Edge-Domsaturation Number of a Graph

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Abstract

The edge-domsaturation number ds′(G) of a graph G = (V, E) is the least positive integer k such that every edge of G lies in an edge dominating set of cardinality k. The connected edge-domsaturation number ds′c(G) of a graph G = (V, E) is the least positive integer k such that every edge of G lies in a connected edge dominating set of cardinality k. In this paper, we obtain several results connecting ds′(G), ds′c(G) and other graph theoretic parameters.

Keywords: edge-dominating set, edge-domination number, edge-domsaturation number, connected edge-domsaturation number.

1 Introduction

Throughout this paper, G denotes a graph with order p and size q. By a graph we mean a finite undirected graph without loops or multiple edges. For graph theoretic terms we refer Harary [2]. In particular, for terminology related to domination theory we refer Haynes et.al [3].

Definition 1.1. Let G = (V, E) be a graph. A subset D of E is said to be an edge dominating set if every edge in E − D is adjacent to at least one edge in D. An edge dominating set D is said to be a minimal edge dominating set if no proper subset of D is a dominating set of G.
Acharya [1] introduced the concept of domsaturation number $ds(G)$ of a graph. Arumugam and Kala [4] observed that for any graph $G$, $ds(G) = \gamma(G)$ or $\gamma(G) + 1$ and obtained several results on $ds(G)$. We now extend the concept of domsaturation to edges.

**Definition 1.2.** The least positive integer $k$ such that every edge of $G$ lies in an edge dominating set of cardinality $k$ is called the edge-domsaturation number of $G$ and is denoted by $ds'(G)$.

**Definition 1.3.** The least positive integer $k$ such that every edge of $G$ lies in a connected edge dominating set of cardinality $k$ is called the connected edge-domsaturation number of $G$ and is denoted by $ds'_c(G)$.

If $G$ is a graph with edge set $E$ and $D$ is a $\gamma'$-set of $G$, then for any edge $e \in E - D, D \cup \{e\}$ is also an edge dominating set and hence $ds'(G) = \gamma'(G)$ or $\gamma'(G) + 1$.

Thus we have the following definition.

**Definition 1.4.** A graph $G$ is said to be of class 1 or class 2 according as $ds'(G) = \gamma'(G)$ or $\gamma'(G) + 1$.

**Definition 1.5.** A tree $T$ of order 3 or more is a caterpillar if the removal of its leaves produces a path.

**Definition 1.6.** A tree containing exactly two vertices that are not leaves (which are necessarily adjacent) is called a double star. Thus a double star is a tree of diameter three.

We use the following theorems.

**Theorem 1.7.** [6] For any tree $T$ of order $p \neq 2$, $\gamma'(G) \leq (p-1)/2$; equality holds if and only if $T$ is isomorphic to the subdivision of a star.

**Theorem 1.8.** [6] Let $T$ be any tree and let $e = uv$ be an edge of maximum degree $\Delta'(T)$. Then $\gamma'(T) = q - \Delta'(T)$ if and only if $\text{diam}(T) \leq 4$ and $\text{deg}_w \leq 2$ for every vertex $w \neq u, v$.

## 2 Main Results

**Theorem 2.1.** The path $P_p$ of order $p$, $p \geq 4$ is of class 1 if and only if $p \equiv 2 \pmod{3}$.

**Proof.** Let $P_p = (1, 2, \ldots, p)$ be of class 1. Let $e_i$ be the edge joining $i$ and $i + 1$. If $p \equiv 0 \pmod{3}$, then $e_3$ does not lie in an edge dominating set of cardinality $\gamma'(G)$. If $p \equiv 1 \pmod{3}$, then either $e_1$ or $e_3$ does not lie in an edge
dominating set of cardinality $\gamma'(G)$. Hence if $p \equiv 0$ or $1(\text{mod } 3)$, then $P_p$ is of class 2.

Conversely, suppose $p = 3k + 2$. Then $\gamma'(G) = k + 1$.

Let $D_1 = \{e_1, e_3, e_6, \ldots, e_{3k}\}$

$D_2 = \{e_2, e_5, e_7, \ldots, e_{3k-1}, e_{3k+1}\}$

and $D_3 = \{e_1, e_4, e_7, \ldots, e_{3k-2}, e_{3k+1}\}$.

Clearly $D_1, D_2$ and $D_3$ are $\gamma'(G)$ sets of $P_p$ and $\bigcup_{i=1}^{3} D_i = E(P_p)$. Hence $ds'(G) = \gamma'(G)$ so that $P_p$ is of class 1.

**Definition 2.2.** Let $T$ be a caterpillar. Two supports $u$ and $v$ of $T$ are said to be consecutive if either $u$ and $v$ are adjacent or every vertex in the $u - v$ path in $T$ has degree 2.

**Theorem 2.3.** Let $T$ be a caterpillar. Then $T$ is of class 1 if and only if every support is adjacent to exactly one pendent vertex and for any two consecutive supports $u$ and $v$, $d(u, v) \equiv 2(\text{mod } 3)$.

**Proof.** Suppose $T$ is a caterpillar of class 1. If there exists two pendent vertices $v_1, v_2$ which are adjacent to $u$, then there is no $\gamma'(G)$-set containing $uv_1$. Hence every support is adjacent to exactly one pendent vertex. Now, let $S$ denote the set of all supports of $T$. Suppose there exists two consecutive supports $u$ and $v$ such that $d(u, v) \equiv 0$ or $1(\text{mod } 3)$. Let $P = (u = u_1, u_2, \ldots, u_k = v)$ be the $u - v$ path in $T$. Then $u_2 u_3$ does not lie in a $\gamma'(G)$-set and hence it follows that for any two consecutive supports $u$ and $v$, $d(u, v) \equiv 2(\text{mod } 3)$.

Conversely, let $T$ be a caterpillar in which every support is adjacent to exactly one pendent vertex and $d(u, v) \equiv 2(\text{mod } 3)$ for any two consecutive supports $u$ and $v$. Let $k$ denote the number of supports in $T$. We prove that $T$ is of class 1 by induction on $k$. If $k = 2$, $T$ is a path $P_2$ with $p \equiv 2(\text{mod } 3)$ vertices and by the theorem [2.1], $T$ is of class 1. Suppose the theorem is true for all caterpillars with $k - 1$ supports. Let $T$ be a caterpillar with $k$ supports $w_1, w_2, \ldots, w_k$ such that $w_i$ and $w_{i+1}$ are consecutive supports. Let $x_i$ be the pendent vertex adjacent to $w_i$. Let $P_1 = (w_1, v_1, \ldots, v_{3m+1}, w_2)$ be the $w_1 - w_2$ path and let $T_1 = T - \{x_1, w_1, v_1, \ldots, v_{3m+1}\}$. Clearly $P_1$ is of class 1 and by induction hypothesis $T_1$ is of class 1. Further the union of any minimum edge dominating set of $P_1$ and any minimum edge dominating set of $T_1$ is a minimum edge dominating set of $T$. Hence $T$ is of class 1.

**Theorem 2.4.** If $G$ is a $k$-regular graph which is edge domatically full, then $G$ is of class 1.

**Proof.** Since $G$ is edge domatically full, $d'(G) = \delta'(G) + 1 = k + 1$.

Let $\{D'_1, D'_2, \ldots, D'_{k+1}\}$ be an edge domatic partition of $G$. Any set $D'_i$ either
contains an edge \( x \) or exactly one of its neighbours. Hence each \( D_i' \) is independent. Also for all \( 1 \leq j \leq k+1 \), \( i \neq j \), every edge in \( D_i' \) is adjacent to exactly one edge in \( D_j' \). Hence all sets \( D_i' \) are of equal cardinality and \( |D_i'| = \gamma'(G) \) so that \( G \) is of class 1.

**Lemma 2.5.** Let \( G \) be a path of even order which is of class 1. Then \( \gamma'(G) + \beta_1(G) = p - 1 \) if and only if \( G \cong P_8 \).

**Proof.** If \( G \cong P_8 \), clearly \( \gamma'(G) + \beta_1(G) = p - 1 \). Conversely, suppose \( \gamma'(G) + \beta_1(G) = n - 1 \). Since \( G \) is a path of even order, obviously it is of class 1. By theorem 2.1, we have \( p = 3k + 2 \). Obviously \( \beta_1(G) = p/2 \). Then \( \gamma'(G) = p - 2/2 \) but \( P_p \) is a path and so \( \gamma'(G) = \lceil \frac{p-1}{3} \rceil \). Now \( \frac{p-2}{2} = \lceil \frac{p-1}{3} \rceil \Rightarrow \frac{3k}{2} = \lceil \frac{3k+1}{3} \rceil \Rightarrow k = 2 \). Therefore \( k=2 \). Hence \( G \cong P_8 \).

**Theorem 2.6.** Let \( G \) be any connected graph which is of class 1. Then \( ds'(G) = q - \beta_1(G) \) (where \( q \) is the number of edges) if and only if \( G \) is isomorphic to \( C_4 \), the subdivision of a star or \( P_8 \).

**Proof.** Suppose \( ds'(G) = q - \beta_1(G) \). Then \( ds'(G) = \gamma'(G) = q - \beta_1(G) \). Since \( \gamma'(G) \leq p/2 \) and \( \beta_1(G) \leq p/2 \), we have \( \gamma'(G) + \beta_1(G) \leq p \) and hence \( q \leq p \). If \( q = p \), then \( p \) is even, \( \gamma = \beta_1 = p/2 \) and \( G \) is unicyclic. Hence it follows from [6] that \( G = C_4 \). If \( q = p - 1 \), then we have the following cases:

**Case(i).** \( p \) is odd. Now \( \gamma'(G) = \beta_1(G) = \frac{(p-1)}{2} \) and \( G \) is a tree. Hence it follows from theorem 2.6 that \( G \) is isomorphic to the subdivision of a star.

**Case(ii) \( p \) is even.**

Now we have \( \gamma'(G) = \frac{(p-2)}{2}, \beta_1(G) = \frac{p}{2} \) and \( G \) is a path. Hence it follows from lemma 2.5 that \( G \) is isomorphic to \( P_8 \). The converse is obvious.

**Theorem 2.7.** For any \((p,q)\) graph \( G \) which is of class 1, \( ds'(G) + d'(G) = q+1 \) if and only if \( G \cong C_3, K_{1,p-1} \) or \( mK_2 \).

**Proof.** Suppose \( ds'(G) + d'(G) = q + 1 \). Since \( G \) is of class 1, we have \( ds'(G) = \gamma'(G) \), i.e. \( \gamma'(G) + d'(G) = q + 1 \). Since \( \gamma'(G)d'(G) \leq q \), we have \( (d'(G) - 1)(q - d'(G)) \leq 0 \). Further, \( d'(G) \geq 1 \) and \( q \geq d'(G) \). So \( (q - d'(G))(d'(G) - 1) = 0 \). Hence \( q = d'(G) \) or \( d'(G) = 1 \). If \( d'(G) = 1 \), then \( G \) is isomorphic to \( mK_2 \). If \( q = d'(G) \), then \( G = C_3 \) or \( K_{1,p-1} \). The converse is obvious.

**Theorem 2.8.** If \( T \) is a bistar, then \( T \) is of class 2.

**Proof.** Since the non-pendent edge of \( T \) is an edge dominating set of \( T \), we have \( \gamma'(T) = 1 \). There is no \( \gamma \)-set containing any of the pendent edges and so \( T \) is of class 2.
Theorem 2.9. Let $T$ be any tree and let $e = uv$ be an edge of maximum degree $\Delta'(T)$. Then $ds'(T) = q - \Delta'(T) + 1$ if and only if $T$ is isomorphic to bistar or $diam(T) = 4$, deg $w \leq 2$ for every vertex $w \neq u, v$ and there exist at least one pair of end vertices which are distant 3 apart.

Proof. By theorem 1.8, it is enough to investigate those graphs that are of class 2. If $diam(T) = 1$ or 2, then obviously $T$ is of class 1. If $diam(T) = 3$, then $T$ has exactly one non-pendent edge. Therefore $T$ is of class 2. If $diam(T) = 4$, then each non-pendent edge of $T$ is adjacent to a pendant edge of $T$ and hence the set of all non-pendent edges of $T$ forms a minimum edge dominating set and $\gamma'(T) = q - \Delta'(T)$. Based on the distance between the pendent vertices, we have the following cases:

Case(i). $d(u, v) \neq 3$, for every $u, v \in S$.
Then $d(u, v) = 1, 2$ or 4. Since $diam(T) = 4$, it is impossible that $d(u, v) = 1$ or 2. Hence there exists $u, v \in S$ with $d(u, v) = 4$. In this case $T$ is of class 1.

Case(ii). There exists $u, v \in S$ with $d(u, v) = 3$.
Let $e, e'$ be the pendent edges incident with $u, v$ respectively. Since $diam(T) = 4$, at least one of $e, e'$ should be adjacent to two non-pendent edges. Without loss of generality let $e$ be adjacent to two non-pendent edges. Then there is no two element edge dominating set containing $e$ so that $T$ is of class 2.

Theorem 2.10. Let $G$ be a graph with $\Delta'(G) = q - 2$. Let $e$ be an edge of degree $q - 2$ and let $f$ be an edge which is non adjacent to $e$. Then $G$ is of class 1 if and only if for every $g_1 \in E(G) \setminus (N[f] \cup \{e\})$, there exists $g_2 \in N[f]$ such that $N[g_1] \cup N[g_2] = E(G)$.

Proof. Suppose $G$ is of class 1. Let $e$ be an edge of degree $q - 2$ and let $f$ be an edge non-adjacent to $e$. Let $g_1 \in E(G) \setminus (N[f] \cup \{e\})$. Since $ds'(G) = \gamma'(G) = 2$, there exists $g_2 \in E(G)$ such that $\{g_1, g_2\}$ is an edge dominating set. Clearly, $g_2 \in N[f]$ and $N[g_1] \cup N[g_2] = E(G)$. The converse is immediate.

Theorem 2.11. Given three positive integers $a, b$ and $c$ with $2 \leq a \leq b \leq c$, there exists a graph $G$ with $\gamma'(G) = a, ds'(G) = a + 1, EIS(G) = b$ and $\beta(G) = c$ if and only if $b \leq 2a - 1$ and $c = b + 1$.

Proof. If there exists a graph $G$ with $\gamma'(G) = a, ds'(G) = a + 1, EIS(G) = b$ and $\beta(G) = b + 1$, then it follows from [5] that $b \leq 2a - 1$ and $c = b + 1$.

Conversely, let $b \leq 2a - 1$ and $c = b + 1$. Let $b = a + k$, where $0 \leq k \leq a - 1$. Construct a graph as follows: Let $\{u_1v_1, u_2v_2, ..., u_av_a\}$ be a set of independent edges. Add vertices $x_1, x_2, ..., x_{k+1}$ and $y_1, y_2, ..., y_{k+1}$ and join $x_i$ with
and $y_i$ with $v_i$ for all $i$, $1 \leq i \leq k + 1$. Also add a vertex $z$ and join $z$ with $u_i$ and $v_i$ for all $i$, $k + 2 \leq i \leq a$.

Clearly $\{u_1v_1, u_2v_2, \ldots, u_av_a\}$ is a minimum edge dominating set of $G$ and hence $\gamma'(G) = a$. But $x_iu_i$ and $y_iv_i$, $1 \leq i \leq k + 2$ does not belong to any $\gamma'$ set. Therefore $ds'(G) = \gamma'(G) + 1$. Therefore $\{x_1u_1, y_1v_1, u_2v_2, \ldots, u_av_a\}$ is an edge-domsaturation set with cardinality $a + 1$.

Also, $I = \{u_1x_1, u_2x_2, \ldots, u_{k+1}x_{k+1}, v_1y_1, v_2y_2, \ldots, v_{k+1}y_{k+1}, u_{k+2}v_{k+2}, \ldots, u_av_a\}$ is a maximum matching in $G$. Hence $\beta_1(G) = a + k + 1 = c$. Since $I_1 = I - \{u_1x_1, v_1y_1\} \cup \{u_1v_1\}$ is a maximum matching containing $u_1v_1$, we have $EIS(u_1v_1) = a + k$ and hence $EIS(G) = \beta_1 - 1 = b$.

3 Connected Edge-Domsaturation Number of a Graph

**Definition 3.1.** Let $G$ be a connected graph. The least positive integer $k$ such that every edge of $G$ lies in a connected edge dominating set of cardinality $k$ is called the connected edge-domsaturation number of $G$ and is denoted by $ds'_c(G)$.

**Example 3.2.** (i) $ds'_c(K_p) = p - 2$
(ii) $ds'_c(P_p) = p - 2$
(iii) $ds'_c(K_{q,p}) = \min\{q, p\}$.

**Observation 3.3.** If $G$ is any connected graph with $\Delta'(G) = q - 1$ and $G \not\cong K_{1,n}$, then $ds'_c(G) = \gamma'_c(G) + 1$.

**Proof.** Since $\Delta'(G) = q - 1$, we have $\gamma'_c(G) = 1$. Further any edge with degree less than $q - 1$ does not lie on a $\gamma'_c(G)$-set. Therefore $ds'_c(G) = \gamma'_c(G) + 1$.

**Observation 3.4.** For any connected graph $G$ with $p \geq 4$ and $\delta'(G) = 1$, we have $ds'_c(G) = \gamma'_c(G) + 1$.

**Proof.** Since no pendent edge lies on a $\gamma'_c(G)$-set, the result follows.

We now find an upper bound on connected edge-domsaturation number for trees and unicyclic graphs.

**Observation 3.5.** For any tree $T$ of order $p \geq 4$, $\gamma'_c(T) = p - 3$ if and only if $T$ is a path or $K_{1,3}$. 
Observation 3.6. For any tree $T$ of order $p \geq 4$, $d_{c}^{t}(T) = p - 2$ if and only if $T$ is a path.

Corollary 3.7. For any tree $T$ of order $p \geq 4$, $d_{c}^{t}(T) + \chi(T) \leq p$ and equality holds if and only if $T$ is a path.

Proof. It follows from observation 3.6 that for any tree $T$, $d_{c}^{t}(G) \leq p - 2$. Also $\chi(G) = 2$. Therefore $d_{c}^{t}(G) + \chi(G) \leq p$. Further $d_{c}^{t}(G) + \chi(G) = p$ if and only if $d_{c}^{t}(G) = p - 2$ or equivalently $T$ is a path.

Theorem 3.8. Let $G$ be a connected unicyclic graph with cycle $C$. Then $d_{c}^{t}(G) = p - 2$ if and only if $G \cong C$ or a cycle $C$ with exactly one pendant edge.

Proof. Let $G$ be a unicyclic graph with $d_{c}^{t}(G) = p - 2$. Let $C$ be the unique cycle in $G$ and suppose $G \neq C$. Let $S$ be the set of all pendant edges of $G$. We observe that $d_{c}^{t}(G) = p - |S|$ if no vertex in $C$ is of degree 2 and $d_{c}^{t}(G) = p - |S| - 1$ otherwise. In the former case, $|S| = 2$. But this is impossible as in this case no vertex in $C$ is of degree 2. Therefore $d_{c}^{t}(G) = p - |S| - 1$. Now $|S| = 1$ and so $G$ has exactly one pendant edge.

Theorem 3.9. For any tree $T$, $T \not\cong K_{1,n}$, $d_{c}^{t}(T) = q - \Delta'(T) + 1$ if and only if $T$ has at most one vertex of degree greater than 2 or exactly two adjacent vertices of degree greater than 2.

Proof. We observe that, $d_{c}^{t}(T) = q - k + 1$, where $k$ is the number of pendant edges of $T$. Hence $d_{c}^{t}(G) = q - \Delta'(G) + 1$ if and only if $\Delta'(G) = k$. However if $T$ has two non-adjacent vertices of degree greater than 2, then $k > \Delta'(G)$ and hence the result follows.

Theorem 3.10. Let $G$ be a connected unicyclic graph with cycle $C$ and $G \not\cong C$. Then $d_{c}^{t}(G) = q - \Delta'(G) + 1$ if and only if one of the following conditions hold.

1. $G$ has exactly one vertex of degree greater than 2
2. $G$ has exactly two vertices $u,v$ of degree greater than 2 and $u,v$ are adjacent
3. $C = C_{3}$, all the vertices of $C$ are of degree $\geq 3$, one vertex of $C$ is of degree 3 and all the vertices not on $C$ have degree one or two.

Proof. Let $G$ be a connected unicyclic graph with $d_{c}^{t}(G) = q - \Delta'(G) + 1$ and as in the proof of theorem 3.8, we have $|S| = \Delta'(G) - 1$ or $|S| = \Delta'(G) - 2$, where $S$ is the number of pendant edges of $T$. 
Case(i). $|S| = \Delta'(G) - 1$.

In this case, every vertex of $C$ is of degree $\geq 3$. Now if $C \neq C_3$, then $G$ has at most $\Delta'(G)$ pendent edges. Thus $C = C_3$. It follows that at most one vertex of $C$ is of degree 3 and all vertices not on $C$ have degree 1 or 2. Hence $G$ is of the form described in (3).

Case(ii). $|S| = \Delta'(G) - 2$

In this case, there exists at least one vertex of degree 2 on $C$. Let $e = uv$ be an edge of maximum degree $\Delta'(G)$. Since $|S| = \Delta'(G) - 2$, at least one of $u, v$ lies on $C$ and all vertices different from $u, v$ have degree one or two. If both $u, v$ have degree at least 3 then $G$ satisfies (2), Otherwise $G$ satisfies (1).

4 Domsaturation Number of a Graph

Theorem 4.1. Let $G$ be any connected graph and let $G'$ be the graph obtained from $G$ by concatenating a vertex of $G$ with the center of a star $k_{1,n_1},n_2$ ($n \geq 2$). Then $ds(G) = \gamma(G) + 1$.

Proof. Let $u \in V(G)$ be the support vertex of a star. Suppose $u$ is not dominated by any vertex of $G$, then clearly $u$ belongs to the $\gamma$-set. Suppose $u$ is dominated by some vertices of $G$. Since number of pendent vertices $\geq 2$. So in this case also $u$ belongs to the $\gamma$-set. In both these cases the pendent vertices does not belong to any $\gamma$-set. So $ds(G) = \gamma(G) + 1$.

Theorem 4.2. Given any three positive integers $a, b, c$ with $3 \leq a \leq b \leq c$, there exists a graph $G$ with $ds(G) = a, IS(G) = b$ and $\Gamma(G) = c$.

Proof. Case(i). $a = 3$

Let $k = \begin{cases} 
0 & \text{if } c \leq 2b - 2 \\
 c - 2b + 2 & \text{if } c > 2b - 2 
\end{cases}$ and let $\alpha = \begin{cases} 
2b - 2 - c & \text{if } c \leq 2b - 2 \\
 0 & \text{if } c > 2b - 2.
\end{cases}$

Let $P_4 = (v_1, v_2, v_3, v_4)$ be a path on 4 vertices. Attach $b - 2$ pendent vertices $u_1, u_2, \ldots, u_{b-2}$ to $v_2$ and $b - 2 + k$ pendent vertices $w_1, w_2, \ldots, w_{b-2+k}$ to $v_3$. Add the edges $u_1w_1, u_2w_2, \ldots, u_{\alpha}w_{\alpha}$. For the resulting graph $G$, we have $\gamma(G) = 2$. But the pendent vertices does not lie in any dominating set of cardinality 2. Thus $ds(G) = 3 = a$.

If $b = c$, then clearly $IS(G) = IS(v_2)$ or $IS(v_3)$.

If $b < c$, then $IS(G) = IS(v_3)$. Since $v_3$ is the only vertex which is the minimum of all $IS(v)$'s, for every $v \in V(G)$. In both the cases, $\{v_3, u_1, u_2, \ldots, u_{b-2}, v_1\}$ is the desired $IS$-set of $G$. Hence $IS(G) = b - 2 + 2 = b$. 


Also \( \{u_1, u_2, \ldots, u_{b-2}, w_{a+1}, w_{a+2}, \ldots, w_{b-2+k}, v_1, v_4\} \) is the maximum cardinality of a minimal dominating set and hence \( \Gamma(G) = 2b - 2 + k - \alpha \).

If \( c \leq 2b - 2 \), then \( 2b - 2 + k - \alpha = 2b - 2 - (2b - 2 - c) = c \).
If \( c > 2b - 2 \), then \( 2b - 2 + k - \alpha = 2b - 2 + c - 2b = c \). Hence \( \Gamma(G) = c \).

\textbf{Case (ii).} \( a \geq 4 \)

Let \( k = \begin{cases} 0 & \text{if } c \leq 2b - a \\ c - 2b + a & \text{if } c > 2b - a \end{cases} \) and

let \( \alpha = \begin{cases} 2b - a - c & \text{if } c \leq 2b - 2 \\ 0 & \text{if } c > 2b - a \end{cases} \)

Let \( P = (v_1, v_2, \ldots, v_a) \) be a path on \( a \) vertices. Attach pendent vertices \( u_1, u_4, \ldots, u_a \) to \( v_1, v_4, \ldots, v_a \) respectively. Attach \( b - (a - 1) \) pendent vertices \( s_1, s_2, \ldots, s_{b-(a-1)} \) to \( v_2 \), attach \( b-(a-1)+k \) pendent vertices \( t_1, t_2, \ldots, t_{b-(a-1)+k} \) to \( v_3 \) add the edge \( u_1u_a \) and the edges \( s_1t_1, s_2t_2, \ldots, sa_{\alpha} \).

Clearly \( \{u_1, v_2, v_3, \ldots, v_{a-1}\} \) is a \( \gamma \)- set. But the pendent vertices adjacent to \( v_2, v_3 \) and the vertices \( v_1, v_a \) does not belong to any \( \gamma \) set. Therefore \( ds(G) = a \).
If \( a = b = c \), then \( k = 0 \) and \( \alpha = 0 \). Hence \( IS(G) = IS(i) = a \) for all \( i \in V \) if \( a < b \) and \( b = c \), then \( IS(v_2) \) or \( IS(v_3) \) is the \( IS \)-set of \( G \). If \( a < b < c \), then \( IS(v_3) \) is the only set having minimum cardinality among all \( IS \)-sets. From these three cases, \( \{v_3, s_1, s_2, \ldots, s_{b-(a-1)}, u_1, u_4, u_5, \ldots, u_{a-1}, v_4\} \) is the desired \( IS \)-set. Hence \( IS(G) = b - (a - 1) + 1 + 1 + a - 3 = b \). Also \( \{s_1, s_2, \ldots, s_{b-(a-1)}, t_{a+1}, t_{a+2}, \ldots, t_{b-(a-1)+k}, u_4, u_5, \ldots, u_{a-1}, v_4, v_1\} \) is a dominating set of maximum cardinality and hence \( \Gamma(G) = 2b - a + k - \alpha \). As in case (i), we have \( \Gamma(G) = c \).

\section*{References}


