Local Existence of the Solution for Stochastic Functional Differential Equations with Infinite Delay

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Abstract

In this paper we present and prove the existence of solution for stochastic functional differential equations with infinite delay in a separable Hilbert space respects to a local Lipchitz condition.

Keywords: Local existence, stochastic functional differential equation, local Lipchitz condition, infinite delay.

1 Introduction

a class of stochastic functional differential equations in a separable Hilbert space $H$ which has the form:

\[
\begin{cases}
    dX(t) = AX(t)dt + f(t, X_t)dt + g(t, X_t)dW(t), & t \geq 0 \\
    X(t) = \varphi(t), & t \leq 0
\end{cases}
\]

where $A : D(A) \subset H \to H$ is a linear (possibly unbound) operator, $\varphi$ is in the phase space $B$, and $X_t$ is defined as

$$X_t(\theta) = X(t + \theta), \quad -\infty < \theta \leq 0,$$

$f : \mathbb{R}_+ \times B \to H$, $g : \mathbb{R}_+ \times B \to L_2^0$ are continuous functions.

In this paper, we present the condition for the local existence of solutions for (1)
2 Preliminaries

2.1 Basic Concepts of Stochastic Analysis

Let $\Omega, \mathcal{F}, \mathbb{P}$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$ i.e. a right continuous, increasing family of sub $\sigma$-fields of $\mathcal{F}$ ($\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$, for all $0 \leq t < s < \infty$).

**Definition 2.1.** [2] An $H$-valued random variable is an $\mathcal{F}$-measurable function $X : \Omega \rightarrow H$ and a collection of random variables $X = \{X(t, \omega) : \Omega \rightarrow H, 0 \leq t \leq T\}$ is called a stochastic process.

**Note.** In this paper, we write $X(t)$ instead of $X(t, \omega)$.

**Definition 2.2.** [2] A stochastic process $X$ is said to be adapted if for every $t$, $X(t)$ is $\mathcal{F}_t$-measurable.

Let $K$ be a separable Hilbert space, $Q$ be a nonnegative definite symmetric trace-class operator on $K$, and $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in $K$, and let the corresponding eigenvalues of $Q$ be $\lambda_n$ i.e $Qe_n = \lambda_n e_n$, for $n = 1, 2, \ldots$. Let $w_n(t)$ be a sequence of real valued independent Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 2.3.** [2] The process

$$W(t) = \sum_{n=1}^\infty \sqrt{\lambda_n} w_n(t) e_n$$

(2)

is called a $Q$-Weiner process in $K$.

Let $K_Q = Q^{1/2}K$ is a Hilbert space equipped with the norm

$$||u||_{K_Q} = ||Q^{1/2}u||_K, \ u \in K_Q$$

Clearly, $K_Q$ is separable with complete orthonormal basis $\{\sqrt{\lambda_n} e_n\}_{n=1}^\infty$.

Now, let $L_Q^0 = L^0_2(K_Q, H)$ be the space of all Hilbert-Schmidt operators from $K_Q$ to $H$. Then $L^0_2$ is a separable Hilbert space with norm

$$||L||_{L^0_2} = \sqrt{tr((LQ^{1/2})(LQ^{1/2})^*)}, \ L \in L^0_2.$$ 

**Remark 2.4.** For $\kappa \in B(K, H)$ this norm reduce to

$$||\kappa||_{L^0_2} = \sqrt{tr(\kappa Q \kappa^*)}$$
Now, for any $T \geq 0$, if $\Phi = \{\Phi(t), t \in [0, T]\}$ be an $\mathcal{F}_t$-adapted, $L^0_2$-valued process such that

$$E \left( \int_0^T tr \left( (\Phi Q^{1/2}) (\Phi Q^{1/2})^* \right) ds \right) < \infty$$

then the stochastic integral $\int_0^t \Phi(s)dW(s) \in H$ be well defined by

$$\int_0^t \Phi(s)dW(s) = \lim_{n \to \infty} \sum_{i=1}^{n} \int_0^t \Phi(s) \sqrt{\lambda_i} e_i dw_i(s) \quad (3)$$

### 2.2 Phase Space

Let $\mathcal{E}$ be a Banach space, we assume that the phase space $(\mathcal{B}, ||.||_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathcal{E}$ satisfying the following fundamental axioms

(A1) For $a > 0$, if $X$ is a function mapping $(-\infty, a]$ into $\mathcal{E}$, such that $X \in \mathcal{B}$ and $X$ is continuous on $[0, a]$, then for every $t \in [0, a]$ the following conditions hold:

(i) $X_t$ is in $\mathcal{B}$;

(ii) $||X(t)|| \leq \mathcal{H}||X_t||_{\mathcal{B}}$;

(iii) $||X_t||_{\mathcal{B}} \leq K(t) \sup_{s \in [0, t]} ||X(s)|| + M(t)||X_0||_{\mathcal{B}}$;

where $\mathcal{H}$ is a possitive constant, $K, M : [0, \infty) \to [0, \infty)$, $K$ is continuous, $M$ is locally bounded, and they are independent of $X$.

(A2) For the function $X$ in (A1), $X_t$ is a $\mathcal{B}$-valued continuous function for $t$ in $[0, a]$.

(A3) The space $\mathcal{B}$ is complete.

**Example 2.5.** We recall some useful phase space $\mathcal{B}$.

(i) Let $BC$ be the space of bounded continuous functions from $(-\infty, 0]$ to $\mathcal{E}$, we define

$$C^0 := \{ \varphi \in BC : \lim_{\theta \to -\infty} \varphi(\theta) = 0 \}$$

and

$$C^\infty := \{ \varphi \in BC : \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } \mathcal{E} \}$$
endowed with the norm
\[ ||\varphi||_B = \sup_{\theta \in (-\infty,0]} ||\varphi(\theta)|| \]

then \( C^0, C^\infty \) satisfies \((A_1) - (A_3)\). However, \( BC \) satisfies \((A_1), (A_3) \) but \((A_2) \) is not satisfied.

(ii) For any real constant \( \gamma \), we define the functional spaces \( C_\gamma \) by
\[ C_\gamma = \left\{ \varphi \in C((-\infty,0], X) : \lim_{\theta \to -\infty} e^{\gamma \theta} \varphi(\theta) \text{ exists in } E \right\} \]
endowed with the norm
\[ ||\varphi|| = \sup_{\theta \in (-\infty,0]} e^{\gamma \theta} ||\varphi(\theta)||. \]

Then conditions \((A_1) - (A_3)\) are satisfied in \( C_\gamma \).

We prefer the reader to [3] for more comprehensive properties of phase space.

3 Main Results

Definition 3.1. [1] For \( \tau > 0 \), a stochastic process \( X \) is said to be a strong solution of \((1)\) on \((-\infty, \tau]\) if the following conditions holds

a) \( X(t) \) is \( \mathcal{F}_t \) - adapted for all \( 0 \leq t \leq \tau \);

b) \( X(t) \) is almost surely continuous in \( t \);

c) for all \( 0 \leq t \leq \tau \), \( X(t) \in D(A) \), \( \int_0^t ||AX(s)|| ds < +\infty \) almost surely, and

\[ X(t) = X(0) + \int_0^t AX(s)ds + \int_0^t f(s,X_s)ds + \int_0^t g(s,X_s)dW(s) \quad (4) \]

with probability one;

d) \( X(t) = \varphi(t) \) with \(-\infty < t \leq 0\) almost surely.

Definition 3.2. [1] For \( \tau > 0 \), a stochastic process \( X \) is said to be a mild solution of \((1)\) on \((-\infty, \tau]\) if the following conditions holds

a) \( X(t) \) is \( \mathcal{F}_t \) - adapted for all \( 0 \leq t \leq \tau \);
b) \( X(t) \) is almost surely continuous in \( t \);

c) for all \( 0 \leq t \leq \tau \), \( X(t) \) is measurable, \( \int_0^t ||X(s)||^2ds < +\infty \) almost surely, and

\[
X(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s,X_s)ds + \int_0^t T(t-s)g(s,X_s)dW(s) \tag{5}
\]

with probability one;

d) \( X(t) = \varphi(t) \) with \( -\infty < t \leq 0 \) almost surely.

**Remark 3.3.** In [4], we proved that if \( A \) generates a strongly semi-group \( (T(t))_{t \geq 0} \) in \( H \) and \( \varphi(0) \in \mathbb{D}(A) \) then (5) can be written as follow

\[
X(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s,X_s)ds + \int_0^t T(t-s)g(s,X_s)dW(s)
\]

This means a strong solution to be a mild one.

We assume that

\((M_1)\) \( A \) generates a strongly semigroup \( (T(t))_{t \geq 0} \) in \( H \).

\((M_2)\) \( f(t,x) \) and \( g(t,x) \) satisfy local Lipchitz conditions respects to second argument i.e. for any \( \alpha > 0 \) be a given real number, there exists \( C_1(\alpha), C_2(\alpha) > 0 \) such that

\[
||f(t,x) - f(t,y)|| \leq C_1(\alpha)||x - y||_B,
\]

\[
||g(t,x) - g(t,y)||_{L^2} \leq C_2(\alpha)||x - y||_B
\]

for all \( t \geq 0 \), \( x, y \in B \) which satisfy \( ||x||_B, ||y||_B \leq \alpha \).

Since Remark 3.3 we have our main result on the local existence of solution for (1).

**Theorem 3.4.** If \((M_1)\) and \((M_2)\) are satisfied then (1) has only local mild solution.

**Proof.** Let \( T > 0 \) be a fixed given real number. Since \( f, g \) satisfy Local Lipchitz condition then for each \( \alpha > 0 \) there exists \( \varphi \in B \) \( (||\varphi||_B \leq \alpha) \), such that

\[
||f(t,\varphi)|| \leq C_1(\alpha)||\varphi||_B + ||f(t,0)|| \leq \alpha C_1(\alpha) + \sup_{s \in [0,T]} ||f(s,0)|| \leq C,
\]

\[
||g(t,\varphi)|| \leq C_2(\alpha)||\varphi||_B + ||g(t,0)|| \leq \alpha C_2(\alpha) + \sup_{s \in [0,T]} ||g(s,0)|| \leq C.
\]
where
\[
C = \max \left\{ \alpha C_1(\alpha) + \sup_{s \in [0,T]} ||f(s,0)||, \alpha C_2(\alpha) + \sup_{s \in [0,T]} ||g(s,0)|| \right\}
\]

For \( \varphi \in \mathcal{B} \), we chose \( \alpha = ||\varphi||_B + 1 \). Let \( C_{ad} \) be a spaces of all functions \( X \) which adapted with \( \{\mathcal{F}_t\}_{t \geq 0} \) such that \( X_0 \in \mathcal{B} \) and \( X : [0,T] \rightarrow H \) is continuous. \( C_{ad} \) is a Banach space with norm
\[
||X||_{ad} = ||X_0||_B + \max_{0 \leq t \leq T} \left( E||X(t)||^2 \right)^{1/2}
\]

Let \( Z \) be a closed subset of \( C_{ad} \) which is defined by
\[
Z = \{ X \in C_{ad} : X(s) = \varphi(s) \text{ for } s \in (-\infty,0] \text{ and } \sup_{0 \leq s \leq T} ||X(s) - \varphi(0)||_H \leq 1 \}
\]

Let \( U : Z \rightarrow Z \) be the operator defined by
\[
U(X)(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s)f(s,X_s)ds + \int_0^t T(t-s)g(s,X_s)dW(s) & \text{for } t \in [0,T] \\ \varphi(t) & \text{for } t \leq 0 \end{cases}
\]

then \( U(Z) \subseteq Z \). Indeed,
\[
||U(X)(t) - \varphi(0)||_H^2 = E||U(X)(t) - \varphi(0)||^2
\]
\[
= E \left( \left\| T(t)\varphi(0) - \varphi(0) + \int_0^t T(t-s)f(s,X_s)ds + \int_0^t T(t-s)g(s,X_s)dW(s) \right\| \right)^2
\]
\[
\leq 3E||T(t)\varphi(0) - \varphi(0)||^2 + 3E \left\| \int_0^t T(t-s)f(s,X_s)ds \right\|^2
\]
\[
+ 3E \left\| \int_0^t T(t-s)g(s,X_s)dW(s) \right\|^2
\]
\[
\leq 3E||T(t)\varphi(0) - \varphi(0)||^2 + 3MT \int_0^t E||f(s,X_s)||^2 ds + 3M \int_0^t E||g(s,X_s)||_{L^2}^2 ds.
\]

Since \( ||X(s) - \varphi(0)|| \leq 1 \) for \( s \in [0,T] \) and \( \alpha = ||\varphi||_B + 1 \) we have \( ||X(s)|| \leq \alpha \), implies \( ||X_s||_B \leq \alpha \) for \( s \in [0,T] \). Furthermore,
\[
||f(s,X_s)|| \leq C \quad \text{and} \quad ||g(t,X_s)|| \leq C.
\]
Hence
\[ ||U(X)(t) - \varphi(0)||^2_{H} \leq 3E||T(t)\varphi(0) - \varphi(0)||^2 + 3MC^2(T^2 + T) \]
where \( M = \sup_{0 \leq t \leq T} ||T(t)||^2 \). If \( T \) is small enough, such that
\[ \sup_{0 \leq s \leq T} \{3E||T(s)\varphi(0) - \varphi(0)||^2 + 3MC^2(T^2 + T)\} \leq 1. \]
then for any \( t \in [0, T] \) we have \( ||U(X)(t) - \varphi(0)|| \leq 1 \). In other words, \( U(Z) \subseteq Z \).

Now, for any \( X, Y \in Z \),
\[ E||U(X)(t) - U(Y)(t)||^2 \]
\[ = E||\int_{0}^{t} T(t-s)[f(s,X_s) - f(s,Y_s)]ds + \int_{0}^{t} T(t-s)[g(s,X_s) - g(s,Y_s)]dW(s)||^2 \]
\[ \leq 2E \left( \int_{0}^{t} ||T(t-s)[f(s,X_s) - f(s,Y_s)]||ds \right)^2 \]
\[ + 2E \left( \int_{0}^{t} ||T(t-s)[g(s,X_s) - g(s,Y_s)]||dW(s) \right)^2 \]
\[ \leq 2ME \left( \int_{0}^{t} ||f(s,X_s) - f(s,Y_s)||ds \right)^2 + 2ME \left( \int_{0}^{t} ||g(s,X_s) - g(s,Y_s)||dW(s) \right)^2 \]
\[ \leq 2MC^2T \int_{0}^{t} E||X(s) - Y(s)||^2ds + 2MC^2 \int_{0}^{t} E||X(s) - Y(s)||^2ds \]
\[ \leq 2MC^2(T + 1) \int_{0}^{t} E||X(s) - Y(s)||^2ds. \]

Now, for any \( a > 0 \), and \( t \in [0, T] \) we have
\[ e^{-at}E||U(X)(t) - U(Y)(t)||^2 \]
\[ \leq 2MC^2(T + 1) \int_{0}^{t} e^{-a(t-s)}e^{-as}E||X(s) - Y(s)||^2ds \]
\[ \leq 2MC^2(T + 1) \max_{0 \leq s \leq t} e^{-as}E||X(s) - Y(s)||^2 \int_{0}^{t} e^{-a(t-s)}ds \]
\[ \leq 2a^{-1}MC^2(T + 1) \max_{0 \leq s \leq t} e^{-as}E||X(s) - Y(s)||^2. \]
Therefore,

$$\max_{0 \leq t \leq T} e^{-at} E\|U(X)(t) - U(Y)(t)\|^2$$

$$\leq 2a^{-1}MC^2(T + 1) \max_{0 \leq s \leq T} e^{-as} E\|X(s) - Y(s)\|^2.$$  

Finally, if $a > 2MC^2(T + 1)$ then $U$ be a contraction mapping on $Z$ respects to the norm

$$|||X||| = ||X_0||_B + \max_{0 \leq t \leq T} (e^{-at} E\|X(t)\|^2)^{1/2}, \quad X \in C_{ad}.$$  

Since the norm $||.||$ is equivalent to the norm $||.||_{ad}$ then by applying fixed point theorem we conclude that (1) has only local mild solution. 

\section{Conclusion}

Our main results is the Theorem 3.4, in which we present and prove the local existence of solution to a class of stochastic functional differential equations with infinite delay in a separable Hilbert space has the form (1). In this Theorem, we can replace Local Lipchitz condition ($M_2$) by some other conditions, for example

$(M_3)$ For any $\alpha > 0$ be a given real number, there exists a constant $C(\alpha) > 0$ such that

$$||f(t, x) - f(t, y)|| + ||g(t, x) - g(t, y)||_{L_0^2} \leq C(\alpha)||x - y||_B$$

or

$(M_3')$ For any $\alpha > 0$ be a given real number, there exists a constant $C(\alpha) > 0$ such that

$$\max\{||f(t, x) - f(t, y)||, ||g(t, x) - g(t, y)||_{L_0^2}\} \leq C(\alpha)||x - y||_B$$

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References

