About Dirichlet’s Transformation and Theoretic-Arithmetic Functions

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Abstract

In this work, we are going to define a transformation from Dirichlet’s series called discrete Dirichlet’s transformation. We will obtain some classical results connected Riemann’s zeta function and theoretic-arithmetic functions. Some probabilistic interpretations are made explicit.

Keywords: Dirichlet’s transformation, Zeta function, Möbius transformation.

1 Introduction

It is well-known (cf. [6]) that the Riemann zeta-function \( \zeta(s) \) is holomorphic in the whole complex plane except for a simple pole at \( s = 1 \) with residue 1.

Riemann discovered the functional equation

\[
\zeta(s) \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} = \zeta(1-s) \Gamma\left(\frac{1-s}{2}\right) \pi^{-(1-s)/2},
\]

where \( \Gamma(s) \) denotes Euler’s Gamma-function.

This equation and the identity

\[
\zeta(s) = \overline{\zeta(\overline{s})}, \quad s \neq 1
\]

show some symmetries of \( \zeta(s) \).
From (1) it follows that \( \zeta(s) \) vanishes at the negative even integers, the so-called trivial zeros of \( \zeta(s) \). It is also known that the other non-trivial zeros lie inside the so-called critical strip \( 0 \leq \Re(s) \leq 1 \), and they are non-real.

The famous, yet open Riemann hypothesis states that every non-trivial zero of \( \zeta(s) \) satisfies \( \Re(s) = \frac{1}{2} \).

In this work, we are going from zeta function and a Dirichlet’s series define one transformation called discrete Dirichlet’s transformation. We obtain some classical results connected Riemann’s zeta function and theoretic-arithmetic functions. Some applications stemmed from number theory and theoretic-arithmetic function are given and probabilistic interpretations are maked explicit.

## 2 Dirichlet’s Transformation

Let \( f : \mathbb{N}^* \rightarrow \mathbb{C} \) be a theoretic-arithmetic function. Associate to this last a Dirichlet series

\[
\sum_{n \geq 1} \frac{f(n)}{n^s}, \quad s \in \mathbb{C},
\]

whose we will denote the abscissa of convergence \( \lambda(f) \) and the abscissa of absolute convergence \( \ell(f) \).

Introduce then \( \mathbb{A} \) a class of theoretic-arithmetic functions such that \( \lambda(f) < +\infty \). So for \( f \in \mathbb{A} \), a Dirichlet series

\[
\sum_{n \geq 1} \frac{f(n)}{n^s}, \quad s \in \mathbb{C},
\]

is convergent for \( \Re(s) > \lambda(f) \) and divergent for \( \Re(s) < \lambda(f) \). It represents a holomorphic function of a complex variable \( s \) in a half plane \( \Re(s) > \lambda(f) \) like that \( \mathbb{A} \) equiped with addition process, multiplication by a scalar and a convolution product \( * \), \((\mathbb{A}, +, \text{mult.by sca.}, *)\) is an algebra of theoretic-arithmetic functions and which is a sub-algebra of Dirichlet’s algebra.

Next we introduce a class denoted \( \mathbb{C} \) of functions of complex variable \( s \), defined on a half-plane \( \Re(s) > a \) where \( a \in [-\infty, +\infty[. \) \( \mathbb{C} \) equiped with operation \( + \), multiplication by a scalar, ordinary product \( \cdot \) : \((\mathbb{C}, +, \text{mult.by sca.}, \cdot)\) is an algebra, called functions algebra.

**Definition 2.1** We call discrete Dirichlet’s transformation a mapping \( \wedge : \mathbb{A} \rightarrow \mathbb{C} \) which to an element \( f \in \mathbb{A} \), associates a function \( \hat{f} \in \mathbb{C} \) defined by
\[ \hat{f}(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}, \quad \Re(s) > \lambda(f). \]  

A function \( \hat{f} \in \mathbb{C} \) is said Dirichlet’s transformation of \( f \).

**Proposition 2.2** A mapping \( \wedge \) is injective if and only if \( f \) and \( g \in \mathbb{A} \) and \( \hat{f} = \hat{g} \implies f = g \).

**Proof.** It is a consequence of uniqueness theorem of a Dirichlet’s series, see ([3], Theorem 3.3) and ([5]). \( \square \)

### 3 Dirichlet’s Transformation as a Homomorphism of Algebra

**Theorem 3.1** A Dirichlet’s transformation of convolution product of two elements of \( \mathbb{A} \) is equal to ordinary product of a Dirichlet’s transformation of these two elements. More precisely, let \( f \) and \( g \in \mathbb{A} \). Put \( h = f \ast g \). Then we have

\begin{itemize}
  \item[a)] \( \ell(h) \leq \max \{ \ell(f), \ell(g) \} < +\infty \), hence \( h \in \mathbb{A} \);
  \item[b)] \( \hat{h}(s) = \hat{f}(s)\hat{g}(s) \) for \( \Re(s) > \max \{ \ell(f), \ell(g) \} \).
\end{itemize}

In shortcut
\[ \hat{f} \ast \hat{g} = \hat{f} \cdot \hat{g}. \]  

**Proof.** Formally, we have
\[
\hat{f}(s)\hat{g}(s) = \left( \sum_{k \geq 1} \frac{f(k)}{k^s} \right) \left( \sum_{m \geq 1} \frac{g(m)}{m^s} \right) = \sum_{k,m \geq 1} \frac{f(k)g(m)}{(km)^s}. \]  

Hence looking terms of same denominator, namely, in fact summing at \( km \) constant we have.
\[
\hat{f}(s)\hat{g}(s) = \sum_{n \geq 1} \frac{1}{n^s} \left( \sum_{km=n} f(k)g(m) \right) = \sum_{n \geq 1} \frac{\hat{h}(n)}{n^s} = \hat{h}(s),
\]

if everyone of series is convergent (absolutely convergent), namely \( h < +\infty \), hence the theorem results. \( \square \)

**Theorem 3.2** Dirichlet’s transformation \( \wedge : \mathbb{A} \longrightarrow \mathbb{C} \) is an homomorphism from algebra \( (\mathbb{A}, +, \text{mult. by scal.}, \ast) \) into algebra \( (\mathbb{C}, +, \text{mult. by scal.}, \cdot) \) :

\begin{itemize}
  \item[a)] \( \hat{f} + \hat{g} = \hat{f} + \hat{g} \);
  \item[b)] \( \alpha \hat{f} = \alpha \hat{f}, \quad \alpha \in \mathbb{N} \);
  \item[c)] \( \hat{f} \ast \hat{g} = \hat{f} \cdot \hat{g} \).
\end{itemize}
4 Dirichlet’s Transformation of Möbius Transformation

Theorem 4.1 Let \( f \in \mathbb{A} \) and \( F \) be its Möbius transformation : \( F = 1 * f \). Then

\[
\hat{F}(s) = \zeta(s) \hat{f}(s), \quad \Re(s) > \text{Max} \{\ell(f), 1\}.
\] (9)

Proof. Apply Theorem 3.1, then we have

\[
\hat{1}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s); \quad \ell(1) = 1
\]

and \( \hat{F} = 1 \hat{f}. \) \( \square \)

Probabilistic Interpretation

Interpret the expression (9) above in probabilistic meaning.

Suppose \( s \) real > \( \text{Max} \{\ell(f), 1\} \), we have

\[
\frac{1}{\zeta(s)} \hat{F}(s) \approx \hat{f}(s),
\]

namely

\[
\frac{1}{\zeta(s)} \sum_{n \geq 1} \frac{F(n)}{n^s} = \hat{f}(s),
\]

and the mathematical expectation \( E_s(F) \) of \( F \) is

\[
E_s(F) = \hat{f}(s).
\]

For the remainder of interpretation see ([3]).
5 Calculus of Dirichlet Transformations

5.1 Direct Calculus

Consider the following cases:

a) \( f(n) = 1 \), then
\[
\hat{f}(s) = \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s), \quad s > 1
\]

b) \( f(n) = n \), then
\[
\hat{f}(s) = \sum_{n \geq 1} \frac{1}{n^{s-1}} = \zeta(s - 1), \quad s > 2.
\]

c) \( f(n) = n^\alpha, \quad \alpha \in \mathbb{R} \), then
\[
\hat{f}(s) = \sum_{n \geq 1} \frac{1}{n^{s-\alpha}} = \zeta(s - \alpha), \quad s > \alpha + 1.
\]

d) \( f(n) = \) indicator function of the set of numbers of perfect squares, then
\[
\hat{f}(s) = \sum_{n \geq 1} \frac{1}{(n^2)^s} = \zeta(2s), \quad s > \frac{1}{2}.
\]

e) \( f(n) = \) indicator function of powers \( k^{th}, \quad k \in \mathbb{N}^* \), then
\[
\hat{f}(s) = \sum_{n \geq 1} \frac{1}{(n^k)^s} = \zeta(ks), \quad s > \frac{1}{k}.
\]

f) \( f(n) = u(n) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } n > 1,
\end{cases} \) then
\[
\hat{u}(s) = \sum_{n \geq 1} \frac{u(n)}{n^s} = 1.
\]

g) Generally, \( f(n) = \delta_a(n) \), where
\[
\delta_a(n) = \begin{cases} 
1 & \text{if } n = a, \\
0 & \text{if } n \neq a,
\end{cases} \quad a \in \mathbb{N}^*,
\]
then
\[
\hat{\delta}_a(s) = \sum_{n \geq 1} \frac{\delta_a(n)}{n^s} = \frac{1}{a^s}.
\]
h) \( f(n) = (-1)^{n-1} \), then

\[
\hat{f}(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s) = \eta(s).
\]

Proof. We have

\[
\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots
\]

\[
= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots - 2(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \ldots)
\]

\[
= \sum_{n \geq 1} \frac{1}{n^s} - 2(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \ldots)
\]

\[
= \zeta(s) - 2\frac{1}{2^s}(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots) = (1 - 2^{1-s})\zeta(s).
\]

\[
\square
\]

Remark 5.1 We have

\[
\eta(s) = (1 - 2^{1-s})\zeta(s).
\]

a) \( \zeta(s) \) is definite for \( s > 1 \);

b) a function

\[
\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}
\]

is definite for \( s > 0 \), (it is an alternate series of abscissa of convergence 0) and

\[
\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}.
\]

We have
6 Calculus of Dirichlet Transformations of Multiplicative Arithmetic Function

A calculus of Dirichlet transformations of multiplicative arithmetic function will be doing in pleasant way using generalized Euler’s identity. Apply generalized Euler identity to the following multiplicative function

\[ \frac{f(n)}{n^s}. \]

**Theorem 6.1** Let \( f : \mathbb{N}^* \rightarrow \mathbb{C} \) be a multiplicative function (no identically zero) and \( s \) be a real number such that series of general term

\[ \sum_{n \geq 1} \frac{f(n)}{n^s} \]

converges, namely, \( \lambda(f) < +\infty \) and \( \Re(s) > \lambda(f) \). Then one has

\[ \hat{f}(s) = \sum_{n \geq 1} \frac{f(n)}{n^s} = \prod_p (1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + ...). \]

(10)

**Corollary 6.2** A necessary and sufficient condition for an arithmetic function \( f \) to be multiplicative is that its Dirichlet’s transformation can be written in the form

\[ h(s) = \prod_p (1 + \frac{c_p}{p^s} + \frac{c_p^2}{p^{2s}} + ...), \]

where \( c_p \) are complex numbers.

We have the following particular cases:
Proposition 6.3 Suppose $f$ strongly multiplicative, namely, for all $p \in \mathbb{P}$, for any $\alpha \in \mathbb{N^*}$, one has $f(p^\alpha) = f(p)$. Then

\[
\hat{f}(s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots\right) = \prod_p \left(1 + \frac{f(p)}{p^s - 1}\right).
\]

Proposition 6.4 If $f$ is completely multiplicative, namely, for all $p \in \mathbb{P}$, for any $\alpha \in \mathbb{N^*}$, one has $f(p^\alpha) = (f(p))^\alpha$. Then

\[
\hat{f}(s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \left(\frac{f(p)}{p^s}\right)^2 + \ldots\right).
\]

Moreover, if for any $p$ such that

\[
\left|\frac{f(p)}{p^s}\right| < 1,
\]

then

\[
\hat{f}(s) = \prod_p \left(1 - \frac{f(p)}{p^s}\right).
\]

Example 6.5 If $f = 1$ then

\[
\hat{f}(s) = \prod_p \left(1 - \frac{1}{1 - \frac{1}{p^s}}\right) = \zeta(s), \text{ for } s > 1.
\]

Indeed, $f = 1$ is an arithmetic function and completely multiplicative, $f(p) = 1$, for any $p$. Hence

\[
\hat{1}(s) = \prod_p \left(1 - \frac{1}{p^s}\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s), \text{ for } s > 1.
\]

Example 6.6 Put $f = \mu$, where

\[
\mu(p^\alpha) = \begin{cases} 
-1 & \text{if } \alpha = 1 \\
0 & \text{if } \alpha > 1.
\end{cases}
\]

Then

\[
\hat{\mu}(s) = \prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}, \text{ for } s > 1.
\]

Indeed, $f = \mu$ is a multiplicative arithmetic function. According to previous example and a definition of $\mu$, one has

\[
\hat{\mu}(s) = \prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}, \text{ for } s > 1.
\]
Remark 6.7 Functions $1$ and $\mu$ are inverse one of the other in Dirichlet’s algebra.

Example 6.8 Let $f = |\mu| = \mu^2$ be an indicator function of square free numbers. Then

$$|\hat{\mu}|(s) = \prod_p (1 + \frac{1}{p^s}) = \frac{\zeta(s)}{\zeta(2s)}, \text{ for } s > 1.$$

Proof. Indeed, one has $f = |\mu| = \mu^2$ the indicator function of square free numbers. It is a multiplicative arithmetic function and

$$|\mu|(p^\alpha) = \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha > 1. \end{cases}$$

Then according to previous examples and a definition of $f$ we have

$$|\hat{\mu}|(s) = \prod_p (1 + \frac{1}{p^s}).$$

But

$$1 + t = \frac{1 - t^2}{1 - t},$$

then

$$|\hat{\mu}|(s) = \prod_p \left( \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p}} \right) = \frac{\zeta(s)}{\zeta(2s)}, \text{ for } s > 1.$$

\[\square\]

Example 6.9 Let $f = \lambda$ be Liouville’s function, where

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^\Omega(n) & \text{if } n > 1, \end{cases}$$

with $\lambda(p) = -1$. Then

$$\hat{\lambda}(s) = \prod_p \left( \frac{1}{1 + \frac{1}{p^s}} \right) = \frac{\zeta(2s)}{\zeta(s)}, \text{ for } s > 1.$$  

Proof. Indeed, let $f = \lambda$ be the Liouville’s arithmetic function. It is a completely multiplicative arithmetic function.
According to previous examples and a definition of $\lambda$ on prime numbers, we have

$$\hat{\lambda}(s) = \prod_p \left( 1 + \frac{1}{p^s} \right).$$

But

$$\frac{1}{1+t} = \frac{1-t}{1-t^2},$$

then

$$\hat{\lambda}(s) = \prod_p \left( 1 - \frac{1}{p^s} \right) = \frac{\zeta(2s)}{\zeta(s)}, \text{ for } s > 1.$$ 

$\square$

**Remark 6.10** We have

$$\hat{\lambda}(s) = \frac{\zeta(2s)}{\zeta(s)}; \quad \left| \hat{\mu}(s) = \frac{\zeta(s)}{\zeta(2s)} \right|$$

imply $|\mu|$ and $\lambda$ are inverse one of the other in Dirichlet’s algebra.

**Example 6.11** Put $f = d$, where $d(n) =$ number of divisors of $n$. Then

$$\hat{d}(s) = \prod_p \left( 1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \ldots + \frac{\alpha + 1}{p^{\alpha s}} \right) = (\zeta(s))^2.$$ 

**Proof.** Let $f = d$ be the arithmetic function number of divisors of $n$. It is a multiplicative arithmetic function.

According to previous examples and a definition of $d$, we have

$$\hat{d}(s) = \prod_p \left( 1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \ldots + \frac{\alpha + 1}{p^{\alpha s}} \right).$$

But

$$1 + 2t + 3t^2 + 4t^3 + \ldots = \frac{1}{(1-t)^2},$$

then

$$\hat{d}(s) = \prod_p \left( 1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \ldots + \frac{\alpha + 1}{p^{\alpha s}} \right) = \prod_p \frac{1}{(1 - \frac{1}{p^s})^2} = (\zeta(s))^2.$$ 

$\square$
Example 6.12 Let $f = d^*$ be a strongly multiplicative projection of $d$ (see [1])

\[ d^*(n) = 2^{\omega(n)} \text{ number of square free divisors of } n. \]

Then

\[ \hat{d}^*(s) = \prod_p \frac{1 - \frac{1}{p^s}}{(1 - \frac{1}{p^s})^2} = \frac{(\zeta(s))^2}{\zeta(2s)}. \]

**Proof.** Indeed, let $f = d^*$ be a strongly multiplicative projection of $d$. Then $f$ is a strongly multiplicative arithmetic function. According to previous examples and a definition of $d^*$, we have

\[ \hat{d}^*(s) = \prod_p (1 + \frac{1 - d^*(p) - 1}{p^s}) = \prod_p \frac{1 + \frac{1}{p^s}}{(1 - \frac{1}{p^s})^2}. \]

But

\[ 1 + t = \frac{1 - t^2}{1 + t}, \]

hence

\[ \hat{d}^*(s) = \prod_p \frac{1}{(1 - \frac{1}{p^s})^2} = \frac{(\zeta(s))^2}{\zeta(2s)}. \]

\[ \square \]

Example 6.13 Let $f = \varphi$ be the totient Euler’s arithmetic function. Then

\[ \hat{\varphi}(s) = \prod_p \left( 1 - \frac{1}{p^s} \right) = \frac{\zeta(s - 1)}{\zeta(s)}, \text{ for } s > 2. \]

**Proof.** Indeed, let $f = \varphi$ be the totient Euler’s arithmetic function. According to ([2]), Theorem 5.1, we have

\[ \varphi(p^a) = p^{a-1}(p - 1). \]

Hence

\[ \hat{\varphi}(s) = \prod_p (1 + \frac{\varphi(p)}{p^s} + \frac{\varphi(p^2)}{p^{2s}} + ...) = \prod_p (1 + \frac{p - 1}{p^s} + \frac{p(p-1)}{p^{2s}} + \frac{p^2(p-1)}{p^{3s}} + ...) \]

\[ = \prod_p (1 + \frac{p - 1}{p^s} (1 + \frac{1}{p^{s-1}} + \frac{1}{p^{2(s-1)}} + ...)), \text{ for } s > 2 \]

\[ = \prod_p (1 + \frac{p - 1}{p^s} \cdot \frac{1}{p^{s-1}}) = \prod_p \frac{1}{(1 - \frac{1}{p^{s-1}})} = \frac{\zeta(s - 1)}{\zeta(s)}, \text{ for } s > 2. \]

\[ \square \]
7 Calculus of Dirichlet Transformations of Möbius Transformations

We have, if $F = 1 * f$, then

$$\hat{F}(s) = \zeta(s).\hat{f}(s).$$

We obtain the following results as propositions:

Proposition 7.1 Put $f = 1$, then

$$\hat{f}(s) = \zeta(s), \text{ for } s > 1.$$ 

Put

$$d(n) = \sum_{d|n} 1.$$ 

We have $d = 1 * 1$. Then

$$\hat{d}(s) = (\zeta(s))^2, \text{ for } s > 1.$$ 

Proposition 7.2 Let $f(n) = n$, then

$$\hat{f}(s) = \sum_{n \geq 1} \frac{n}{n^s} = \zeta(s - 1), \text{ for } s > 2 \text{ and } F(n) = \sum_{k|n} k.$$ 

We have

$$\sigma(n) = \sum_{k|n} k; \quad \sigma = 1 * f$$

and

$$\hat{\sigma}(s) = \zeta(s)\zeta(s - 1), \text{ for } s > 2.$$ 

Proposition 7.3 Generally, let $f(n) = n^\alpha$, $\alpha \in \mathbb{R}$, then

$$\hat{f}(s) = \zeta(s - \alpha), \text{ for } s > \alpha + 1.$$ 

We have

$$\sigma_\alpha(n) = \sum_{k|n} k^\alpha; \quad \sigma_\alpha = 1 * f$$

and

$$\hat{\sigma_\alpha}(s) = \zeta(s)\zeta(s - \alpha), \text{ for } s > \max(\alpha + 1, 1).$$

We obtain some particular cases in the following corollary:
Corollary 7.4 a) if $\alpha = 0$ then $\sigma_0 = \alpha$;
b) if $\alpha = 1$ then $\sigma_1 = \sigma$;
c) if $\alpha = -1$ then
\[ \sigma_{-1}(n) = \sum_{k|n} \frac{1}{k}, \]
and $\tilde{\sigma}_{-1}(s) = \zeta(s)\zeta(s + 1)$, for $s > 1$.

Proposition 7.5 Let $f = \varphi$, where $\varphi$ is a totient Euler’s arithmetic function. Then
\[ \hat{\varphi}(s) = \frac{\zeta(s - 1)}{\zeta(s)}, \text{ for } s > 2. \tag{11} \]

Proof. Indeed, we have
\[ \hat{f}(s) = \sum_{n \geq 1} \frac{\varphi(n)}{n^s} \leq \sum_{n \geq 1} \frac{1}{n^s - 1} = \zeta(s - 1) \tag{12} \]
and
\[ n = \sum_{k|n} \varphi(k) \text{ (Möbius’s transformation)} \tag{13} \]
implies $\zeta(s - 1) = \zeta(s)\hat{\varphi}(s)$, for $s > 2$, hence
\[ \hat{\varphi}(s) = \frac{\zeta(s - 1)}{\zeta(s)}, \text{ for } s > 2. \]

\[
\]

Proposition 7.6 Let $f = \Lambda$ be von Mangoldt’s arithmetic function introduced in ([6]). Then we have
\[ \hat{\Lambda}(s) = -\frac{\zeta'(s)}{\zeta(s)}, \text{ for } s > 1. \]

Proof. Indeed, put $f = \Lambda$, then we have
\[ \hat{f}(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}, \text{ for } s > 1 \]
with
\[ \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \leq \sum_{n \geq 1} \frac{\text{Log} n}{n^s} < +\infty, \text{ for } s > 1 \]
and
\[ \text{Log} n = \sum_{k|n} \lambda(k), \]
hence
\[ \sum_{n \geq 1} \frac{\log n}{n^s} = \zeta(s) \hat{\Lambda}(s), \text{ for } s > 1. \]

But
\[ \sum_{n \geq 1} \frac{\log n}{n^s} = -\frac{d}{ds} \left( \sum_{n \geq 1} \frac{1}{n^s} \right) = -\zeta'(s), \text{ for } s > 1, \]

hence
\[ \hat{\Lambda}(s) = -\frac{\zeta'(s)}{\zeta(s)}, \text{ for } s > 1. \]

\[ \square \]

**Proposition 7.7** Let \( f = \mu \), then

\[ \hat{\mu}(s) = \frac{1}{\zeta(s)}, \text{ for } s > 1. \]

**Proof.** We have
\[ \hat{f}(s) = \sum_{n \geq 1} \frac{\mu(n)}{n^s}, \]

with
\[ \sum_{n \geq 1} \left| \frac{\mu(n)}{n^s} \right| \leq \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s), \text{ for } s > 1. \]

But
\[ \sum_{k | n} \mu(k) = u(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \]

hence
\[ \zeta(s) \hat{\mu}(s) = 1, \text{ for } s > 1. \]

\[ \square \]

We find the same results in ([6]) but in analytical way using real and complex analysis in its proofs.

**References**


