Analytic Functions in Their Debye’s Form

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(Received 18.10.2010, Accepted 5.11.2010)

Abstract

We show that any analytic function in complex variable admits a splitting in the same sense as Debye potentials in electromagnetic theory.

Keywords: Cauchy-Riemann conditions; Debye potentials.
200 MSC No. 03.30 ; 03.50De ; 41.20

For the electromagnetic field, the solution of Maxwell equations without sources can be written [2,5,11-13,15] in terms of scalar generators (Debye potentials), $\psi_E$ and $\psi_M$, which satisfy the wave equation:

$$\Box \psi_E = \Box \psi_M = 0, \quad \Box = \frac{\partial^2}{c^2 \partial t^2} - \nabla^2,$$

(1.a)

in according to:

$$\phi = -c \frac{\vec{r}}{r} \cdot \nabla (r \psi_E), \quad \vec{A} = -\vec{r} \times \nabla \psi_M + \frac{\vec{r}}{c \partial t} \frac{\partial \psi_E}{\partial t}.$$

(1.b)

up to gauge transformations, where:
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\[ \mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad r = \sqrt{x^2 + y^2}, \quad \nabla^\prime = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}. \] (1.c)

The existence of \( \psi_E \) and \( \psi_M \) follows from results of several authors [6,8-10,14].

If \( f \) is an analytic function of the complex variable \( z = x + iy \), then it has the form \( f(z) = u(x, y) + iv(x, y) \), and besides are valid the Cauchy-Riemann relations [1]:

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \] (2.a)

that is [7]:

\[ \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)(u + iv) = 0. \] (2.b)

We note that (2.b) is very interesting because it permits to generalize easily the conditions (2.a) to four dimensions [3,7] via quaternions. The functions \( u \) and \( v \) are harmonic because they verify the Laplace equation:

\[ \nabla^2 u = \nabla^2 v = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \] (3)

The conditions (2.a,3) allow to deduce a splitting of \( f(z) \) which has strong similarity with the Debye expressions (1.b) for the electromagnetic potentials. In fact, we introduce the notation:

\[ [\mathbf{r} \times \nabla g]^3 \equiv \frac{\partial g}{\partial y} x - \frac{\partial g}{\partial x} y, \] (4.a)

then:

\[ \frac{\mathbf{r}}{r} \cdot \nabla (ru) = u x + y \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}, \quad [\mathbf{r} \times \nabla v]^3 \equiv \frac{\partial v}{\partial y} x - \frac{\partial v}{\partial x} y, \] (2.4b)

therefore:

\[ \frac{\mathbf{r}}{r} \cdot \nabla (ru) - [\mathbf{r} \times \nabla v]^3 = u + x \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + y \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^{(2.a)} = u; \] (4.b)

besides:
\[ \frac{\bar{r}}{r} \cdot \nabla (rv) - \left[ \bar{r} \times \nabla u \right]_3 = v + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) = v. \] 

(4.c)

We see that (4.b,c) have a similar structure to (1.b), and they imply the following non-trivial splitting for the analytic function \( f(z) = u + iv \):

\[ f(z) = \frac{\bar{r}}{r} \cdot \nabla (rf(z)) + i \left[ \bar{r} \times \nabla f(z) \right]_3, \]

(5.a)

which means that \( f(z) \) is a Debye potential for itself, and where each term is a harmonic function:

\[ \nabla^2 \left[ \frac{\bar{r}}{r} \cdot \nabla (rf) \right] = \nabla^2 \left[ \bar{r} \times \nabla f \right]_3 = 0. \]

(5.b)

The expression (5.a) is a reformulation [4] of the Cauchy-Riemann conditions (2.a), and it is a strong motivation for the existence of Debye generators in electromagnetic theory.

References