Common Fixed Point Theorem of Compatible Mappings of Type (K) and Property (E.A.) in Fuzzy 2-Metric Space

Mala Hakwadiya¹, R.K. Gujetiya² and Dheeraj Kumari Mali³

¹,³Research Scholar, Pacific Academy of Higher Education and Research University Udaipur, Rajasthan, India
²Associate Professor & Head, Department of Mathematics Govt. P.G. College, Neemuch, M.P., India

E-mail: manuvi9the@gmail.com
E-mail: dheerajmali@gmail.com
E-mail: gujetiya71@gmail.com

(Received: 14-8-14 / Accepted: 18-9-14)

Abstract

In this paper we prove a common fixed point theorem in fuzzy 2-metric space on six self-mappings using the concept of compatible of type (K) and Property (E.A.).

Keywords: Compatible of type (K), Fixed point, Fuzzy-2 metric space, Property (E.A.).

1 Introduction

In 1965, L.A. Zadeh [8] introduced the concept of fuzzy sets which became active field of research for many researchers. In 1975, Kramosil and Michalek [5] came in front with the concept of Fuzzy metric space based on fuzzy sets which were
further modified by George and Veermani [2] with the help of t-norms. Many authors did good work and are still doing in proving fixed point theorems in Fuzzy metric space. Singh and Chauhan [4] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric spaces. Manandhar al. [6] introduced the concept of compatible maps of type (k) in Fuzzy metric space and proved fixed point theorems. Recently, many authors [1, 7, 9, 3] have also studied the fixed point theory in the fuzzy 2-metric spaces.

2 Preliminaries

Definition 2.1: [7] A binary operation ∗ : [0,1] × [0,1] × [0,1] → [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$ whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ for all $a_1, b_1, c_1, a_2, b_2, c_2$ are in [0,1].

Definition 2.2: [1] The 3-tuple $(X, M,*)$ is called a fuzzy 2-metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm, and $M$ is a fuzzy set in $X \times X \times [0, \infty)$ satisfying the following conditions.

1. $M(x, y, a, 0) = 0$.
2. $M(x, y, a, t) = 1$, for all $t > 0$ if and only if at least two of them are equal.
3. $M(x, y, a, t) = M(y, a, x, t)$. (Symmetric)
4. $M(x, y, a, r+s+t) \geq M(x, y, z, r) * M(x, z, a, s) * M(z, y, a, t)$ for all $x, y, z, a \in X$ and $r, s, t > 0$.
5. $M(x, y, a, .) : [0, \infty) \rightarrow [0,1]$ is left continuous for all $x, y, z, a \in X$ and $t > 0$.
6. $\lim_{n \to \infty} M(x, y, a, t) = 1$ for all $x, y, z, a \in X$ and $t > 0$.

Definition 2.3: [9] Self-mappings $S$ and $T$ of a fuzzy 2-metric space $(X, M, *)$ are said to be compatible if and only if $M(STx_n, TSx_n, z, t) \to 1 \forall t > 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $Tx_n, Sx_n \to p$ for some $p$ in $X$ as $n \to \infty$.

Definition 2.4: [1] A Fuzzy 2-metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in $X$ converges in $X$.

Definition 2.5: [7] Let $(X, M, *)$ be a fuzzy 2-metric space. A sequence $\{x_n\}$ in fuzzy 2-metric space $X$ is said to be convergent to a point $x \in X$, $\lim_{n \to \infty} M(x_n, x, a, t) = 1$ for all $a \in X$, and $t > 0$.

Definition 2.6: [7] A sequence $\{x_n\}$ in fuzzy 2-metric space $X$ is called a Cauchy sequence, if $\lim_{n \to \infty} M(x_{n+p}, x_n, a, t) = 1$ for all $a \in X$, and $t, p > 0$.

Definition 2.7: [7] A function $M$ is continuous in a Fuzzy 2-metric space, if and only if whenever for all $a > X$ and $t > 0$, $x_n \to x$, $y_n \to y$, then $\lim_{n \to \infty} M(x_n, y_n, a, t) = M(x, y, a, t)$ for all $a > X$ and $t > 0$. 

\[\text{Mala Hakwadiya et al.}\]
Definition 2.8. [6] The self maps A and S of a fuzzy metric space \((X, M, \ast)\) are said to be compatible of type \((K)\) iff \(\lim_{n \to \infty} A x_n = x\) and \(\lim_{n \to \infty} S x_n = x\) for some \(x \in X\) and \(t > 0\).

Definition 2.9: [3] Two pairs of self mappings \((A, S)\) and \((B, T)\) defined on a fuzzy metric space \((X, M, \ast)\) are said to share the common property \((E. A)\) if there exist a sequence \(\{x_n\}\) and \(\{y_n\}\) in X such that

\[
\lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} T y_n = z \text{ for some } z > X.
\]

Definition 2.10: [9] Self-maps \(S\) and \(T\) of a fuzzy 2-metric space \((X, M, \ast)\) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points that is if \(S p = T p\) for some \(p \in X\) then \(S T p = T S p\).

Lemma: [9] \(M(x, y, z, .)\) is non-decreasing for all \(x, y, z \in X\).

Lemma: [9] Let \((X, M, \ast)\) be a fuzzy 2-metric space. If there exists \(k \in (0, 1)\) such that \(M(x, y, z, kt) \geq M(x, y, z, t)\) for all \(x, y, z \in X\) with \(z \neq x, z \neq y\) and \(t > 0\), then \(x = y\).

3 Main Result

Theorem 3.1 : Let \((X, M, \ast)\) be a complete Fuzzy 2-metric space and \(A, B, P, Q, S\) and \(T\) be a self mapping of \(X\) satisfying the following condition:

(i) \(P(X) \subseteq BT(X)\) and \(Q(X) \subseteq SA(X)\)
(ii) \(SA\) and \(BT\) are continuous.
(iii) \((P, SA)\) and \((Q, BT)\) compatible of type of \((K)\)
(iv) \([1 + aM(SA x, Px, a, kt)] \ast M(Px, Qy, a, kt) \geq aM(Px, SA x, a, kt) \ast M(BT y, Qy, a, kt) \ast M(BT y, Px, a, kt) + M(BT y, SA x, a, t) \ast M(Px, SA x, a, t)] \ast M(Qy, BT y, a, (2−\alpha)t) \ast M(Qy, SA x, a, t) \ast M(Px, BT y, a, (2−\alpha)t)\)

For all \(x, y, a \in X, \alpha \in (0,2), a \geq 0\) and \(t > 0\)
(v) \((P, SA)\) and \((BT, Q)\) are commute,

Then \(A, B, P, Q, S\) and \(T\) have a unique common fixed point.

Proof: Since \(P(X) \subseteq BT(X)\) and \((Q(X) \subseteq SA(X)\), so for any \(x_0 \in X\), there exists \(x_1 \in X\) such that \(P x_0 = BT x_1\) and for this \(x_1\), there exists \(x_2 \in X\) such that \(BT x_1 = SA x_2\). Inductively, we define a sequences \(\{y_n\}\) in \(X\) such that

\[
y_{2n+1} = P x_{2n} = BT x_{2n+1} \quad \text{and} \quad y_{2n+2} = Q x_{2n+1} = SA x_{2n+2} \quad \text{for all } n=1,2,3,\ldots
\]

Putting \(x = x_{2n}\) and \(y = x_{2n+1}\) with \(\alpha = 1\) Form (iv), we get

\[
[1 + aM(SA x_{2n}, Px_{2n}, a, kt)] \ast M(Px_{2n}, Qx_{2n+1}, a, kt) \geq aM(Px_{2n}, SA x_{2n}, a, kt) \ast M(BT x_{2n+1}, Qx_{2n+1}, a, kt) \ast M(BT x_{2n+1}, Px_{2n}, a, kt)]
\]
\[ + M(Bx_{2n+1}, Ax_{2n}, a, t) \cdot M(Px_{2n}, Ax_{2n}, a, t) \cdot M(Qx_{2n+1}, Bx_{2n+1}, a, t) \\
\cdot M(Qx_{2n+1}, Ax_{2n}, a, t) \cdot M(Px_{2n}, Bx_{2n+1}, a, t) \\
[1 + aM(y_{2n+1}, y_{2n+2}, a, kt)] \cdot M(y_{2n+1}, y_{2n+2}, a, kt) \]
\[ \geq a[M(y_{2n+1}, y_{2n}, a, kt) \cdot M(y_{2n+1}, y_{2n+2}, a, kt) \cdot M(y_{2n+1}, y_{2n+1}, a, kt)] \]
\[ + M(y_{2n+1}, y_{2n}, a, t) \cdot M(y_{2n+1}, y_{2n}, a, t) \cdot M(y_{2n+2}, y_{2n+1}, a, t) \]
\[ \cdot M(y_{2n+2}, y_{2n}, a, t) \cdot M(y_{2n+1}, y_{2n+1}, a, t) \]
\[ M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n+1}, y_{2n}, a, t) \cdot M(y_{2n+1}, y_{2n}, a, t) \cdot \\
M(y_{2n+2}, y_{2n+1}, a, t) \cdot M(y_{2n+2}, y_{2n+1}, a, t) \cdot M(y_{2n+1}, y_{2n}, a, t) \]
\[ M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n+1}, y_{2n}, a, t) \cdot M(y_{2n+2}, y_{2n+1}, a, t) \]

Similarly, we also have
\[ M(y_{2n+2}, y_{2n+3}, a, kt) \geq M(y_{2n+2}, y_{2n+1}, a, t) \cdot M(y_{2n+3}, y_{2n+2}, a, t) \]

In general for \( m = 1, 2, 3, \ldots \)
\[ M(y_{m+1}, y_{m+2}, a, kt) \geq M(y_{m+1}, y_{m}, a, t) \cdot M(y_{m+2}, y_{m+1}, a, t) \]

Consequently, it follows that for \( m = 1, 2, 3, \ldots \) and \( p = 1, 2, 3, \ldots \)
\[ M(y_{m+1}, y_{m+2}, a, kt) \geq M(y_{m+1}, y_{m}, a, t) \cdot M\left(\frac{y_{m+2}, y_{m+1}, a, t}{t}\right) \]

We have \( M(y_{m+1}, y_{m+2}, a, kt) \geq M\left(\frac{y_{m+1}, y_{m}, a, t}{t}\right) \)
\[ \geq M\left(y_{m}, y_{m-1}, a, \frac{t}{k}\right) \geq \cdots \geq M\left(y_{2}, y_{1}, a, \frac{t}{k^{m}}\right) \rightarrow \infty \]

As \( n \rightarrow \infty \), so \( M(y_{m+1}, y_{m}, a, t) \rightarrow 1 \) for any \( t>0 \). For each \( \varepsilon>0 \) and each \( t>0 \),

we can choose \( m_{0} \in N \) such that \( M(y_{m+1}, y_{m}, a, t) > 1 - \varepsilon \) for all \( m > m_{0} \) for \( m_{1}, m_{0} \in N \). Then \( M(y_{m+1}, y_{m+2}, a, kt) \geq M(y_{m}, y_{m+1}, a, t) \)

Hence by lemma \( \{y_{n}\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete then \( \{y_{n}\} \)
converges to some point \( z \in X \), and so that \( \{Px_{2n}\}, \{Bx_{2n+1}\}, \{Qx_{2n+1}\} \) and \( \{Ax_{2n+2}\} \) also converges to \( z \). Since \( (P, A) \) and \( (Q, BT) \) are compatible of type (K), we have
Putting $x = P_{x_{2n}}$ and $y = Q_{x_{2n+1}}$ with $\alpha = 1$ Form (iv), we get

$$[1 + a M(SA(P_{x_{2n}}), P(P_{x_{2n}}), a, k_r)] M(P(P_{x_{2n}}), Q(Q_{x_{2n+1}}), a, k_r)$$

$$\geq a \left[ M(BT(Q_{x_{2n+1}}), Q(Q_{x_{2n+1}}), a, k_r) M(BT(Q_{x_{2n+1}}), P(P_{x_{2n}}), a, k_r) \right] +$$

$$M(BT(Q_{x_{2n+1}}), SA(P_{x_{2n}}), a, t) M(P(P_{x_{2n}}), SA(P_{x_{2n}}), a, t) * M(Q(Q_{x_{2n+1}}), BT(Q_{x_{2n+1}}), a, t) * M(Q(Q_{x_{2n+1}}), SA(P_{x_{2n}}), a, t)$$

Letting $n \to \infty$, we have

$$[1 + a M(SAz, SAz, a, k_r)] M(SAz, BTz, a, k_r) \geq a [M(SAz, SAz, a, k_r) *$$

$$M(BTz, BTz, a, k_r) M(BTz, SAz, a, k_r)] + M(BTz, SAz, a, t) * M(SAz, BTz, a, t)$$

$$M(SAz, BTz, a, k_r) \geq M(BTz, SAz, a, t) M(BTz, SAz, a, t) M(SAz, BTz, a, t)$$

Which implies that $M(SAz, BTz, a, k_r) \geq M(BTz, SAz, a, t)$

Therefore by lemma, we have $SAz = BTz$. (3.1)

Putting $x = z$ and $y = Q_{x_{2n+1}}$ with $\alpha = 1$ Form (iv), we get

$$[1 + a M(SAz, Pz, a, k_r)] M(Pz, Q(Q_{x_{2n+1}}), a, k_r) \geq a [M(Pz, SAz, a, k_r) *$$

$$M(BT(Q_{x_{2n+1}}), Q(Q_{x_{2n+1}}), a, k_r) M(BT(Q_{x_{2n+1}}), Pz, a, k_r)] +$$

$$M(BT(Q_{x_{2n+1}}), SAz, a, t) M(Pz, SAz, a, t) M(Q(Q_{x_{2n+1}}), BT(Q_{x_{2n+1}}), a, t)$$

Letting $n \to \infty$, we have

$$[1 + a M(BTz, Pz, a, k_r)] M(Pz, BTz, a, k_r) \geq a [M(Pz, BTz, a, k_r) *$$

$$M(BTz, BTz, a, k_r) M(BTz, Pz, a, k_r)] + M(BTz, BTz, a, t)$$

$$M(Pz, BTz, a, k_r) \geq M(BTz, BTz, a, t) M(BTz, BTz, a, t) M(Pz, BTz, a, t)$$

Which implies that $M(Pz, BTz, a, k_r) \geq M(Pz, BTz, a, t)$. 
Therefore by lemma, we have \( P_z = B T_z \) \hspace{1cm} (3.2)

Putting \( x = z \) and \( y = z \), using (3.1), (3.2) with \( \alpha = 1 \) Form (iv), we get

\[
[1 + aM(SA_z, P_z, a, kt)] \ast M(P_z, Q_z, a, kt) \geq a[M(P_z, SA_z, a, kt) \ast M(BT_z, Q_z, a, kt) \ast M(BT_z, P_z, a, kt) \ast M(BT_z, SA_z, a, t) \ast M(P_z, SA_z, a, t) \ast M(P_z, BT_z, a, t)]
\]

\[
[1 + aM(P_z, P_z, a, kt)] \ast M(P_z, Q_z, a, kt) \geq a[M(P_z, P_z, a, kt) \ast M(P_z, Q_z, a, kt) \ast M(P_z, P_z, a, kt) \ast M(P_z, P_z, a, t) \ast M(Q_z, P_z, a, t) \ast M(Q_z, P_z, a, t) \ast M(P_z, P_z, a, t)]
\]

Which implies that \( M(P_z, Q_z, a, kt) \geq M(P_z, Q_z, a, t) \).

Therefore by lemma, we have \( P_z = Q_z \) \hspace{1cm} (3.3)

Therefore form (3.1), (3.2) and (3.3), we have \( SA_z = BT_z = P_z = Q_z \) \hspace{1cm} (3.4)

Now we show that \( Q_z = z \). Putting \( x = x_{2^n} \) and \( y = z \) with \( \alpha = 1 \) Form (iv), we get

\[
[1 + aM(SA_{2^n}, P_{2^n}, a, kt)] \ast M(P_{2^n}, Q_z, a, kt) \geq a[M(P_{2^n}, SA_{2^n}, a, kt) \ast M(BT_z, Q_z, a, kt) \ast M(BT_z, P_{2^n}, a, kt) \ast M(BT_z, SA_{2^n}, a, t) \ast M(P_{2^n}, SA_{2^n}, a, t) \ast M(Q_z, BT_z, a, t) \ast M(Q_z, SA_{2^n}, a, t) \ast M(P_{2^n}, BT_z, a, t) \ast M(P_{2^n}, BT_z, a, t)]
\]

Letting \( n \to \infty \), we have

\[
[1 + aM(z, z, a, kt)] \ast M(z, Q_z, a, kt)
\]

\[
\geq a[M(z, z, a, kt) \ast M(Q_z, Q_z, a, kt) \ast M(Q_z, z, a, kt)]
\]

\[
+ M(Q_z, z, a, t) \ast M(z, z, a, t) \ast M(Q_z, Q_z, a, t) \ast M(Q_z, z, a, t) \ast M(z, Q_z, a, t) \ast M(z, Q_z, a, t)
\]

Which implies that \( M(z, Q_z, a, kt) \geq M(z, Q_z, a, t) \).

Therefore by lemma, we have \( z = Q_z \).

Hence by (3.4), we have \( SA_z = BT_z = P_z = Q_z = z \) \hspace{1cm} (3.5)

Now to prove \( A_z = z \), putting \( x = A_z \), \( y = z \) with \( \alpha = 1 \) in (iv), we obtain

\[
[1 + aM(SA(A_z), P(A_z), a, kt)] \ast M(P(A_z), Q_z, a, kt)
\]
\[
\geq a \{ M(P(Az), SA(Az), a, kt) \ast M(BTz, Qz, a, kt) \ast M(BTz, P(Az), a, kt) \}
+ M(BTz, SA(Az), a, t) \ast M(P(Az), SA(Az), a, t) \ast M(Qz, BTz, a, t)
\ast M(Qz, SA(Az), a, t) \ast M(P(Az), BTz, a, t)
\]

\[
[1 + a M(Az, Az, a, kt)] \ast M(Az, z, a, kt)
\geq a \{ M(Az, Az, a, kt) \ast M(z, z, a, kt) \ast M(Az, z, a, t) \}
+ M(z, Az, a, t) \ast M(Az, Az, a, t) \ast M(z, Az, a, t) \ast M(Az, z, a, t)
\]

Which implies that \( M(Az, z, a, kt) \geq M(Az, z, a, t). \)

Therefore by lemma, we have \( z = Az. \) Since \( SAz = z \) which implies that \( Sz = z. \)

Again, Now to prove \( Tz = z, \) putting \( x = z, y = Tz \) with \( \alpha = 1 \) in (iv), we obtain

\[
[1 + a M(SAz, Pz, a, kt)] \ast M(Pz, Q(Tz), a, kt)
\geq a \{ M(Pz, SAz, a, kt) \ast M(BT(Tz), Q(Tz), a, kt) \ast M(BT(Tz), Pz, a, kt) \}
+ M(BT(Tz), SAz, a, t) \ast M(Pz, SAz, a, t) \ast M(Q(Tz), BT(Tz), a, t) \ast
\]

\[
M(Q(Tz), SAz, a, t) \ast M(Pz, BT(Tz), a, t)
\]

\[
[1 + a M(z, z, a, kt)] \ast M(z, Tz, a, kt)
\geq a \{ M(z, z, a, kt) \ast M(Tz, Tz, a, kt) \ast M(Tz, z, a, t) \}
+ M(Tz, z, a, t) \ast M(z, z, a, t) \ast M(Tz, Tz, a, t) \ast M(Tz, z, a, t)
\]

Which implies that \( M(z, Tz, a, kt) \geq M(Tz, z, a, t). \)

Therefore by lemma, we have \( z = Tz. \) Since \( BTz = z \) which implies that \( Bz = z. \)

Thus combining all the above result, we have \( Az = Bz = Pz = z = Qz = Sz = Tz, \)

Hence \( z \) is common fixed point of \( A, B, P, Q, S \) and \( T. \)

**Uniqueness:** let \( u \) be an another common fixed point of \( A, B, P, Q, S \) and \( T. \)

putting \( x = z, y = u \) with \( \alpha = 1 \) in (iv), we obtain

\[
[1 + a M(SAz, Pz, a, kt)] \ast M(Pz, Qu, a, kt) \geq a [ M(Pz, SAz, a, kt) \ast
\]

\[
M(BTu, Qu, a, kt) \ast M(BTu, Pz, a, kt) \ast M(BTu, SAz, a, t) \ast
\]

\[
M(Pz, SAz, a, t) \ast M(Qu, BTu, a, t) \ast M(Qu, SAz, a, t) \ast M(Pz, BTu, a, t)
\]
\[ [1 + aM(z, z, a, kt)] * M(z, u, a, kt) \geq a[M(z, z, a, kt) * M(u, u, a, kt) * M(u, z, a, kt)] \]
\[ + M(u, z, a, t) * M(z, z, a, t) * M(u, u, a, t) * M(u, z, a, t) * M(z, u, a, t) \]

Which implies that \( M(z, u, a, kt) \geq M(u, z, a, t) \).

Therefore by lemma, we have \( z = u \).

Hence \( z \) is unique common fixed point of \( A, B, P, Q, S \) and \( T \).

**Corollary:** Let \((X, M, \ast)\) be a complete Fuzzy 2-metric space and \( A, B, P \) and \( Q \) be a self mapping of \( X \) satisfying the following condition:

(i) \( P(X) \subset B(X) \) and \( Q(X) \subset A(X) \)
(ii) \( A \) and \( B \) are continuous
(iii) \( (P, A) \) and \( (Q, B) \) compatible of type of \( (K) \)
(iv) \[ 1 + aM(Ax, Px, a, kt) \ast M(Px, Qy, a, kt) \geq \]
\[ a[M(Px, Ax, a, kt) * M(By, Qy, a, kt) * M(By, Px, a, kt)] + M(By, Ax, a, t) * \]
\[ M(Px, Ax, a, \alpha t) * M(Qy, By, a, (2-\alpha) t) * M(Qy, Ax, a, \alpha t) \]
\[ * M(Px, By, a, (2-\alpha) t) \]

For all \( x, y \in X, \alpha \in (0, 2), a \geq 0 \) and \( t > 0 \)
(v) \( (P, A) \) and \( (B, Q) \) are commute.

Then \( A, B, P \) and \( Q \) have a unique common fixed point.

**Example:** Let \( X = [4, 20] \) with the metric \( d \) defined by \( d(x, y) = |x - y| \) define \( M(x, y, t) = \frac{t}{d(x, y)} \) for all \( x, y \in X, t > 0 \) clearly \((X, M, \ast)\) is a complete fuzzy metric space define \( A, B, P, Q, S \) and \( T : X \rightarrow Y \) as follows \( Px = 2 \) if \( x \leq 6 \), \( Px = 6 \) if \( x > 6 \), \( Qx = 4 \) if \( x \leq 6 \) and \( Qx = 6 \) if \( x > 10 \) and \( Sax, BTx = x \) for all \( x \in X \). The \( A, B, P, Q, S \) and \( T \) satisfy all the conditions of the above theorem and have a unique common fixed point \( x = 4 \).

**Theorem 3.2:** Let \((X, M, \ast)\) be a Fuzzy 2-metric space and \( A, B, P, Q, S \) and \( T \) be a self mapping of \( X \) satisfying the following condition:

(i) \( P(X) \subset BT(X) \) and \( Q(X) \subset SA(X) \)
(ii) \( (P, SA) \) and \( (Q, BT) \) weakly compatible.
(iii) \[ [1 + aM(SAx, Px, a, kt)] * M(Px, Qy, a, kt) \geq \]
\[ a[M(Px, SAx, a, kt) * M(BTy, Qy, a, kt) * M(BTy, Px, a, kt)] + M(BTy, Ax, a, t) * \]
\[ M(Px, SAx, a, \alpha t) * M(Qy, BTy, a, (2-\alpha) t) * M(Qy, Ax, a, \alpha t) \]
\[ * M(Px, BTy, a, (2-\alpha) t) \]

For all \( x, y \in X, \alpha \in (0, 2), a \geq 0 \) and \( t > 0 \)
(iv) The pair \( (P, SA) \) and \( (BT, Q) \) are commute.
(v) The pair (P, SA) and (BT, Q) satisfy E.A. Property.
(vi) One of SA(X) or BT(X) is closed subset of X.

Then A, B, P, Q, S and T have a unique common fixed point.

**Proof:** We assume that the pair (Q, BT) satisfy the E.A. property. Then there exists a sequence \( \{x_n\} \) in X such that \( \lim_{n \to \infty} Qx_n = \lim_{n \to \infty} BTx_n = z \) for some \( z \in X \). Since \( Q(X) \subset SA(X) \), there exists a sequence \( \{y_n\} \) in X such that \( Qx_n = SAy_n \). Hence \( \lim_{n \to \infty} SAy_n = z \). Also \( P(X) \subset BT(X) \), there exists a sequence \( \{y'_n\} \) in X such that \( Py'_n = BTx_n \). Hence \( \lim_{n \to \infty} Py'_n = z \). Suppose that \( SA(X) \) is a closed subset of X. Then \( z = SAu \) for some \( u \in X \). Subsequently, we have \( \lim_{n \to \infty} Qx_n = \lim_{n \to \infty} BTx_n = \lim_{n \to \infty} Py'_n = \lim_{n \to \infty} SAy_n = z = SAu \). For some \( u \in X \). Now, To prove that \( Pu = SAu \). From (3) putting \( x = u \) and \( y = x_n \) with \( \alpha = 1 \).

\[
[1 + aM(SAu, Pu, a, \kappa t)] \ast M(Pu, Qx_n, a, \kappa t) \geq a[M(Pu, SAu, a, \kappa t) \ast M(BTx_n, Qx_n, a, \kappa t) \ast M(BTx_n, Pu, a, \kappa t)] \\
+ M(BTx_n, SAu, a, t) \ast M(Pu, SAu, a, t) \ast M(Qx_n, BTx_n, a, t) \\
* M(Qx_n, SAu, a, t) \ast M(Pu, BTx_n, a, t)
\]

Letting \( n \to \infty \), we have

\[
[1 + aM(z, Pu, a, \kappa t)] \ast M(Pu, z, a, \kappa t) \\
\geq a[M(Pu, z, a, \kappa t) \ast M(z, z, a, \kappa t) \ast M(z, Pu, a, \kappa t)] \\
+ M(z, z, a, t) \ast M(Pu, z, a, t) \ast M(z, z, a, t) \ast M(z, z, a, t) \ast M(Pu, z, a, t)
\]

Which implies that \( M(Pu, z, a, \kappa t) \geq M(Pu, z, a, t) \).

Therefore by lemma, we have \( Pu = z \) and hence \( Pu = SAu = z \).

Since \( P(X) \subset BT(X) \), there exists a point \( v \in X \) such that \( Pu = z = BTv \).

Now, we claim that \( BTv = Qv \). From (3) putting \( x = u \) and \( y = v \) with \( \alpha = 1 \),

we have

\[
[1 + aM(SAu, Pu, a, \kappa t)] \ast M(Pu, Qv, a, \kappa t) \\
\geq a[M(Pu, SAu, a, \kappa t) \ast M(BTv, Qv, a, \kappa t) \ast M(BTv, Pu, a, \kappa t)] + M(BTv, SAu, a, t) \\
* M(Pu, SAu, a, t) \ast M(Qv, BTv, a, t) \ast M(Qv, SAu, a, t) \ast M(Pu, BTv, a, t)
\]

\[
[1 + aM(z, z, a, \kappa t)] \ast M(z, Qv, a, \kappa t) \\
\geq a[M(z, z, a, \kappa t) \ast M(z, Qv, a, \kappa t) \ast M(z, z, a, \kappa t)] \\
+ M(z, z, a, t) \ast M(z, z, a, t) \ast M(Qv, z, a, t) \ast M(Qv, z, a, t) \ast M(z, z, a, t)
\]

Which implies that \( M(z, Qv, a, \kappa t) \geq M(Qv, z, a, t) \).
Therefore by lemma, we have $z = Qv$. Hence we have $BTv = Qv$. Thus $Pu = SAu = BTv = Qv = z$. Since the pairs $(P, SA)$ and $(Q, BT)$ are weakly compatible points, respectively, we obtain $Pz = P(SAu) = SA(Pu) = SAz$ and $Qz = Q(BTv) = BT(Qv) = BTz$. Now to prove that $Pz = z$. From (3) putting $x = z$ and $y = v$ with $\propto = 1$, we have

$$[1 + aM(SAz, Pz, a, kt)] \geq a[M(Pz, SAz, a, kt) + M(BTv, Pz, a, kt)] + M(BTv, SAz, a, t) * M(BTv, Pz, a, t)$$

$$1 + aM(Pz, Pz, a, kt)\geq a[M(Pz, Pz, a, kt) + M(z, z, a, kt)] + M(z, Pz, a, t) * M(Pz, z, a, t)$$

Which implies that $M(Pz, z, a, t) \geq M(Pz, z, a, t)$. Therefore by lemma, we have $z = Pz$. Since $Pz = SAz$, which implies that $SAz = z$.

Now to prove $Qz = z$, from (3) putting $x = z$ and $y = z$ with $\propto = 1$, we have

$$[1 + aM(SAz, Pz, a, kt)] \geq a[M(Pz, SAz, a, kt) + M(BTv, Pz, a, kt)] + M(BTv, SAz, a, t) * M(BTv, Pz, a, t)$$

$$1 + aM(z, z, a, kt)\geq a[M(z, z, a, kt) + M(Qz, z, a, kt)] + M(z, z, a, t) * M(z, z, a, t)$$

Which implies that $M(z, z, a, t) \geq M(z, z, a, t)$. Therefore by lemma, we have $z = z$. Since $Qz = BTz$, which implies that $BTz = z$.

Now to prove $Az = z$, from (3) putting $x = Az$ and $y = z$ with $\propto = 1$, we have

$$[1 + aM(SA(Az), P(Az), a, kt)] \geq a[M(P(Az), SA(Az), a, kt) + M(BTv, P(Az), a, kt)] + M(BTv, SA(Az), a, t) * M(P(Az), SA(Az), a, t) * M(Qz, BTz, a, t)$$

$$1 + aM(Az, Az, a, kt)\geq a[M(Az, Az, a, kt) + M(Qz, Az, a, kt)] + M(Az, Az, a, t) * M(Az, Az, a, t) * M(Az, Az, a, t)$$

Which implies that $M(Az, Az, a, kt) \geq M(Az, Az, a, t)$. Therefore by lemma, we have $Az = z$. Since $SAz = z$, which implies that $Sz = z$. 

Now to prove $Tz = z$, from (3) putting $x = z$ and $y = Tz$ with $\alpha = 1$, we have

$$[1 + aM(SAz, Pz, a, kt)] * M(Pz, Q(Tz), a, kt) \geq a[M(Pz, SAz, a, kt) * M(BT(Tz), Q(Tz), a, kt) * M(BT(Tz), Pz, a, kt)]
+ M(BT(Tz), SAz, a, t) * M(Pz, SAz, a, t) * M(Q(Tz), BT(Tz), a, t)
* M(Q(Tz), SAz, a, t) * M(Pz, BT(Tz), a, t)

[1 + aM(z, z, a, kt)] * M(z, Tz, a, kt)
\geq a[M(z, z, a, kt) * M(Tz, Tz, a, kt) * M(Tz, z, a, kt)]
+ M(Tz, z, a, t) * M(z, z, a, t) * M(z, Tz, a, t) * M(Tz, z, a, t) * M(z, Tz, a, t)

Which implies that $M(z, Tz, a, kt) \geq M(z, z, a, kt)$.

Therefore by lemma, we have $z = Tz$. Since $z = BTz$ which implies that $Bz = z$.

Thus combining all the above result, we have $Az = Bz = Pz = z = Qz = Sz = Tz$

Hence $z$ is common fixed point of $A, B, P, Q, S$ and $T$.

**Uniqueness:** Let $u$ be an another common fixed point of $A, B, P, Q, S$ and $T$.

Putting $x = z, y = u$ with $\alpha = 1$ in (iv), we obtain

$$[1 + aM(SAz, Pz, a, kt)] * M(Pz, Qu, a, kt) \geq a[M(Pz, SAz, a, kt) * M(BTu, Qu, a, kt) * M(BTu, Pz, a, kt) + M(BTu, SAz, a, t) * M(Pz, SAz, a, t) * M(Qu, BTu, a, t) * M(Qu, SAz, a, t) * M(Pz, BTu, a, t)

[1 + aM(z, z, a, kt)] * M(u, u, a, kt)
\geq a[M(z, z, a, kt) * M(u, u, a, kt) * M(u, z, a, kt)]
+ M(u, z, a, t) * M(z, z, a, t) * M(u, u, a, t) * M(u, z, a, t) * M(z, u, a, t)

Which implies that $M(z, u, a, kt) \geq M(u, z, a, t)$.

Therefore by lemma, we have $z = u$.

Hence $z$ is unique common fixed point of $A, B, P, Q, S$ and $T$.

**Corollary:** Let $(X, M, *)$ be a Fuzzy 2-metric space and $P, Q, S$ and $T$ be a self mapping of $X$ satisfying the following condition:

(i) $P(X) \subset T(X)$ and $Q(X) \subset S(X)$
(ii) $(P, S)$ and $(Q, T)$ weakly compatible.
(iii) $[1 + aM(Sx, Px, a, kt)] * M(Px, Qy, a, kt) \geq a[M(Px, Sx, a, kt) * M(Ty, Qy, a, kt) * M(Ty, Px, a, kt) + M(Ty, Sx, a, \alpha t) * M(Px, Sx, a, \alpha t)]
* M(Qy, Ty, a, (2-\alpha)t) * M(Qy, Sx, a, \alpha t) * M(Px, Ty, a, (2-\alpha)t)$

For all $x, y \in X, \alpha \in (0,2), a \geq 0$ and $t > 0$
(iv) The pair \((P, S)\) and \((T, Q)\) are commute.
(v) The pair \((P, S)\) and \((T, Q)\) satisfy E.A. Property.
(vi) One of \(S(X)\) or \(T(X)\) is closed subset of \(X\)

Then \(P, Q, S\) and \(T\) have a unique common fixed point.

4 Conclusion

In this paper, we have presented common fixed point theorem for six self mappings in Fuzzy 2-metric spaces through concept of compatible of type (K) and Property (E.A.).

Acknowledgement: The Authors are thankful to the anonymous referees for his valuable suggestions for the improvement of this paper.

References