On the Fine Spectrum of the Forward Difference Operator on the Hahn Space

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Abstract

The main purpose of this paper is to determine the fine spectrum with respect to Goldberg’s classification of the difference operator over the sequence space $h$. As a new development, we give the approximate point spectrum, defect spectrum and compression spectrum of the difference operator on the sequence space $h$.

Keywords: Spectrum of an operator, spectral mapping theorem, the Hahn sequence space, Goldberg’s classification, forward difference operator.

1 Introduction

Spectral Theory is one of the most important mathematics fields that can be applied to branches of science and technology. It has been seen to be a very useful tool because of its convenient and easy applicability of the different fields such as numerical analysis, probability theory, quantum mechanics, structural mechanics, aeronautics, electrical engineering, ecology etc. We can give some examples of the spectral theory of how to use these and other disciplines: The flow over a wing may have determined whether laminar or turbulent by the spectral values in aeronautics. The frequency response of an amplifier or
the reliability of a power system may have identified by the Spectral Theory in electrical engineering. The rate of convergence of a Markov process may have computed by the Spectral Theory in the Probability Theory. A food network may have identified whether or not settle into a steady equilibrium by the spectral values in ecology. The atomic energy levels and thus, the frequency of a laser or the spectral signature of a star may have determined by the Spectral Theory in quantum mechanics. Also, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge, in numerical analysis and it may determine whether an automobile is too noisy or whether a building will collapse in an earthquake, in structural mechanics.

There are several studies about the spectrum of the linear operators defined by some triangle matrices over certain sequence spaces. This long-time behavior was intensively studied over many years, starting with the work by Wenger [30], who established the fine spectrum of the integer power of the Cesàro operator in $c$. The generalization of [30] to the weighted mean methods is due to Rhoades [27]. The study of the fine spectrum of the operator on the sequence space $\ell_p$, $(1 < p < \infty)$ was initiated by González [14]. The method of the spectrum of the Cesàro operator prepared by Reade [26], Akhmedov and Başar [1], and Okutoyi [22], respectively, whose established to this idea on the sequence spaces $c_0$ and $bv$. In [32], the fine spectrum of the Rhaly operators on the sequence spaces $c_0$ and $c$ was given. The spectrum and fine spectrum for $p$-Cesàro operator acting on the space $c_0$ was studied by Coşkun [9]. The investigation of the spectrum and the fine spectrum of the difference operator on the sequence spaces $s_r$ and $c_0$, $c$ was made by Malafosse [21] and Altay and Başar [4], respectively, where $s_r$ denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k \in \mathbb{N}} |x_k|/r^k$, $(r > 0)$. The idea of the fine spectrum applied to the Zweier matrix which is a band matrix as an operator over the sequence spaces $\ell_1$ and $bv$ by Altay and Karakus [5]. Let $\Delta_\nu$ is double sequential band matrix on $\ell_1$ such that $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_k)$. The spectra and the fine spectra of matrix $\Delta_\nu$ were determined by Srivastava and Kumar [28]. Afterwards, these results of the double sequential band matrix $\Delta_\nu$ generalized to the double sequential band matrix $\Delta_{uv}$ such that defined by $\Delta_{uv}x = (u_nx_n + v_{n-1}x_{n-1})_{n \in \mathbb{N}}$ for all $n \in \mathbb{N}$ (see [29]). In [6], the fine spectra of the Toeplitz operators represented by an upper and lower triangular $n$-band infinite matrices, over the sequence spaces $c_0$ and $c$ was computed. The fine spectra of upper triangular double-band matrices over the sequence spaces $c_0$ and $c$ was obtained by Karakaya and Altun [16]. Let $\Delta_{a,b}$ is a double band matrix with the convergent sequences $\tilde{a} = (a_k)$ and $\tilde{b} = (b_k)$ having certain properties, over the sequence space $c$. The fine spectrum of the matrix $\Delta_{a,b}$
was examined by Akhmedov and El-Shabrawy [3]. The approach to the fine spectrum with respect to Goldberg’s classification studied with of the operator $B(r, s, t)$ defined by a triple band matrix over the sequence spaces $l_p$ and $bv_p$, $(1 < p < \infty)$ by Furkan et al. [11]. Quite recently, the fine spectrum with respect to Goldberg’s classification of the operator defined by the lambda matrix over the sequence spaces $c_0$ and $c$ computed by Yeşilkayagil and Başar [31].

Hahn sequence space is defined as

$$h = \left\{ x : \sum_{k=1}^{\infty} k|\Delta x_k| < \infty \text{ and } \lim_{k \to \infty} x_k = 0 \right\},$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. Here and after $\Delta$ denotes the forward difference operator. The space $h$ is a $BK$-space with the norm

$$\|x\|_h = \sum_k k|\Delta x_k| + \sup_k |x_k|$$

The space $h$ was firstly defined by Hahn [15]. Goes and Goes [12] and Rao [23], [24], [25] constructed and studied some important properties of Hahn space.

Hahn proved that the space $h$ is a Banach space, $h \subset \ell_1 \cap c_0$ and $h_{\beta} = \rho_{\infty}$, where $\int \lambda = \{ x = (x_k) \in \omega : (kx_k) \in \lambda \}$ and $\rho_{\infty} = \{ x = (x_k) \in \omega : \sup_n n^{-1} \left| \sum_{k=1}^{n} x_k \right| < \infty \}$. Functional analytic properties of the $BK$-space $bv_0 \cap d\ell_1$ was studied by Goes and Goes [12], where $d\ell_1 = \{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \frac{1}{k}|x_k| < \infty \}$. Also, in [12], the arithmetic means of sequences in $bv_0$ and $bv_0 \cap d\ell_1$ were considered, and used the fact that the Cesàro transform $(n^{-1} \sum_{k=1}^{n} x_k)$ of order one $x \in bv_0$ is a quasiconvex null sequence.

Rao [23] studied some geometric properties of Hahn sequence space and gave the characterizations of some classes of matrix transformations. Also, in [24] and [25], Rao examined the different properties of Hahn sequence space.

Balasubramanian and Pandiarani [8] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that $\beta-$ and $\gamma-$duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences.

After these works, it weren’t done any study on Hahn space until the Kirişçi’s works ([17], [18], [19]). In [17], it was compiled all studies about to Hahn sequence space and in [18], new Hahn space was constructed with the Cesaro mean. Also, Kirişçi have defined p-Hahn sequence space in [19].

In the present paper, our propose is to investigate the fine spectrum of the difference operator $\Delta$ on the sequence space $h$. And also, we define the
approximate point spectrum, defect spectrum and compression spectrum of
the difference operator on the sequence space $h$, as a new approach.

2 Preliminaries and Definition

Let $X$ and $Y$ be Banach spaces, $T : X \to Y$ be a bounded linear operator.
We give

\[ R(T) = \{ y \in Y : y = Tx, \ x \in X \}. \]

$B(X,Y) = \{ T : X \to Y : T \text{ is continuous and linear} \}$ and $B(X) = B(X,X)$.

The sets $R(T)$, $B(X,Y)$ and $B(X)$ are called the range of the operator $T$, the
set of all bounded linear operators from $X$ to $Y$ and the set of all bounded
linear operators from $X$ to $X$, respectively.

**Definition 2.1.** Let $X$ be any Banach space and $T \in B(X)$. Then $T^*$ of $T$
is said to be adjoint, if $T^*$ is a bounded linear operator on the dual $X^*$ of $X$such that $(T^*f)(x) = f(Tx)$ for all $f \in Y^*$ and $x \in X$.

Let $X \neq \{ \theta \}$ be a non-trivial complex normed space, $D(T) \subseteq X$ and $T$is a linear operator such that $T : D(T) \to X$. Let $D(T)$ is not dense in $X$ or
$T$ hasn’t closed graph $\{(x,Tx) : x \in D(T)\} \subseteq X \times X$. If we say $T$ is in-
vertible, then we have to mention that there exists a bounded linear operator
$S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $R(T) = X$; such that $S = T^{-1}$is necessarily uniquely determined, and linear. Again if we say the boundedness
of $S$ then we have to understand that $T$ must be bounded below, in the sense
that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$.

If $T_\alpha = \alpha I - T$ such that associated with each complex number $\alpha$, then $T_\alpha$
is called the perturbed operator defined on the same domain $D(T)$ as $T$.

**Definition 2.2.** The spectrum is

\[ \sigma(T,X) = \{ \alpha \in \mathbb{C} : T_\alpha \text{ is not invertible} \}. \]

The resolvent set of $T$ is $\rho(T,X) = \mathbb{C} \backslash \sigma(T,X)$, the complement of spectrum.
The resolvent of $T$ is the function defined on $\rho(T,X)$ by $\alpha \mapsto T_\alpha^{-1}$.

**Definition 2.3.** The set of all sequences denotes with $\omega := \mathbb{C}^\mathbb{N} := \{ x = (x_k) : x : \mathbb{N} \to \mathbb{C}, k \to x_k := x(k) \}$ where $\mathbb{C}$ denotes the complex field and $\mathbb{N}$ is the set of positive integers. Each linear subspace of $\omega$ (with the induced addition and scalar multiplication) is called a sequence space.
The following subsets of \( \omega \) are obviously sequence spaces:

\[
\ell_\infty = \{ x = (x_k) \in \omega : \sup_k |x_k| < \infty \} \quad c = \{ x = (x_k) \in \omega : \lim_k x_k \text{ exists} \}
\]

\[
c_0 = \{ x = (x_k) \in \omega : \lim_k x_k = 0 \} \quad bs = \{ x = (x_k) \in \omega : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \}
\]

\[
 cs = \{ x = (x_k) \in \omega : \left( \sum_{k=1}^{n} x_k \right) \in c \}
\]

\[
 \ell_p = \{ x = (x_k) \in \omega : \sum_k |x_k|^p < \infty, \quad 1 \leq p < \infty \}
\]

These sequence spaces are Banach space with the norms; \( \|x\|_{\ell_\infty} = \sup_k |x_k| \), \( \|x\|_{bs} = \|x\|_{cs} = \sup_n \left| \sum_{k=1}^{n} x_k \right| \) and \( \|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p} \) as usual, respectively. Let \( X \) is one of the above mentioned sequence spaces.

Let \( A = (a_{nk}) \) be an infinite matrix of complex numbers \( a_{nk} \) and \( x = (x_k) \in \omega \), where \( k, n \in \mathbb{N} \). Then the sequence \( Ax \) is called as the \( A \)-transform of \( x \) defined by the usual matrix product. Hence, we transform the sequence \( x \) into the sequence \( Ax = \{(Ax)_n\} \) where

\[
(Ax)_n = \sum_k a_{nk}x_k
\]

for each \( n \in \mathbb{N} \), provided the series on the right hand side of (1) converges for each \( n \in \mathbb{N} \).

3 Subdivision of the Spectrum

In this section, we define the parts called point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

3.1 The Point Spectrum, Continuous Spectrum and Residual Spectrum

Let \( \alpha \in \mathbb{C} \). Because of \( (\alpha I - T)^{-1} \) provides convenience to the solution of the equation \( T_\alpha x = y \), there is no problem with use the word resolvent. Therefore, the existence of \( T_\alpha^{-1} \) is ensured by \( x = T_\alpha^{-1}y \). Furthermore, if we will investigate of properties of \( T_\alpha^{-1} \), then it will help us obtained information about the
operator $T$ itself. Let’s also note that many properties of $T_\alpha$ and $T^{-1}_\alpha$ which will be investigated, depends on $\alpha$. Indeed, we know that the Spectral Theory is relevant to the investigation of such properties. In this case, if $T^{-1}_\alpha$ exists, then we will study the set of all $\alpha$’s in the complex plane. Also, we will deal another important property which name is boundedness of $T^{-1}_\alpha$. We will also ask for what $\alpha$’s the domain of $T^{-1}_\alpha$ is dense in $X$, to name just a few aspects.

Now, we will give some definitions for use in the study of some properties of $T$, $T_\alpha$ and $T^{-1}_\alpha$ (see Kreyszig [20, pp. 370-371]).

**Definition 3.1.** A regular value $\alpha$ of $T$ is a complex number such that $T^{-1}_\alpha$ exists and is bounded whose domain is dense in $X$.

The resolvent set $\rho(T,X)$ of $T$ is the set of all regular values $\alpha$ of $T$.

The point (discrete) spectrum $\sigma_p(T,X)$ is the set such that $T^{-1}_\alpha$ does not exist. An $\alpha \in \sigma_p(T,X)$ is called an eigenvalue of $T$.

The continuous spectrum $\sigma_c(T,X)$ is the set such that $T^{-1}_\alpha$ exists and is unbounded, and the domain of $T^{-1}_\alpha$ is dense in $X$.

The residual spectrum $\sigma_r(T,X)$ is the set such that $T^{-1}_\alpha$ exists (and may be bounded or not) but the domain of $T^{-1}_\alpha$ is not dense in $X$.

Therefore, these three subspectra form a disjoint subdivision such that

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X).$$

(2)

To avoid trivial misunderstandings, let us say that some of the sets defined above may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite-dimensional case.

### 3.2 The Approximate Point Spectrum, Defect Spectrum and Compression Spectrum

In this subsection, three more subdivision of the spectrum called the *approximate point spectrum*, *defect spectrum* and *compression spectrum* have been defined as in Appell et al. [7].

**Definition 3.2.** Let $X$ is a Banach space and $T$ is a bounded linear operator. A $(x_k) \in X$ Weyl sequence for $T$ defined by $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$. 


The approximate point spectrum of $T$ is the set
\[ \sigma_{ap}(T, X) := \{ \alpha \in \mathbb{C} : \text{there exists a Weyl sequence for } \alpha I - T \} \] (3)

The subspectrum
\[ \sigma_\delta(T, X) := \{ \alpha \in \mathbb{C} : \alpha I - T \text{ is not surjective} \} \] (4)
is called defect spectrum of $T$.

The two subspectra given by (3) and (4) form a (not necessarily disjoint) subdivision
\[ \sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_\delta(T, X) \]
of the spectrum. There is another subspectrum,
\[ \sigma_{co}(T, X) = \{ \alpha \in \mathbb{C} : \overline{R(\alpha I - T)} \neq X \} \]
which is often called compression spectrum in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition
\[ \sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X) \]
of the spectrum. Clearly, $\sigma_p(T, X) \subseteq \sigma_{ap}(T, X)$ and $\sigma_{co}(T, X) \subseteq \sigma_\delta(T, X)$. Moreover, comparing these subspectra with those in (2) we note that
\[ \sigma_r(T, X) = \sigma_{co}(T, X) \setminus \sigma_p(T, X), \]
\[ \sigma_c(T, X) = \sigma(T, X) \setminus [\sigma_p(T, X) \cup \sigma_{co}(T, X)] \].

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints are also useful.

**Proposition 3.3.** [7, Proposition 1.3, p. 28] The following relations on the spectrum and subspectrum of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ hold:

(a) $\sigma(T^*, X^*) = \sigma(T, X)$.
(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.
(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$.
(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$.
(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.
(f) $\sigma_{col}(T^*, X^*) \supseteq \sigma_p(T, X)$.

(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum and the point spectrum is dual to the compression spectrum. The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if $X$ is a Hilbert space and $T$ is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see Appell et al. [7]).

### 3.3 Goldberg’s Classification of Spectrum

If $X$ is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(A) $R(T) = X$.

(B) $R(T) \neq \overline{R(T)} = X$.

(C) $\overline{R(T)} \neq X$.

and

(1) $T^{-1}$ exists and is continuous.

(2) $T^{-1}$ exists but is discontinuous.

(3) $T^{-1}$ does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$. If an operator is in state $C_2$ for example, then $\overline{R(T)} \neq X$ and $T^{-1}$ exists but is discontinuous (see Goldberg [13]).
If $\alpha$ is a complex number such that $T_{\alpha} \in A_1$ or $T_{\alpha} \in B_1$, then $\alpha \in \rho(T, X)$. All scalar values of $\alpha$ not in $\rho(T, X)$ comprise the spectrum of $T$. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of $T$. That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, $C_3\sigma(T, X)$. For example, if $T_{\alpha}$ is in a given state, $C_2$ (say), then we write $\alpha \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivision (2) in the following table:

<table>
<thead>
<tr>
<th></th>
<th>C_1</th>
<th>C_2</th>
<th>C_3</th>
<th>B_1</th>
<th>B_2</th>
<th>B_3</th>
<th>A_1</th>
<th>A_2</th>
<th>A_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>T*</td>
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</tr>
</tbody>
</table>

Table 1.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space $X$
One can observe by the closed graph theorem that in the case $A_2$ cannot occur in a Banach space $X$. If we are not in the third column of Table 1.2, i.e., if $\alpha$ is not an eigenvalue of $T$, we may always consider the resolvent operator $T^{-1}_\alpha$ (on a possibly thin domain of definition) as algebraic inverse of $\alpha I - T$.

The forward difference operator $\Delta$ is represented by the matrix

$$
\Delta = \begin{bmatrix}
1 & -1 & 0 & 0 & \ldots \\
0 & 1 & -1 & 0 & \ldots \\
0 & 0 & 1 & -1 & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

Corollary 3.4. $\Delta : h \to h$ is a bounded linear operator.

4 On the Fine Spectrum of the Forward Difference Operator on the Hahn Space

In this section, we determine the spectrum and fine spectrum of the forward difference operator $\Delta$ on the Hahn space $h$ and calculate the norm of the operator

Theorem 4.1. $\sigma(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| \leq 1\}$.

Proof. Let $|1 - \alpha| > 1$. Since $\Delta - \alpha I$ is triangle, $(\Delta - \alpha I)^{-1}$ exists and solving the matrix equation $(\Delta - \alpha I)x = y$ for $x$ in terms of $y$ gives the matrix $(\Delta - \alpha I)^{-1} = B = (b_{nk})$, where

$$
b_{nk} = \begin{cases}
\frac{1}{|1-\alpha|^{n+1}}, & 0 \leq k \leq n, \\
0, & k > n
\end{cases}.
$$
for all $k, n \in \mathbb{N}$. Thus, we observe that

$$\| (\Delta - \alpha I)^{-1} \|_{(h,h)} = \sum_{n=1}^{\infty} n|b_{nk} - b_{n+1,k}|$$

$$\leq \sum_{n=1}^{\infty} n|b_{nk}| + \sum_{n=1}^{\infty} n|b_{n+1,k}|$$

$$= \sum_{n=1}^{\infty} \frac{n}{|1 - \alpha|^{n+1}} + \sum_{n=1}^{\infty} \frac{n}{|1 - \alpha|^{n+2}}.$$  

From the ratio test, we have

$$\| (\Delta - \alpha I)^{-1} \|_{(h,h)} < \infty,$$

that is, $(\Delta - \alpha I)^{-1} \in (h : h)$. But for $|1 - \alpha| \leq 1$,

$$\| (\Delta - \alpha I)^{-1} \|_{(h,h)} = \infty,$$

that is, $(\Delta - \alpha I)^{-1}$ is not in $B(h)$. This completes the proof.

**Theorem 4.2.** $\sigma_p(\Delta, h) = \emptyset$.

**Proof.** Suppose that $\Delta x = \alpha x$ for $x \neq \theta$ in $h$. Then, by solving the system of linear equations

$$
x_0 = \alpha x_0, \\
-x_0 + x_1 = \alpha x_1, \\
-x_1 + x_2 = \alpha x_2, \\
\vdots \\
-x_{n-1} + x_n = \alpha x_n,
$$

we find that if $x_{n_0}$ is the first nonzero entry of the sequence $x = (x_n)$, then $\alpha = 1$. From the equality $-x_{n_0} + x_{n_0+1} = \alpha x_{n_0+1}$ we have $x_{n_0}$ is zero. This contradicts the fact that $x_{n_0} \neq 0$, which completes the proof.

**Theorem 4.3.** $\sigma_p(\Delta^*, h^*) = \{ \alpha \in \mathbb{C} : |1 - \alpha| < 1 \}$.

**Proof.** Suppose that $\Delta^* x = \alpha x$ for $x \neq \theta$ in $h^* \cong \sigma_\infty$. Then, by solving the system of linear equations

$$
x_0 - x_1 = \alpha x_0, \\
x_1 - x_2 = \alpha x_1, \\
\vdots \\
x_{n-1} - x_n = \alpha x_n,
$$
we observe that $x_n = (1 - \alpha)^n x_0$. Therefore, sup $\frac{|x_0|}{n} \sum_{k=1}^{n} |1 - \alpha|^k < \infty$ if and only if $|1 - \alpha| < 1$. This step concludes the proof.

If $T \in B(h)$ with the matrix $A$, then it is known that the adjoint operator $T^* : h^* \to h^*$ is defined by the transpose $A^t$ of the matrix $A$. It should be noted that the dual space $\sigma^*$ is unbounded. Also $\Delta_{\sigma}$

Proof.\(\quad (a)\) Since $\sigma$ is one. But $\Delta_{\sigma}$ is not one to one by Theorem 4.3. Therefore by Lemma 4.4, $R(\Delta - \alpha I) \neq h$ and this step concludes the proof.

Theorem 4.6. $\sigma_c(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| = 1\}$.

Proof. For $|1 - \alpha| < 1$, the operator $\Delta - \alpha I$ is triangle, so has an inverse but is unbounded. Also $\Delta^* - \alpha I$ is one to one by Theorem 4.3. By Lemma 4.4, $R(\Delta - \alpha I) = h$. Thus, the proof is completed.

Theorem 4.7. $A_3 \sigma(\Delta, h) = B_3 \sigma(\Delta, h) = C_3 \sigma(\Delta, h) = \emptyset$.

Proof. From Theorem 4.2 and Table 1.2., $A_3 \sigma(\Delta, h) = B_3 \sigma(\Delta, h) = C_3 \sigma(\Delta, h) = \emptyset$ is observed.

Theorem 4.8. $C_1 \sigma(\Delta, h) = \emptyset$ and $\alpha \in \sigma_r(\Delta, h) \cap C_2 \sigma(\Delta, h)$.

Proof. We know $C_1 \sigma(\Delta, h) \cup C_2 \sigma(\Delta, h) = \sigma_r(\Delta, h)$ from Table 1.2. For $\alpha \in \sigma_r(\Delta, h)$, the operator $(\Delta - \alpha I)^{-1}$ is unbounded by Theorem 4.1. So $C_1 \sigma(\Delta, h) = \emptyset$. This completes the proof.

Theorem 4.9. The following results hold:

(a) $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h)$.

(b) $\sigma_{b}(\Delta, h) = \sigma(\Delta, h)$.

(c) $\sigma_{co}(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| < 1\}$.

Proof. (a) Since $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h) \setminus C_1 \sigma(\Delta, h)$ from Table 1.2. and $C_1 \sigma(\Delta, h) = \emptyset$ by Theorem 4.8, we have $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h)$.

(b) Since $\sigma_{b}(\Delta, h) = \sigma(\Delta, h) \setminus A_3 \sigma(\Delta, h)$ from Table 1.2 and $A_3 \sigma(\Delta, h) = \emptyset$ by Theorem 4.7, we have $\sigma_{b}(\Delta, h) = \sigma(\Delta, h)$.

(c) Since the equality $\sigma_{co}(\Delta, h) = C_1 \sigma(\Delta, h) \cup C_2 \sigma(\Delta, h) \cup C_3 \sigma(\Delta, h)$ holds from Table 1.2, we have $\sigma_{co}(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| < 1\}$ by Theorems 4.8 and 4.9.
The next corollary can be obtained from Proposition 2.1.

**Corollary 4.10.** The following results hold:

(a) \( \sigma_{ap}(\Delta^*, \ell_1) = \sigma(\Delta, h) \).

(b) \( \sigma_{\delta}(\Delta^*, \ell_1) = \{ \alpha : |\alpha - (2 - \delta)^{-1}| = (1 - \delta)(2 - \delta) \} \cup E \).

(c) \( \sigma_p(\Delta^*, \ell_1) = \{ \alpha \in \mathbb{C} : |\alpha - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta) \} \cup S \).

5 Conclusion

Hahn [15] defined the space \( h \) and gave its some general properties. Goes and Goes [12] studied the functional analytic properties of the space \( h \). The study on the Hahn sequence space was initiated by Rao [23] with certain specific purpose in Banach space theory. Also Rao [23] emphasized on some matrix transformations. Rao and Srinivasalu [24] introduced a new class of sequence space called the semi replete space. Rao and Subramanian [25] defined the semi Hahn space and proved that the intersection of all semi Hahn spaces is Hahn space. Balasubramanian and Pandiarani [8] defined the new sequence space \( h(F) \) called the Hahn sequence space of fuzzy numbers and proved that \( \beta- \) and \( \gamma- \) duals of \( h(F) \) is the Cesàro space of the set of all fuzzy bounded sequences. The sequence space \( h \) was introduced by Hahn [15] and Goes and Goes [12], and Rao [23, 24, 25] investigated some properties of the space \( h \). Quite recently, Kirisci [17] has defined a new Hahn sequence space by using Cesàro mean, in [18].

The difference matrix \( \Delta \) was used for determining the spectrum or fine spectrum acting as a linear operator on any of the classical sequence spaces \( c_0 \) and \( c, \ell_1 \) and \( bv, \ell_p \) for \( (1 \leq p < \infty) \), respectively in [4], [10] and [2].

As a natural continuation of this paper, one can study the spectrum and fine spectrum of the Cesàro operator, Weighted mean operator or another known operators in the sequence space \( h \).

Conflict of Interests

The authors declare that there are no conflict of interests regarding the publication of this paper.

References


