A Note on the Generalized Shift Map

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Abstract

In this short note we have discussed generalized shift map in the symbol space $\Sigma_2$. Some stronger chaotic properties have been proved. Some special properties are discussed in a different section. In the last section we have also provided few examples.

Keywords: Symbolic dynamics, Shift map, Generalized shift map, Strong sensitive dependence on initial conditions, Periodic points.

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1 Introduction

A dynamical system is sometimes defined as a pair $(X, f)$ consisting of a set $X$ together with a continuous map $f$ from $X$ into itself. Chaotic dynamical systems constitute a special class of dynamical systems. Symbolic dynamics is also an example of chaotic dynamical systems. In particular, there are several works on symbolic dynamics such as [1, 2, 3, 4, 5, 8, 9, 11, 13]. Of particular interest is the space $\Sigma_2$ which has been considered in a large number of works. Devaney [6] have given vivid description of the space $\Sigma_2$. By symbolic dynamical system we mean here the space of sequences $\Sigma_2 = \{\alpha : \alpha = (\alpha_0\alpha_1\ldots), \alpha_i = 0$ or 1$\}$ along with the shift map defined on it. It is known that $\Sigma_2$ is a compact
metric space by the metric
\[ d(s, t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^{i+1}}, \]
where \( s = (s_0s_1\ldots) \) and \( t = (t_0t_1\ldots) \) are any two points of \( \Sigma_2 \).

The present authors extended the idea of the shift map into the generalized shift map in [1] and proved that it is chaotic both in the sense of Devaney [6] and Li-Yorke [10]. It is also chaotic in the sense of Du [5, 7].

In this short note we have proved some stronger chaotic properties of the generalized shift map. A comparative study of dynamics of the shift map and the generalized shift map is given. Lastly, we have given some examples.

We now give some definitions and lemmas which are required for this note.

**Definition 1.1** (Shift map [6]) The shift map \( \sigma : \Sigma_2 \to \Sigma_2 \) is defined by \( \sigma(\alpha_0\alpha_1\ldots) = (\alpha_1\alpha_2\ldots) \), where \( \alpha = (\alpha_0\alpha_1\ldots) \) is any point of \( \Sigma_2 \).

**Definition 1.2** (Generalized shift map [1]) The generalized shift map \( \sigma_n : \Sigma_2 \to \Sigma_2 \) is defined by \( \sigma_n(s) = (s_n s_{n+1} s_{n+2} \ldots) \), where, \( s = (s_0 s_1 \ldots s_n \ldots) \) is any element of \( \Sigma_2 \). For \( n = 1 \), the generalized shift map reduces to the shift map and \( n \geq 1 \) is a finite positive integer.

**Definition 1.3** (Sensitive dependence on initial conditions [6]) Let \((S, \rho)\) be a compact metric space. A continuous map \( f : S \to S \) is said to have sensitive dependence on initial conditions if there exists \( \delta > 0 \) such that, for any \( x \in S \) and any neighborhood \( N(x) \) of \( x \) there exist \( y \in N(x) \) and \( n \geq 0 \) such that \( \rho(f^n(x), f^n(y)) > \delta \).

In the following we give a stronger version of the above definition.

**Definition 1.4** (Strong sensitive dependence on initial conditions [4]) Let \((S, \rho)\) be a compact metric space. A continuous map \( f : S \to S \) has strong sensitive dependence on initial conditions if for any \( x \in S \) and any non empty open set \( U \) of \( S \) (not necessarily an open neighborhood of \( x \)), there exist \( y \in U \) and \( n \geq 0 \) such that \( \rho(f^n(x), f^n(y)) \) is maximum in \( S \).

It is obvious that if a map has strong sensitive dependence on initial conditions, it has also sensitive dependence on initial conditions. At the end of this paper we give an example to establish that the converse is not necessarily true.

**Definition 1.5** (Totally transitive [12]) Let \((X, \rho)\) be a compact metric space. A continuous map \( f : X \to X \) is called totally transitive if \( f^n \) is topologically transitive for all \( n \geq 1 \).

**Definition 1.6** (Transitive point [6]) In the symbol space \( \Sigma_2 \) there are points whose orbit comes arbitrarily close to any given sequence of \( \Sigma_2 \), that is, the point with dense orbit. Those points are called transitive points.
Definition 1.7 (Fixed point [6]) Let \( f : I \to I \) be a continuous map. If a point \( a \in I \) is such that \( f(a) = a \), then \( a \) is called a fixed point of \( f \).

Definition 1.8 (Periodic point [6]) Let \( f : I \to I \) be a continuous map. The point \( x \in I \) is called a periodic point of least period \( n \) if \( f^n(x) = x \) and \( f^m(x) \neq x \), for all \( m < n \) where \( m \) and \( n \) are positive integers.

We also require the following lemma.

Lemma 2.1 [6] Let \( s, t \in \Sigma_2 \) and \( s_i = t_i \), for \( i = 0, 1, \ldots, m \). Then \( d(s, t) < \frac{1}{2^m} \) and conversely if \( d(s, t) < \frac{1}{2^m} \) then \( s_i = t_i \), for \( i = 0, 1, \ldots, m \).

2 The Main Results

Theorem 2.1 Let \( \sigma_n : \Sigma_2 \to \Sigma_2 \) be the generalized shift map. Then for any point \( x \) of \( \Sigma_2 \) and any open neighborhood \( U \) of \( x \), there exist two non empty subsets \( K \) and \( L \) of \( U \), which satisfy the following conditions:

i) both \( K \) and \( L \) are countable,

ii) \( K \cap L = \emptyset \) and

iii) \( d(\sigma_n^m(k), \sigma_n^m(x)) = 1 \), for all \( k \in K \) and \( d(\sigma_n^m(l), \sigma_n^m(x)) = 0 \), for all \( l \in L \), where \( n \) and \( m \) are different for different points of \( K \) and \( L \) and depend on the minimum distance of \( x \) from the boundary \( U \).

Proof. Let \( x = (x_0x_1 \ldots) \) be any point of \( \Sigma_2 \) and \( U \) be any open neighborhood of \( x \) such that minimum distance of \( x \) from the boundary of \( U \) is \( \varepsilon > 0 \). We now choose \( p \geq 5 \) as an integer such that \( \frac{1}{2^p} < \varepsilon \), for all \( n \geq 1 \). We now consider the two sets \( K = \{ k_i : k_i = (x_0x_1 \ldots x_{2ni-1}x_{2ni}x_{2ni+1} \ldots), i \geq p \} \) and \( L = \{ l_j : l_j = (x_0x_1 \ldots x_{2nj-1}x_{2nj} \ldots), j \geq p \} \).

Now by our construction we see that all \( k_i \)'s of \( K \) agree with \( x \) at least up to \( x_{np} \). Hence by Lemma 2.1 we get that \( d(x, k_i) < \frac{1}{2^p} \), for all \( k_i \in K \), that is, \( d(x, k_i) < \varepsilon \), for all \( k_i \in K \). So \( k_i \in U \), for all \( i \geq p \) and we get that \( K \) is a non empty subset of \( U \). Similarly, we can show that \( L \) is a non empty subset of \( U \). Again by our construction we see that both \( K \) and \( L \) are countable. This proves i).

We now observe the two sets \( K \) and \( L \) and see that, after the \((2ni+2n)\)-th (or \((2nj+2n)\)-th) term all terms of \( k_i \) (or \( l_i \)) are mutually complementary terms for all \( i \) (or \( j \)). Hence \( l_j \neq k_i \), for all \( i \) and \( j \), that is, \( K \cap L = \emptyset \). This proves ii).

Now, \( d(\sigma_n^{2i}(k_i), \sigma_n^{2i}(x)) = d((x_{2ni}x_{2ni+1} \ldots), (x_{2ni}x_{2ni+1} \ldots)) = \frac{1}{2} + \frac{1}{2^2} + \ldots \ldots = 1 \), for all \( k_i \in K \), and

\[ d(\sigma_n^{2j+2}(l_j), \sigma_n^{2j+2}(x)) = d((x_{2nj+2n}x_{2nj+2n+1} \ldots), (x_{2nj+2n}x_{2nj+2n+1} \ldots)) = \frac{1}{2} + \frac{1}{2^2} + \ldots \ldots = 0 \], for all \( l_j \in L \).
Also by our constructions of $K$ and $L$ we see that $i \geq p$ and $j \geq p$, for $K$ and $L$ respectively and $p$ is depending on $\varepsilon$, where $\varepsilon$ is the minimum distance of $x$ from the boundary of $U$. So we conclude that $n_j$'s and $m_j$'s are depending on the minimum distance of $x$ from the boundary of $U$. This proves iii).

Hence the theorem is proved.

**Theorem 2.2** The generalized shift map $\sigma_n : \Sigma_2 \to \Sigma_2$ has strong sensitive dependence on initial conditions.

**Proof.** Let $x = (x_0x_1\ldots\ldots)$ be any point of $\Sigma_2$ and $U$ be any non empty open set of $\Sigma_2$. Hence we can take an open ball $V$ with radius $\varepsilon > 0$ and center at $\alpha = (\alpha_0\alpha_1\ldots\ldots)$, such that $V \subset U$. Let $p > 0$ be an integer, such that $\frac{1}{2^{np-1}} < \varepsilon$. We now consider the point $y = (\alpha_0\alpha_1\ldots\ldots; \alpha_{np-1}x'_{np}x'_{np+1}\ldots\ldots)$.

Then the point $y$ agrees with $\alpha$ up to $\alpha_{np-1}$ and after that all terms of $y$ are the complementary terms of the point $x$ starting with $x'_{np}$.

By the application of Lemma 2.1 above we get that $d(\alpha, y) < \frac{1}{2^{np-1}} < \varepsilon$. Hence $y \in V$, that is, $y \in U$ also.

Again we get $d(\sigma_n^p(x), \sigma_n^p(y)) = d((x_{np}x_{np+1}\ldots\ldots), (x'_{np}x'_{np+1}\ldots\ldots))$

$= \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots$

$= 1$, (this is maximum)

that is, $y \in U$ such that $d(\sigma_n^p(x)\sigma_n^p(y)) = 1$, where $U$ is an arbitrary open set of $\Sigma_2$.

Hence the generalized shift map $\sigma : \Sigma_2 \to \Sigma_2$ has strong sensitive dependence on initial conditions.

### 3 Some Special Properties

In this section we discuss some basic differences of dynamics of the generalized shift map and the shift map. We also present a comparative study between the shift map and the generalized shift map.

We know that transitive points play a big role in any Devaney’s chaotic system. For the shift map $\sigma$, if a point of $\Sigma_2$ which contains every finite sequence of 0’s and 1’s, the point is a transitive point. But there is a different situation for the generalized shift map $\sigma_n$. A point of $\Sigma_2$ which contains every finite sequence of 0’s and 1’s with a power $n$ is a transitive point with respect to the generalized shift map. The following is an example. If we consider a point $a$ of $\Sigma_2$ as given below,

\[
a = (\text{1 block} \quad (0)^n(1)^n \quad \text{2 block} \quad (00)^n(01)^n(10)^n(11)^n \quad \text{3 block} \quad (000)^n(001)^n \quad \text{4 block} \quad (0000)^n) \ldots\ldots)
\]

then obviously $a \in \Sigma_2$ is a transitive point with respect to the generalized shift.
map. But
\[ b = (01 \ 00 \ 01 \ 10 \ 11 \ 000 \ 001 \ldots \ 0000 \ldots \ldots ) \]
is a transitive point for \( \sigma \). Hence we can say that all transitive points of the generalized shift map \( \sigma_n \) are also transitive points of the shift map \( \sigma \), but not conversely. Note that, at this time \( (\alpha_0\alpha_1\alpha_2)^n = (\alpha_0\alpha_1\alpha_2)(\alpha_0\alpha_1\alpha_2)\ldots\ldots\ldots(\alpha_0\alpha_1\alpha_2) \) \( n \)-times.

We now discuss the periodic points of the generalized shift map. Throughout this paper periods mean prime periods. If \( \sigma : \Sigma_2 \to \Sigma_2 \) is the shift map then we all know that any repeating sequence of 0’s and 1’s is always a periodic point of \( \sigma \). For example, \( \beta = (\beta_0\beta_1\ldots\ldots\beta_{n-1}\beta_0\beta_1\ldots\ldots\beta_{n-1}) \) is a periodic point of period \( n \) of \( \sigma \), for all \( n \geq 1 \). But \( \sigma_n(\beta) = \beta \), that is, \( \beta \) is a fixed point of \( \sigma_n \). On the other hand if we consider the points \( O = (0000\ldots\ldots) \) and \( I = (1111\ldots\ldots) \) of \( \Sigma_2 \). These are the only fixed points of \( \sigma \). The above two points are fixed points of \( \sigma_n \) also, but there exist other fixed points of \( \sigma_n \) in \( \Sigma_2 \). For example, \( x = (x_0x_1\ldots\ldots x_{n-1}x_0x_1\ldots\ldots x_{n-1}) \) is a fixed points of \( \sigma_n \), where \( x_i \)'s are not all 0 or 1 at the same time.

Hence we conclude that periodic points of \( \sigma \) and \( \sigma_n \) are not same in general.

4 Conclusions

In this note we have proved some stronger chaotic properties of the generalized shift map. Since the generalized shift map is chaotic in the sense of Devaney, it is topologically transitive on \( \Sigma_2 \). Hence we can say that the shift map is totally transitive on \( \Sigma_2 \). So a question arises whether all topologically transitive maps are totally transitive? The answer is no. In the following we give an example to establish this fact.

**Example 4.1.** Let \( f(x) \) be a continuous map from \([0,1]\) onto itself defined by

\[
    f(x) = \begin{cases} 
        4x + \frac{1}{3}, & 0 \leq x \leq \frac{1}{5} \\
        -4x + \frac{5}{6}, & \frac{1}{6} \leq x \leq \frac{1}{3} \\
        -\frac{1}{2}x + \frac{1}{3}, & \frac{1}{3} \leq x \leq 1 
    \end{cases}
\]

It can be easily proved that the function \( f \) is topologically transitive on \([0,1]\). On the other hand it is not totally transitive, since the subintervals \([0, \frac{1}{5}]\) and \([\frac{1}{3}, 1]\) are invariant under \( f^2 \), so \( f^2 \) is not topologically transitive on \([0,1]\). Hence \( f(x) \) is not totally transitive on \([0,1]\).

As noted earlier that if a continuous map has strong sensitive dependence on initial conditions then it has sensitive dependence on initial conditions, but the converse is not always true. The following example establishes this fact.
**Example 4.2.** Let \( f : [-1, 1] \rightarrow [-1, 1] \) be a map defined by

\[
\begin{align*}
  f(x) &= \begin{cases} 
    \frac{3}{2}x + \frac{3}{2}, & -1 \leq x \leq -\frac{1}{3} \\
    -3x, & -\frac{1}{3} \leq x \leq 0 \\
    -x, & 0 \leq x \leq 1
  \end{cases}
\end{align*}
\]

The function defined above is obviously a continuous map. Also it can be easily proved that the function has sensitive dependence on initial conditions. Note that maximum distance between any two points of \([-1,1]\) is equal to 2. We now consider the point \(-\frac{2}{5}\) and the open interval \(U = (0, 1)\). Then there exists no point \(y \in U\) such that \(d(f^n(x), f^n(y)) = 2\), for any \(n \geq 0\). Hence \(f(x)\) does not have not strong sensitive dependence on initial conditions.

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**References**


