).

Definition 2.4 \([12]\) A map \(f : X \to Y\) from a topological space \(X\) into a topological space \(Y\) is called \(\beta^*\)-irresolute if the inverse image of every \(\beta^*\)-closed set in \(Y\) is \(\beta^*\)-closed in \(X\).
Lemma 2.5 Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \zeta) \) be two functions between topological spaces.

(i) If \( f \) and \( g \) are \( \beta^* \)-irresolute, then the composition \( g \circ f \) is also \( \beta^* \)-irresolute.

(ii) The identity function \( 1_X : (X, \tau) \to (Y, \sigma) \) is \( \beta^* \)-irresolute.

Proof: The proofs are obvious from definitions.

Theorem 2.6 Let \( f : (X, \tau) \to (Y, \sigma) \) be a function then every homeomorphism is \( \beta^* \)-continuous.

Proof: Let \( f \) be a homeomorphism. Then, \( f^{-1} : (Y, \sigma) \to (X, \tau) \) is also a homeomorphism. By definition, it is shown that \( f = (f^{-1})^{-1} \) is continuous. By Theorem 3.2 in [12], it is shown that \( f \) is \( \beta^* \)-continuous.

3 More on Functions Preserving \( \beta^* \)-Closed Sets

Definition 3.1 A bijection \( f : (X, \tau) \to (Y, \sigma) \) is called \( \beta^* \)-homeomorphism [13] if \( f \) is both \( \beta^* \)-continuous and \( \beta^* \)-closed (or \( f^{-1} \) is \( \beta^* \)-continuous).

For a topological space \((X, \tau)\), we introduce the following:

1. \( h(X; \tau) = \{ f | f : (X, \tau) \to (X, \tau) \text{ is a homeomorphism} \} \).
2. \( \beta^* h(X; \tau) = \{ f | f : (X, \tau) \to (X, \tau) \text{ is a \( \beta^* \)-homeomorphism} \} \).

Theorem 3.2 For a topological spaces \((X, \tau)\) then \( h(X; \tau) \subseteq \beta^* h(X; \tau) \).

Proof: Let \( f \in h(X; \tau) \). Then by definition \( f \) and \( f^{-1} \) are continuous. By Theorem 3.2 in [12], it is shown that \( f \) and \( f^{-1} \) are \( \beta^* \)-continuous and so \( f \) is \( \beta^* \)-homeomorphism, i.e., \( f \in \beta^* h(X; \tau) \).

Theorem 3.3 Let \((X, \tau)\) be a topological space. Then, we have the following properties.

(i) The collection \( \beta^* h(X; \tau) \) forms a group under the composition of functions.

(ii) The homeomorphism group \( h(X; \tau) \) is a subgroup of the group \( \beta^* h(X; \tau) \).

Proof: (i) A binary operation \( \eta_X : \beta^* h(X; \tau) \times \beta^* h(X; \tau) \to \beta^* h(X; \tau) \) is well defined by \( \eta_X(a, b) = b \circ a \), where \( b \circ a : X \to X \) is the composite function of the functions \( a \) and \( b \) such that \( (b \circ a)(x) = b(a(x)) \) for every point \( x \in X \). Indeed, by Lemma 2.5 (i), it is shown that, for every \( \beta^* \)-homeomorphisms \( a \) and \( b \), the composition \( b \circ a \) is also \( \beta^* \)-homeomorphism. Namely, for every pair \((a, b) \in \beta^* h(X; \tau), \eta_X(a, b) = b \circ a \in \beta^* h(X; \tau) \). Then, it is claimed that
the binary operation $\eta_X : \beta^* h(X; \tau) \times \beta^* h(X; \tau) \to \beta^* h(X; \tau)$ satisfies the axiom of group. Namely, putting $a.b = \eta_X(a, b)$, the following properties hold $\beta^* h(X; \tau)$.

(1) $((a.b).c) = (a.(b.c))$ holds for every elements $a, b, c \in \beta^* h(X; \tau)$;

(2) for all element $a \in \beta^* h(X; \tau)$, there exists an element $e \in \beta^* h(X; \tau)$ such that $a.e = e.a = a$ hold in $\beta^* h(X; \tau)$;

(3) for each element $a \in \beta^* h(X; \tau)$, there exists an element $a_1 \in \beta^* h(X; \tau)$ such that $a.a_1 = a_1.a = e$ hold in $\beta^* h(X; \tau)$.

Indeed, the proof of (1) is obvious; the proof of (2) is obtained by taking $e = 1_X$, where $1_X$ is the identity function on $X$ and using Lemma 2.5 (ii); the proof of (3) is obtained by taking $a_1 = a^{-1}$ for each $a \in \beta^* h(X; \tau)$ and Definition 3.1, where $a^{-1}$ is the inverse function of $a$. Therefore, by definition of groups, the pair $(\beta^* h(X; \tau), \eta_X)$ forms a group under the composition of functions. i.e., $\beta^* h(X; \tau)$ is a group.

(ii) It is obvious that $1_X : (X, \tau) \to (X, \tau)$ is a homeomorphism and so $h(X; \tau) \neq \phi$. It follows from Theorem 3.2 that $h(X; \tau) \subseteq \beta^* h(X; \tau)$. Let $a, b \in h(X; \tau)$. Then we have that $\eta_X(a, b^{-1}) = b^{-1} \circ a \in h(X; \tau)$, here $\eta_X : \beta^* h(X; \tau) \times \beta^* h(X; \tau) \to \beta^* h(X; \tau)$ is the binary operation (cf. Proof of Theorem 3.3 (i)). Therefore, the group $h(X; \tau)$ is a subgroup of $\beta^* h(X; \tau)$.

**Theorem 3.4** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. If $(X, \tau)$ and $(Y, \sigma)$ are homeomorphism, then there exist isomorphisms: i.e., $\beta^* h(X; \tau) \cong \beta^* h(Y; \sigma)$.

**Proof:** It follows from assumption that there exists a homeomorphism, say $f : (X, \tau) \to (Y, \sigma)$. We define a function $f_* : \beta^* h(X, \tau) \to \beta^* h(Y, \sigma)$ by $f_*(a) = f \circ a \circ f^{-1}$ for every element $a \in \beta^* h(X, \tau)$; by Theorem 2.6 (or Theorem 3.2) and Lemma 2.5(i), the bijections $f \circ a \circ f^{-1}$ and $(f \circ a \circ f^{-1})^{-1}$ are $\beta^*$-closed and so $f_*$ is well defined. The induced function $f_*$ is a homeomorphism. Indeed, $f_*(\eta_X(a, b)) = f \circ b \circ f^{-1} \circ f \circ a \circ f^{-1} = (f_*(b)) \circ (f_*(a)) = \eta_X(f_*(a), f_*(a))$ hold. Obviously, $f_*$ is bijective. Thus, we have (i), i.e., $f_*$ is an isomorphism.

4 More on the Groups including the Homeomorphism Group $h(X; \tau)$ as Subgroup

**Definition 4.1** For a topological spaces $(X, \tau)$ and $(Y, \sigma)$, we define the following functions. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra $\beta^*$-irresolute if $f^{-1}(V)$ is $\beta^*$-closed in $(X, \tau)$ for every $\beta^*$-open set $V$ of $(Y, \sigma)$.

For these we can immediately see the following lemma.
Lemma 4.2 Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two functions between topological spaces.

(i) If \( f \) and \( g \) are contra \( \beta^* \)-irresolute, then the composition \( g \circ f \) is also \( \beta^* \)-irresolute.

(ii) If \( f \) is \( \beta^* \)-irresolute (resp. contra \( \beta^* \)-irresolute) and \( g \) are contra \( \beta^* \)-irresolute (resp. \( \beta^* \)-irresolute), then the composition \( g \circ f \) is contra \( \beta^* \)-irresolute.

Definition 4.3 For a topological space \((X, \tau)\), we define the collection of functions con-\( \beta^* \)h\((X, \tau)\) = \{\( f \mid f : (X, \tau) \to (X, \tau) \) is a contra \( \beta^* \)-irresolute bijection and \( f^{-1} \) is contra \( \beta^* \)-irresolute \}.

For a topological space \((X, \tau)\), we construct alternative groups, say \( \beta^* \)h\((X; \tau)\) \(\cup\) con-\( \beta^* \)h\((X; \tau)\).

Theorem 4.4 Let \( (X, \tau) \) be a topological space. Then, we have the following properties.

(i) The union of two collections, \( \beta^* \)h\((X; \tau)\) \(\cup\) con-\( \beta^* \)h\((X; \tau)\), forms a group under the composition of functions.

(ii) The homeomorphism group \( h(X; \tau) \) is a subgroup of \( \beta^* \)h\((X; \tau)\) \(\cup\) con-\( \beta^* \)h\((X; \tau)\).

Proof: (i) Let \( B_X = \beta^* \)h\((X; \tau)\) \(\cup\) con-\( \beta^* \)h\((X; \tau)\). A binary operation \( w_X : B_X \times B_X \to B_X \) is well defined by \( w_X(a, b) = b \circ a \), where \( b \circ a : X \to X \) is the composite function of the functions \( a \) and \( b \). Indeed, let \( (a, b) \in B_X \); if \( a \in \beta^* \)h\((X; \tau)\) and \( b \in \) con-\( \beta^* \)h\((X; \tau)\), then \( b \circ a : (X, \tau) \to (X, \tau) \) a contra \( \beta^* \)-irresolute bijection and \( (b \circ a)^{-1} \) is also contra \( \beta^* \)-irresolute and so \( w_X(a, b) = b \circ a \in \beta^* \)h\((X; \tau)\) \(\subset B_X \) (cf. Lemma 4.2 (ii)) if \( a \in \beta^* \)h\((X; \tau)\) and \( b \in \beta^* \)h\((X; \tau)\) then \( b \circ a : (X, \tau) \to (X, \tau) \) is a \( \beta^* \)-irresolute bijection and so \( w_X(a, b) = b \circ a \in \beta^* \)h\((X; \tau)\) \(\subset B_X \) (cf. Lemma 2.5 (i)), if \( a \in \) con-\( \beta^* \)h\((X; \tau)\) and \( b \in \) con-\( \beta^* \)h\((X; \tau)\), then \( b \circ a : (X, \tau) \to (X, \tau) \) is a \( \beta^* \)-irresolute bijection and \( (b \circ a)^{-1} \) is also \( \beta^* \)-irresolute and so \( w_X(a, b) = b \circ a \in \beta^* \)h\((X; \tau)\) \(\subset B_X \) (cf. Lemma 4.2(i)) if \( a \in \) con-\( \beta^* \)h\((X; \tau)\) and \( b \in \beta^* \)h\((X; \tau)\) then \( b \circ a : (X, \tau) \to (X, \tau) \) is a contra \( \beta^* \)-irresolute bijection and \( (b \circ a)^{-1} \) is also \( \beta^* \)-irresolute and so \( w_X(a, b) = b \circ a \in \beta^* \)h\((X; \tau)\) \(\subset B_X \) (cf. Lemma 4.2 (ii)). By the similar arguments of Theorem 3.3, it is claimed that the binary operation \( w_X : B_X \times B_X \to B_X \) satifies the axiom of group; for the identity element \( e \) of \( B_X \), \( e = 1_X : (X, \tau) \to (X, \tau) \) (the identity function). Thus the pair \( (B_X, w_X) \) forms a group under the composition of functions, i.e., \( \beta^* \)h\((X; \tau)\) \(\cup\) con-\( \beta^* \)h\((X; \tau)\) is a group.
(ii) By Theorem 3.3 (ii) above, it is shown that $h(X; \tau)$ is a subgroup of $\beta^*h(X; \tau) \cup \text{con-}^*h(X; \tau)$.

The groups of Theorem 4.4 are also invariant concepts under homeomorphisms between topological spaces (c.f. Theorem 3.4).

**Theorem 4.5** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. If $(X, \tau)$ and $(Y, \sigma)$ are homeomorphic, then there exist isomorphisms i.e., $\beta^*h(X; \tau) \cup \text{con-}^*h(X; \tau) \cong \beta^*h(Y; \sigma) \cup \text{con-}^*h(Y; \sigma)$.

**Proof:** Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism. We put $B_X = \beta^*h(X; \tau) \cup \text{con-}^*h(X; \tau)$ (resp. $B_Y = \beta^*h(Y; \sigma) \cup \text{con-}^*h(Y; \sigma)$) for a topological space $(X, \tau)$ (resp. $(Y, \sigma)$). First we have a well defined function $f_* : B_X \to B_Y$ by $f_*(a) = f \circ a \circ f^{-1}$ for every element $a \in B_X$. Indeed by Theorem 2.6(ii) (or Theorem 3.2), $f$ and $f^{-1}$ are $\beta^*$-irresolute. By Lemma 2.5(i) and Lemma 4.2(ii), the bijections $f \circ a \circ f^{-1}$ and $(f \circ a \circ f^{-1})^{-1}$ are $\beta^*$-irresolute or contra $\beta^*$-irresolute and so $f_*$ is well defined. The induced function $f_*$ is a homeomorphism. Indeed, $f_*(w_X(a,b)) = f \circ b \circ f^{-1} \circ f \circ a \circ f^{-1} = (f_*(b)) \circ (f_*(a)) = w_Y(f_*(a)) \circ (f_*(a))$ hold, $w_X : B_X \times B_X \to B_X$ and $w_Y : B_Y \times B_Y \to B_Y$ are the binary operations defined in Proof of Theorem 4.4(i). Obviously, $f_*$ is bijective. Thus, we have the isomorphisms.

**References**


