Some Results on Intuitionistic Fuzzy Ideals in BCK-Algebras

B. Satyanarayana¹ and R. Durga Prasad²

¹,²Department of Applied Mathematics, Acharya Nagarjuna University Campus, Nuzvid-521201, Krishna (District), Andhra Pradesh, India
¹E-mail: drbsn63@yahoo.co.in
²E-mail: durgaprasad.fuzzy@gmail.com

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Abstract

In this paper, we give some results on the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals, intuitionistic fuzzy commutative ideals.

Keywords: BCK-algebra, Fuzzy (implicative, positive implicative and commutative) ideal.

1 Introduction

After the introduction of the concept of fuzzy sets by Zadeh [12] several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanassov [1, 2] as a generalization of the notion of fuzzy set. The first author (together with Hong, Kim, Meng, Roh and Song) [3, 5, 6, 7] considered the fuzzification of ideals and sub-algebras in BCK-algebras (cf. [3, 4, 5, 6]). In this paper we give some results on the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals, intuitionistic fuzzy commutative ideals.
2 Preliminaries

First we present the fundamental definitions. By a BCK-algebra (see [7, 8, 9]) we mean a nonempty set $X$ with a binary operation $*$ and a constant 0 satisfying the axioms:

1. **(BCK-1)** $(x * y) * (x * z) \leq (z * y)$,
2. **(BCK-2)** $x * (x * y) \leq y$,
3. **(BCK-3)** $x \leq x$,
4. **(BCK-4)** $x \leq y$ and $y \leq x$ imply that $x = y$,
5. **(BCK-5)** $0 \leq x$

for all $x, y, z \in X$.

A partial ordering “≤” on $X$ can be defined by $x \leq y$ if and only if $x * y = 0$. In any BCK-algebra $X$ the following holds:

1. **(P1)** $x * 0 = x$
2. **(P2)** $x * y \leq x$
3. **(P3)** $(x * y) * z = (x * z) * y$
4. **(P4)** $(x * z) * (y * z) \leq x * y$
5. **(P5)** $x * (x * (x * y)) = x * y$
6. **(P6)** $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$, for all $x, y, z \in X$.

A BCK-algebra $X$ is said to be implicative if $x = x * (y * x)$, for all $x, y \in X$.

A BCK-algebra $X$ is said to be positive implicative if $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

A BCK-algebra $X$ is said to be commutative if $x * (x * y) = y * (y * x)$ for all $x, y, z \in X$.

A non-empty subset $I$ of a BCK-algebra $X$ is called an ideal of $X$, if it satisfies:

1. **(I1)** $0 \in I$
2. **(I2)** $x * y \leq x$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

A non-empty subset $I$ of a BCK-algebra $X$ is said to be sub-algebra of $X$ if $x * y \in X$ whenever $x, y \in X$.

A non-empty subset $I$ of a BCK-algebra $X$ is called an implicative ideal of $X$ if it satisfies **(I1)** and **(I2)**.

A non-empty subset $I$ of a BCK-algebra $X$ is called a commutative ideal of $X$ if it satisfies **(I1)** and **(I2)**.

A non-empty subset $I$ of a BCK-algebra $X$ is said to be positive implicative ideal of $X$ if it satisfies **(I1)** and **(I2)**.

Let $\mu$ and $\lambda$ be the fuzzy sets in a set $X$. For $s, t \in [0, 1]$, the set $U(\mu, s) = \{ x \in X / \mu(x) \geq s \}$ is called a upper level of $\mu$ and the set $L(\lambda, t) = \{ x \in X / \lambda(x) \leq t \}$ is called a lower level of $\lambda$. 
An intuitionistic fuzzy set $A$ in a non-empty set $X$ is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x))| x \in X\}$, where the function $\mu_A : X \rightarrow [0,1]$ and $\lambda_A : X \rightarrow [0,1]$ denoted the degree of membership (namely $\mu(x)$) and the degree of non membership (namely $\lambda(x)$) of each element $x \in X$ to the set $A$ respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (X, \mu_A, \lambda_A)$ or $A = (\mu_A, \lambda_A)$.

**Definition 2.1.** Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in $X$. Then

(i) $A = \{(x, \mu_A(x), \mu_A(x))| x \in X\}$

(ii) $\Diamond A = \{(x, \lambda_A(x), \lambda_A(x))| x \in X\}$.

In what follows, let $X$ denote a BCK-algebra unless otherwise specified.

**Definition 2.2.** An IFS $A = (X, \mu_A, \lambda_A)$ in $X$ is an intuitionistic fuzzy sub-algebra of $X$ if it satisfies

(IFS 1) $\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}$

(IFS 2) $\lambda_A(x \ast y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in X$.

**Example 2.3.** Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $A = (X, \mu_A, \lambda_A)$ be an IFS in $X$ defined by

$\mu_A(0) = \mu_A(a) = \mu_A(c) = 0.7 > 0.3 = \mu_A(b)$

and

$\lambda_A(0) = \lambda_A(a) = \lambda_A(c) = 0.2 < 0.5 = \lambda_A(b)$.

Then $A = (X, \mu_A, \lambda_A)$ is an IF subalgebra of $X$.

**Proposition 2.4.** Let $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy sub-algebra of $X$, then

$\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in X$. 

**Definition 2.5.** An IF \( A = (X, \mu_A, \lambda_A) \) in \( X \) is an intuitionistic fuzzy ideal (IF-ideal) of \( X \) if it satisfies

\begin{align*}
(\text{IF1}) & \quad \mu_A(0) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \lambda_A(x) \\
(\text{IF2}) & \quad \mu_A(x) \geq \min\{\mu_A(x \ast y), \mu_A(x)\} \\
(\text{IF3}) & \quad \lambda_A(x) \leq \min\{\lambda_A(x \ast y), \lambda_A(y)\}, \text{ for all } x, y \in X.
\end{align*}

**Theorem 2.6.** \([4]\) Let \( A = (X, \mu_A, \lambda_A) \) be an intuitionistic fuzzy ideal of \( X \). If \( x \leq y \) in \( X \), then

\[ \mu_A(x) \geq \mu_A(y), \quad \lambda_A(x) \leq \lambda_A(y), \]

that is \( \mu_A \) is order-reversing and \( \lambda_A \) is order-preserving.

**Theorem 2.7.** \([4]\) Every intuitionistic fuzzy ideal of \( X \) is an intuitionistic fuzzy subalgebra of \( X \).

**Theorem 2.8.** \([4]\) \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( X \) if and only if for \( x, y, z \in X, x \ast y \leq z \Rightarrow \mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\} \) and \( \lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z)\} \).

**Proposition 2.9.** \([4]\) \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( X \) if and only if the non-empty upper \( s \)-level cut \( U(\mu_A; s) \) and the non-empty lower \( t \)-level cut \( L(\lambda_A; t) \) are ideals of \( X \), for any \( s, t \in [0,1] \).

**Corollary 2.10.** \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy subalgebra of \( X \) if and only if the non-empty upper \( s \)-level cut \( U(\mu_A; s) \) and the non-empty lower \( t \)-level cut \( L(\lambda_A; t) \) are sub-algebras of \( X \), for any \( s, t \in [0,1] \).

**Proposition 2.11.** \([11]\) In a BCK-algebra \( X \), the following holds, for all \( x, y, z \in X \).

\begin{align*}
(i) & \quad ((x \ast z) \ast z) \ast (y \ast z) \leq (x \ast y) \ast z. \\
(ii) & \quad (x \ast z) \ast (x \ast (x \ast z)) = (x \ast z) \ast z \\
(iii) & \quad (x \ast (y \ast (y \ast x))) \ast (y \ast (x \ast (y \ast (y \ast x)))) \leq x \ast y.
\end{align*}

### 3 Main Results

In this section we present the results on the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals and intuitionistic fuzzy commutative ideals.
Definition 3.1. [11] An IFS \((X, \mu_A, \lambda_A)\) in a BCK-algebra \(X\) is an intuitionistic fuzzy implicative ideal (IFI-ideal) of \(X\) if it satisfies

\begin{align*}
(IFI 1) \quad & \mu_A(0) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \lambda_A(x) \\
(IFI 2) \quad & \mu_A(x) \geq \min\{\mu_A((x \ast (y \ast x)) \ast z), \mu_A(z)\} \\
(IFI 3) \quad & \lambda_A(x) \leq \max\{\lambda_A((x \ast (y \ast x)) \ast z), \lambda_A(z)\}, \text{ for all } x, y, z \in X.
\end{align*}

Definition 3.2. [11] An IFS \((X, \mu_A, \lambda_A)\) in \(X\) is an intuitionistic fuzzy commutative ideal (IFCI-ideal) of \(X\) if it satisfies

\begin{align*}
(IFCI 1) \quad & \mu_A(0) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \lambda_A(x) \\
(IFCI 2) \quad & \mu_A(x \ast (y \ast (y \ast x))) \geq \min\{\mu_A((x \ast y) \ast z), \mu_A(z)\} \\
(IFCI 3) \quad & \lambda_A(x \ast (y \ast (y \ast x))) \leq \max\{\lambda_A((x \ast y) \ast z), \lambda_A(z)\}, \text{ for all } x, y, z \in X.
\end{align*}

Definition 3.3. [11] An IFS \((X, \mu_A, \lambda_A)\) in a BCK-algebra \(X\) is an intuitionistic fuzzy positive implicative ideal (IFPI-ideal) of \(X\) if it satisfies

\begin{align*}
(IFPI 1) \quad & \mu_A(0) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \lambda_A(x) \\
(IFPI 2) \quad & \mu_A(x \ast z) \geq \min\{\mu_A((x \ast y) \ast z), \mu_A(y \ast z)\} \\
(IFPI 3) \quad & \lambda_A(x \ast z) \leq \max\{\lambda_A((x \ast y) \ast z), \lambda_A(y \ast z)\}, \text{ for all } x, y, z \in X.
\end{align*}

Theorem 3.4. An intuitionistic fuzzy ideal \(A = (X, \mu_A, \lambda_A)\) of \(X\) is an intuitionistic fuzzy implicative if and only if \(A\) is both intuitionistic commutative and intuitionistic fuzzy positive implicative.

**Proof:** Assume that \(A = (X, \mu_A, \lambda_A)\) is an intuitionistic fuzzy implicative ideal of \(X\). By (2.11(i) and 2.8), we have

\[
\min\{\mu_A((x \ast y) \ast z), \mu_A(y \ast z)\} \leq \mu_A((x \ast z) \ast z)
\]

\[
= \mu_A((x \ast z) \ast (x \ast (x \ast z))) \quad \text{ (by 2.11(ii))}
\]

\[
= \mu_A(x \ast z) \quad \text{ (by [11, 3.7(iii)])}
\]

and

\[
\max\{\lambda_A((x \ast y) \ast z), \lambda_A(y \ast z)\} \geq \lambda_A((x \ast z) \ast z)
\]

\[
= \lambda_A((x \ast z) \ast (x \ast (x \ast z)))
\]

\[
= \lambda_A(x \ast z), \text{ for all } x, y, z \in X.
\]

Then \(A = (X, \mu_A, \lambda_A)\) is an intuitionistic fuzzy positive implicative ideal of \(X\). And by theorem 2.6, 2.11(iii) and 3.7(iii),

\[
\mu_A(x \ast y) \leq \mu_A((x \ast (y \ast (y \ast x))) \ast (y \ast (x \ast (y \ast x)))) = \mu_A(x \ast (y \ast (y \ast x)))
\]

and
\[ \lambda_A(x \ast y) \geq \lambda_A((x \ast (y \ast (y \ast x))) \ast (y \ast (x \ast (y \ast x)))) = \lambda_A(x \ast (y \ast (y \ast x))). \]

It follows from [11, 4.6] that \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy commutative ideal of \( X \). Conversely, suppose that \( A = (X, \mu_A, \lambda_A) \) is both intuitionistic fuzzy positive implicative and intuitionistic fuzzy commutative ideal of \( X \). Since, \((y \ast (y \ast x)) \ast (y \ast x) \leq x \ast (y \ast x)\), it follows from theorem 2.6.

\[ \mu_A(y \ast (y \ast x)) \geq \mu_A(x \ast (y \ast x)) \text{ and } \lambda_A(y \ast (y \ast x)) \leq \lambda_A(x \ast (y \ast x)). \]

Using [11, 5.8], we have
\[ \mu_A(y \ast (y \ast x)) = \mu_A(y \ast (y \ast x)) \]
and
\[ \lambda_A(y \ast (y \ast x)) = \lambda_A(y \ast (y \ast x)). \]
Therefore
\[ \mu_A(x \ast (y \ast x)) \leq \mu_A(y \ast (y \ast x)) \text{ and } \lambda_A(x \ast (y \ast x)) \geq \lambda_A(y \ast (y \ast x)) \quad \text{... (1)} \]

On the other hand since \( x \ast y \leq x \ast (y \ast x) \), we have, by theorem 2.6
\[ \mu_A(x \ast y) \geq \mu_A(x \ast (y \ast x)) \text{ and } \lambda_A(x \ast y) \leq \lambda_A(x \ast (y \ast x)). \]

Since \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy commutative ideal of \( X \), by [11, 4.7] we have
\[ \mu_A(x \ast y) = \mu_A(x \ast (y \ast (y \ast x))) \text{ and } \lambda_A(x \ast y) = \lambda_A(x \ast (y \ast (y \ast x))). \]
Hence
\[ \mu_A(x \ast (y \ast x)) \leq \mu_A(x \ast (y \ast (y \ast x))) \text{ and } \lambda_A(x \ast (y \ast x)) \geq \lambda_A(x \ast (y \ast (y \ast x))) \quad \text{... (2)} \]

Combining (1) and (2), we obtain
\[ \mu_A(x \ast (y \ast x)) \leq \min\{\mu_A(x \ast (y \ast (y \ast x))), \mu_A(y \ast (y \ast x))\} \leq \mu_A(x) \]
and
\[ \lambda_A(x \ast (y \ast x)) \geq \max\{\lambda_A(x \ast (y \ast (y \ast x))), \lambda_A(y \ast (y \ast x))\} \geq \lambda_A(x). \]
So \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy implicative ideal of \( X \). The proof is complete.

**Theorem 3.5.** If \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy ideal of \( X \) with the following conditions holds

(i) \( \mu_A(x \ast y) \geq \min\{\mu_A((x \ast y) \ast z), \mu_A(z)\} \)

(ii) \( \lambda_A(x \ast y) \leq \max\{\lambda_A(((x \ast y) \ast z), \lambda_A(z)\} \), for all \( x, y, z \in X \). Then \( A \) is intuitionistic fuzzy positive implicative ideal of \( X \).
Proof: Suppose $A = (X, \mu_A, \lambda_A)$ is intuitionistic fuzzy ideal of $X$.
with condition (i) and (ii). Using (P3) and (P4), we have

$((x \ast z) \ast (y \ast z)) \leq (x \ast z) \ast y = (x \ast y) \ast z$, for all $x, y, z \in X$.

therefore by theorem 2.6

$\mu_A(((x \ast z) \ast (y \ast z))) \geq \mu_A((x \ast y) \ast z)$

And

$\lambda_A(((x \ast z) \ast (y \ast z))) \leq \lambda_A((x \ast y) \ast z)$.

Now

$\mu_A(x \ast z) \geq \min\{\mu_A(((x \ast z) \ast (y \ast z)), \mu_A(y \ast z)\}$

$\geq \min\{\mu_A((x \ast y) \ast z), \mu_A(y \ast z)\}$, for all $x, y, z \in X$

and

$\lambda_A(x \ast z) \leq \max\{\lambda_A(((x \ast z) \ast (y \ast z)), \lambda_A(y \ast z)\}$

$\leq \max\{\lambda_A((x \ast y) \ast z), \lambda_A(y \ast z)\}$, for all $x, y, z \in X$.

Hence $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of $X$.

Lemma 3.6. Let $A = (X, \mu_A, \lambda_A)$ be a fuzzy ideal of $X$, then $A$ is an intuitionistic fuzzy positive implicative ideal of $X$ if and only if

$\mu_A((x \ast z) \ast (y \ast z)) \geq \mu_A((x \ast y) \ast z)$ and $\lambda_A((x \ast z) \ast (y \ast z)) \leq \lambda_A((x \ast y) \ast z)$,

for all $x, y, z \in X$.

Proof: Suppose that $A = (X, \mu_A, \lambda_A)$ is a fuzzy ideal of $X$ and

$\mu_A((x \ast z) \ast (y \ast z)) \geq \mu_A((x \ast y) \ast z)$ and $\lambda_A((x \ast z) \ast (y \ast z)) \leq \lambda_A((x \ast y) \ast z)$,

for all $x, y, z \in X$. Therefore

$\mu_A(x \ast z) \geq \min\{\mu_A((x \ast z) \ast (y \ast z)), \mu_A(y \ast z)\} \geq \min\{\mu_A((x \ast y) \ast z), \mu_A(y \ast z)\}$

$\lambda_A(x \ast z) \leq \max\{\lambda_A((x \ast z) \ast (y \ast z)), \lambda_A(y \ast z)\} \leq \max\{\lambda_A((x \ast y) \ast z), \lambda_A(y \ast z)\}$,

for all $x, y, z \in X$. Thus $A$ is an intuitionistic fuzzy positive implicative ideal of $X$.
Conversely, assume that $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of $X$ implies that $A = (X, \mu_A, \lambda_A)$ is an IF-ideal of $X$.

Let $a = x \ast (y \ast z)$ and $b = x \ast y$,
Since \(((x*(y*z))*y)\) \(\leq y*(y*z)\),

we have that
\[
\mu_A((a*b)*z) = \mu_A(((x*(y*z))*(x*y)*z) \geq \mu_A((y*(y*z))*z) = \mu_A(0)
\]
and so,
\[
\mu_A((x*z)*(y*z)) = \mu_A((x*(y*z))*z) = \mu_A(a*z)
\]
\[
\geq \min\{\mu_A((a*b)*z), \mu_A(b*z)\} \geq \min\{\mu_A(0), \mu_A(b*z)\}
\]
\[
= \mu_A(b*z) = \mu_A((x*y)*z).
\]

Therefore
\[
\mu_A((x*z)*(y*z)) \geq \mu_A((x*y)*z), \text{ for all } x, y, z \in X.
\]

And
\[
\lambda_A((a*b)*z) = \lambda_A(((x*(y*z))*(x*y)*z) \leq \lambda_A((y*(y*z))*z) = \lambda_A(0)
\]

And so,
\[
\lambda_A((x*z)*(y*z)) = \lambda_A((x*(y*z))*z) = \lambda_A(a*z)
\]
\[
\leq \max\{\lambda_A((a*b)*z), \lambda_A(b*z)\} \leq \max\{\lambda_A(0), \lambda_A(b*z)\}
\]
\[
= \lambda_A(b*z) = \lambda_A((x*y)*z).
\]

Therefore
\[
\lambda_A((x*z)*(y*z)) \leq \lambda_A((x*y)*z), \text{ for all } x, y, z \in X.
\]

Thus
\[
\mu_A((x*z)*(y*z)) \geq \mu_A((x*y)*z)), \lambda_A((x*z)*(y*z)) \leq \lambda_A((x*y)*z)),
\]

for all \(x, y, z \in X\).

**Theorem 3.7.** If \(A = (X, \mu_A, \lambda_A)\) is intuitionistic fuzzy positive implicative ideal of \(X\) then (PI 1) for any
\[
x, y, a, b \in X, ((x*y)*y)*a \leq b \Rightarrow \mu_A(x*y) \geq \min\{\mu_A(a), \mu_A(b)\}
\]

and
\[
\lambda_A(x*y) \leq \max\{\lambda_A(a), \lambda_A(b)\}.
\]

(PI 2) For any
\[
x, y, z, a, b \in X, ((x*y)*z)*a \leq b \Rightarrow \mu_A((x*z)*(y*z)) \geq \min\{\mu_A(a), \mu_A(b)\}
\]

and
\[
\lambda_A((x*z)*(y*z)) \leq \max\{\lambda_A(a), \lambda_A(b)\}.
\]

**Proof:** Suppose, \(A = (X, \mu_A, \lambda_A)\) is intuitionistic fuzzy positive implicative ideal of \(X\).

(PI1). Let \(x, y, z \in X\) be such that \(((x*y)*y)*a \leq b\). Using 2.6,

we have
\[ \mu_A((x \ast y) \ast y) \leq \min\{\mu_A(a),\mu_A(b)\} \text{ and } \lambda_A((x \ast y) \ast y) \leq \max\{\lambda_A(a),\lambda_A(b)\}. \]

It follows that

\[ \mu_A(x \ast y) \geq \min\{\mu_A((x \ast y) \ast y),\mu_A(y \ast y)\} = \min\{\mu_A((x \ast y) \ast y),\mu_A(0)\} \]

\[ = \mu_A((x \ast y) \ast y) \geq \min\{\mu_A(a),\mu_A(b)\}. \]

And

\[ \lambda_A(x \ast y) \leq \max\{\lambda_A((x \ast y) \ast y),\lambda_A(y \ast y)\} \]

\[ = \max\{\lambda_A((x \ast y) \ast y),\lambda_A(0)\} = \lambda_A((x \ast y) \ast y) \leq \max\{\lambda_A(a),\lambda_A(b)\}. \]

(ii) Now let \( x, y, z \in X \) be such that \( (x \ast y) \ast z \ast a \leq b \).

Since \( A = (X, \mu_A, \lambda_A) \) intuitionistic fuzzy positive implicative ideal of \( X \), it follows from known lemma 3.6,

\[ \mu_A((x \ast z) \ast (y \ast z)) \geq \mu_A((x \ast y) \ast z) \geq \min\{\mu_A(a),\mu_A(b)\} \]

and

\[ \lambda_A((x \ast z) \ast (y \ast z)) \leq \lambda_A((x \ast y) \ast z) \leq \max\{\lambda_A(a),\lambda_A(b)\} \]

This completes the proof.

**Theorem 3.8.** Let \( A = (X, \mu_A, \lambda_A) \) be IFS in \( X \) satisfying the condition

\[ ((x \ast y) \ast y) \ast a \leq b \implies \mu_A(x \ast y) \geq \min\{\mu_A(a),\mu_A(b)\} \]

and

\[ \lambda_A(x \ast y) \leq \max\{\lambda_A(a),\lambda_A(b)\}, \]

for any \( x, y, a, b \in X \). Then \( A = (X, \mu_A, \lambda_A) \) intuitionistic fuzzy positive implicative ideal of \( X \).

**Proof:** First we prove that \( A = (X, \mu_A, \lambda_A) \) is an IF-ideal of \( X \).

Let \( x, y, z \in X \) be such that \( x \ast y \leq z \).

Then \( (((x \ast 0) \ast 0) \ast y) \ast z = (x \ast y) \ast z = 0 \), that is \( (((x \ast 0) \ast 0) \ast y) \leq z \)

Since, for \( x, y, a, b \in X \),

\[ ((x \ast y) \ast y) \ast a \leq b \implies \mu_A(x \ast y) \geq \min\{\mu_A(a),\mu_A(b)\} \]

and

\[ \lambda_A(x \ast y) \leq \max\{\lambda_A(a),\lambda_A(b)\}. \]
Put \( y = 0, a = y, b = z \),

we get

\[
\mu_A(x) = \mu(x \ast 0) \geq \min\{\mu_A(y), \mu_A(z)\}
\]

and

\[
\lambda_A(x) = \lambda_A(x \ast 0) \leq \max\{\lambda_A(y), \lambda_A(z)\}.
\]

It follows that \( A = (X, \mu_A, \lambda_A) \) is IF- ideal of \( X \).

Note that

\[
((x \ast y) \ast y) \ast ((x \ast y) \ast y) \ast 0 = 0
\]

implies

\[
((x \ast y) \ast y) \ast ((x \ast y) \ast y) \leq 0, \forall x, y \in X.
\]

From hypothesis we have

\[
\mu_A(x \ast y) \geq \min\{\mu_A((x \ast y) \ast y), \mu_A(0)\} = \mu_A((x \ast y) \ast y)
\]

and

\[
\lambda_A(x \ast y) \leq \max\{\lambda_A((x \ast y) \ast y), \lambda_A(0)\} = \lambda_A((x \ast y) \ast y).
\]

And so \( A = (X, \mu_A, \lambda_A) \) is intuitionistic fuzzy positive implicative ideal of \( X \).

**Theorem 3.9.** Let \( A = (X, \mu_A, \lambda_A) \) be an IFS in \( X \) satisfying \( ((x \ast y) \ast z) \ast a \leq b \) imply \( \mu_A((x \ast y) \ast (y \ast z)) \geq \min\{\mu_A(a), \mu_A(b)\} \) and \( \lambda_A((x \ast y) \ast (y \ast z)) \leq \max\{\lambda_A(a), \lambda_A(b)\} \) for any \( x, y, z, a, b \in X \).

Then \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy positive implicative ideal of \( X \).

**Proof:** Let \( x, y, a, b \in X \) be such that \( ((x \ast y) \ast y) \ast a \leq b \), that is

\[
((x \ast y) \ast y) \ast a \ast b = 0
\]

therefore

\[
\mu_A(x \ast y) = \mu_A((x \ast y) \ast 0) = \mu_A((x \ast y) \ast (y \ast y)) \geq \min\{\mu_A(a), \mu_A(b)\}
\]

And

\[
\lambda_A(x \ast y) = \lambda_A((x \ast y) \ast 0) = \lambda_A((x \ast y) \ast (y \ast y)) \geq \min\{\lambda_A(a), \lambda_A(b)\}.
\]

It follows from 3.8, \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy positive implicative ideal of \( X \).

**Theorem 3.10.** Let \( A = (X, \mu_A, \lambda_A) \) be an intuitionistic fuzzy positive implicative ideal of BCK-algebra \( X \), then so is \( A = (X, \mu_A, \lambda_A) \).

**Proof:** We have \( \mu_A(0) \geq \mu_A(x) \Rightarrow 1 - \overline{\mu_A}(0) \geq 1 - \overline{\mu_A}(x) \Rightarrow \overline{\mu_A}(0) \leq \overline{\mu_A}(x), \forall x \in X \).

Consider for any \( x, y, z \in X \),
\[ \mu_A(x \ast z) \geq \min\{\mu_A((x \ast y) \ast z), \mu_A(y \ast z)\} \]
\[ \Rightarrow 1 - \overline{\mu}_A(x \ast z) \geq \min\{1 - \overline{\mu}_A((x \ast y) \ast z), 1 - \overline{\mu}_A(y \ast z)\} \]
\[ \Rightarrow \overline{\mu}_A(x \ast z) \leq 1 - \min\{1 - \overline{\mu}_A((x \ast y) \ast z), 1 - \overline{\mu}(y \ast z)\} \]
\[ \Rightarrow \overline{\mu}_A(x \ast z) \leq \max\{\overline{\mu}_A((x \ast y) \ast z), \overline{\mu}_A(y \ast z)\} \]

Hence \( A=(X, \mu_A, \overline{\mu}_A) \) is an intuitionistic fuzzy positive implicative ideal of BCK-algebra \( X \).

**Theorem 3.11.** Let \( A = (X, \mu_A, \lambda_A) \) be an intuitionistic fuzzy positive implicative ideal of BCK-algebra \( X \) then so is \( \Diamond A = (X, \lambda_A, \overline{\lambda}_A) \).

**Proof:** We have \( \lambda_A(0) \leq \lambda_A(x) \Rightarrow 1 - \overline{\lambda}_A(0) \leq 1 - \overline{\lambda}_A(x) \Rightarrow \lambda_A(0) \geq \lambda_A(x), \forall x \in X \).

Consider for any \( x, y, z \in X \)
\[ \lambda_A(x \ast z) \leq \max\{\lambda_A((x \ast y) \ast z), \lambda_A(y \ast z)\} \]
\[ \Rightarrow 1 - \overline{\lambda}_A(x \ast z) \leq \max\{1 - \overline{\lambda}_A((x \ast y) \ast z), 1 - \overline{\lambda}_A(y \ast z)\} \]
\[ \Rightarrow \overline{\lambda}_A(x \ast z) \leq 1 - \max\{1 - \overline{\lambda}_A((x \ast y) \ast z), 1 - \overline{\lambda}_A(y \ast z)\} \]
\[ \Rightarrow \overline{\lambda}_A(x \ast z) \geq \min\{\lambda_A((x \ast y) \ast z), \overline{\lambda}_A(y \ast z)\} \].

Hence \( \Diamond A = (X, \lambda_A, \overline{\lambda}_A) \) is an intuitionistic fuzzy positive implicative ideal of BCK-algebra \( X \).

**Theorem 3.12.** \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy positive implicative ideal of BCK-algebra \( X \) if and only if \( A=(X, \mu_A, \overline{\mu}_A) \) and \( \Diamond A = (X, \lambda_A, \overline{\lambda}_A) \) are intuitionistic fuzzy positive implicative ideal of BCK-algebra.

**Theorem 3.13.** \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy positive implicative ideal of BCK-algebra \( X \) if and only if the non-empty upper \( s \)-level cut \( U(\mu_A; s) \) and the non-empty lower \( t \)-level cut \( L(\lambda_A; t) \) are PI-ideals of \( X \), for any \( s, t \in [0,1] \).

**Proof:** Suppose \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy positive implicative ideal of \( X \) and \( U(\mu_A; s) \neq \emptyset \) for any \( s \in [0,1] \). It is clear that for any \( x \in X \),
\[ \mu_A(0) \geq \mu_A(x) \Rightarrow \mu_A(0) \geq \mu_A(x) \geq s \Rightarrow \mu_A(0) \geq s \text{ implies } 0 \in U(\mu_A; s). \]

Furthermore if \( (x \ast y) \ast z \in U(\mu_A; s), y \ast z \in U(\mu_A; s) \) implies \( \mu_A((x \ast y) \ast z)) \geq s \) and \( \mu_A(y \ast z) \geq s \).
Therefore
\[ \mu_A(x * z) \geq \min\{\mu_A((x * y) * z), \mu_A(y * z)\} \geq \min\{s, s\} = s \]
implies \( x * z \in U(\mu_A; s) \).

This shows that \( U(\mu_A; s) \) is positive implicative ideal of \( X \).
Similarly, we can prove \( L(\lambda_A, t) \) is positive implicative ideal of \( X, \forall s, t \in [0,1] \)
Conversely, assume that for any \( s, t \in [0,1] \), \( U(\mu_A; s) \) and \( L(\lambda_A, t) \) are either empty or positive implicative ideals of \( X \).

Put \( \mu_A(x) = s, \lambda_A(x) = t \) for any \( x \in X \).

Since \( 0 \in U(\mu_A; s) \Rightarrow \mu_A(0) \geq s = \mu_A(x) \) and \( 0 \in L(\lambda_A, t) \Rightarrow \lambda_A(0) \leq t = \lambda_A(x) \)
thus
\[ \mu_A(0) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \lambda_A(x) \text{ for all } x \in X. \]

Now we only need to show that (IFPI 3),
then take \( s_1 = \min\{\mu_A((x * y) * z), \mu_A(y * z)\} \Rightarrow (x * y) * z, y * z \in U(\mu_A; s_1). \)
Since \( U(\mu_A; s_1) \) is implicative ideal of \( X \)
we have
\[ y * z \in U(\mu_A; s_1) \Rightarrow \mu_A(x * z) \geq s_1 = \min\{\mu_A((x * y) * z), \mu_A(y * z)\}. \]
Therefore
\[ \mu_A(x * z) \geq \min\{\mu_A((x * y) * z), \mu_A(y * z)\} \text{ for all } x, y, z \in X \]
Similarly we can prove \( \lambda_A(x * z) \leq \min\{\lambda_A((x * y) * z), \lambda_A(y * z)\} \) for all \( x, y, z \in X \).

Hence \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy positive implicative ideal of BCK-algebra \( X \).

**Theorem 3.14.** \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy implicative or commutative ideals of BCK-algebra \( X \) if and only if the non-empty upper \( s \)-level cut \( U(\mu_A; s) \) and the non-empty lower \( t \)-level cut \( L(\lambda_A, t) \) are implicative or commutative ideals of \( X \), for any \( s, t \in [0,1] \).

**Corollary 3.15.** \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy implicative ideal of BCK-algebra \( X \) if and only if the non-empty upper \( s \)-level cut \( U(\mu_A; s) \) and the non-empty
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lower t-level cut \( L(\lambda_A; t) \) are both commutative and positive ideals of \( X \), for any \( s, t \in [0, 1] \).

**Corollary 3.16.** \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy commutative and intuitionistic fuzzy positive implicative ideals of BCK-algebra \( X \) if and only if the non-empty upper s-level cut \( U(\mu_A; s) \) and the non-empty lower t-level cut \( L(\lambda_A; t) \) are implicative ideals of \( X \), for any \( s, t \in [0, 1] \).

**Theorem 3.17.** Let \( A = (X, \mu_A, \lambda_A) \) be an IFS of a BCK-algebra. If \( A \) is an intuitionistic fuzzy positive implicative ideal of \( X \) then the set \( J = \{ x \in X/ \mu_A(x) = \mu_A(0) \} \) and \( K = \{ x \in X/ \lambda_A(x) = \lambda_A(0) \} \) are an PI-ideal of \( X \).

**Proof:** Assume that \( A = (X, \mu_A, \lambda_A) \) intuitionistic fuzzy positive implicative ideal of \( X \). Since, \( \mu_A(0) = \mu_A(0) \Rightarrow 0 \in J \).

If \((x * y) * z, y * z \in J \Rightarrow \mu_A((x * y) * z) = \mu_A(0) \) and \( \mu_A(y * z) = \mu_A(0) \).

Since
\[
\mu_A(x * z) \geq \min\{\mu_A((x * y) * z), \mu_A(y * z)\} = \min\{\mu_A(0), \mu_A(0)\} = \mu_A(0),
\]
but,
\[
\mu_A(x * z) \leq \mu_A(0). \text{ Therefore, } \mu_A(x * z) = \mu_A(0) \Rightarrow x * z \in J.
\]

Thus, \( J \) is an implicative ideal of \( X \) and \( \lambda_A(0) = \lambda_A(0) \Rightarrow 0 \in K \).

If \((x * y) * z, y * z \in K \)

Then
\[
\lambda_A((x * y) * z) = \lambda_A(0)
\]
And
\[
\lambda_A(y * z) = \lambda_A(0).
\]

Since,
\[
\lambda_A(x * z) \leq \max\{\lambda_A((x * y) * z), \lambda_A(y * z)\} = \max\{\lambda_A(0), \lambda_A(0)\} = \lambda_A(0)
\]
but,
\[
\lambda_A(x * z) \geq \lambda_A(0).
\]

Therefore, \( \lambda_A(x * z) = \lambda_A(0) \Rightarrow x * z \in K \).
Thus, \( K \) is an implicative ideal of \( X \).

**Theorem 3.18.** (Extension property for intuitionistic fuzzy positive implicative ideals)
Let \( A = (X, \mu_A, \lambda_A) \) and \( B = (X, \mu_B, \lambda_B) \) are two fuzzy ideals of \( X \) such that \( A(0) = B(0) \) and \( A \subseteq B \) (that is \( \mu_A(0) = \mu_B(0), \lambda_A(0) = \lambda_B(0) \) and \( \mu_A(x) \leq \mu_B(x) \),
\( \lambda_A(x) \geq \lambda_B(x), \forall x \in X \). If \( A = (X, \mu_A, \lambda_A) \) is an intuitionistic fuzzy positive implicative ideal of \( X \), then so is \( B \).

**Proof:** Suppose that \( A = (X, \mu_A, \lambda_A) \) is intuitionistic fuzzy positive implicative ideal of \( X \)

\[
\mu_B(((x \ast z) \ast (y \ast z)) \ast ((x \ast y) \ast z)) = \mu_B(((x \ast z) \ast ((x \ast y) \ast z)) \ast (y \ast z)) \quad \text{(by P2)}
\]

\[
= \mu_B(((x \ast ((x \ast y) \ast z)) \ast z) \ast (y \ast z)) \quad \text{(by P2)}
\]

\[
\geq \mu_A(((x \ast ((x \ast y) \ast z)) \ast z) \ast (y \ast z)) \quad \text{(Since } \mu_A \subseteq \mu_B) \]

\[
\geq \mu_A(((x \ast ((x \ast y) \ast z)) \ast y) \ast z) \quad \text{(by lemma 3.6)}
\]

\[
= \mu_A(((x \ast y) \ast ((x \ast y) \ast z)) \ast z) \quad \text{(by P2)}
\]

\[
= \mu_A(((x \ast y) \ast z) \ast ((x \ast y) \ast z)) \quad \text{(by P2)}
\]

\[
= \mu_A(0) = \mu_B(0) \quad \text{(by BCK-3 ).}
\]

It follows from (F1) and (F2) that

\[
\mu_B((x \ast z) \ast (y \ast z)) \geq \min\{\mu_B(((x \ast z) \ast (y \ast z)) \ast ((x \ast y) \ast z)), \mu_B((x \ast y) \ast z)\}
\]

\[
\geq \min\{\mu_B(0), \mu_B((x \ast y) \ast z)\} = \mu_B((x \ast y) \ast z) \text{ for all } x, y, z \in X.
\]

Therefore, for any \( x, y, z \in X \), \( \mu_B((x \ast z) \ast (y \ast z)) \geq \mu_B((x \ast y) \ast z) \) and

\[
\lambda_B(((x \ast z) \ast (y \ast z)) \ast ((x \ast y) \ast z)) = \lambda_B(((x \ast z) \ast ((x \ast y) \ast z)) \ast (y \ast z)) \quad \text{(by P2)}
\]

\[
= \lambda_B(((x \ast ((x \ast y) \ast z)) \ast z) \ast (y \ast z)) \quad \text{(by P2)}
\]

\[
\leq \lambda_A(((x \ast ((x \ast y) \ast z)) \ast z) \ast (y \ast z)) \quad \text{(Since } \lambda_B \subseteq \lambda_A) \]

\[
\leq \lambda_A(((x \ast z) \ast ((x \ast y) \ast z)) \ast y) \ast z) \quad \text{(by 3.6)}
\]

\[
= \lambda_A(((x \ast y) \ast ((x \ast y) \ast z)) \ast z)
\]

\[
= \lambda_A(((x \ast y) \ast z) \ast ((x \ast y) \ast z))
\]

\[
= \lambda_A(0) = \lambda_B(0) \quad \text{(by BCK-3)}
\]

It follows from (F1) and (F2) that

\[
\lambda_B((x \ast z) \ast (y \ast z)) \leq \max\{\lambda_B(((x \ast z) \ast (y \ast z) \ast ((x \ast y) \ast z)), \lambda_B((x \ast y) \ast z)\}
\]

\[
\leq \max\{\lambda_B(0), \lambda_B((x \ast y) \ast z)\} = \lambda_B((x \ast y) \ast z) \text{ for all } x, y, z \in X.
\]

Therefore \( \lambda_B((x \ast z) \ast (y \ast z)) \leq \lambda_B((x \ast y) \ast z) \), for all \( x, y, z \in X \).

Hence \( B = (X, \mu_B, \lambda_B) \) is an intuitionistic fuzzy positive implicative ideal of \( X \).
References