On $\alpha$-$\eta$-$\varphi$-Contraction in Fuzzy Metric Space and its Application

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Abstract

The aim of this paper is to introduce new concepts of $\alpha$-$\eta$-complete fuzzy metric space and $\alpha$-$\eta$-continuous function and establish fixed point results for $\alpha$-$\eta$-$\varphi$-contraction function in $\alpha$-$\eta$-complete fuzzy metric space. As an application, we derive some Suzuki type fixed point theorems, fixed point in orbitally f-complete. Moreover, we introduce concept $\alpha$-$\psi$-$\phi$-contraction function and application on $\alpha$-$\eta$-$\varphi$-contraction.

Keywords: $\alpha$-$\eta$-complete, $\alpha$-$\eta$-continuous, $\alpha$-$\eta$-$\varphi$-contraction, orbitally f-complete, Suzuki Type Fixed Point Result, $\alpha$-$\psi$-$\phi$-Contraction Function, Application on $\alpha$-$\eta$-$\varphi$-Contraction.

1 Introduction

The study of fixed points of functions in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. In 1965, the concept of fuzzy sets was introduced by Zadeh [10]. With the concept of fuzzy sets, the fuzzy metric space was introduced by I. Kramosil and J. Michalek

This paper we introduce new concepts of $\alpha$-$\eta$-complete fuzzy metric space and $\alpha$-$\eta$-continuous function and establish fixed point results for $\alpha$-$\eta$-$\phi$-contraction function in $\alpha$-$\eta$-complete fuzzy metric space. As an application, we derive some Suzuki type fixed point theorems, fixed point in orbitally $f$-complete. Moreover, we introduce concept $\alpha$-$\psi$-$\phi$-contraction function and application on $\alpha$-$\eta$-$\phi$-contraction.

2 Preliminaries

**Definition 2.1 [4]:** Let $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be two function, $f$ is said to be $\alpha$-admissible function if

$$\alpha(x,y) \geq 1 \implies \alpha(f(x),f(y)) \geq 1 \text{ for } x,y \in X.$$  

**Definition 2.2 [6]:** Let $f : X \rightarrow X$ and $\alpha, \beta : X \times X \rightarrow [0, \infty)$ be two function, $f$ is said to be $(\alpha, \beta)$-admissible function if

$$\begin{align*}
\alpha(x,y) \geq 1 & \implies \alpha(f(x),f(y)) \geq 1 \\
\beta(x,y) \geq 1 & \implies \beta(f(x),f(y)) \geq 1
\end{align*} \text{ for all } x,y \in X.$$  

**Definition 2.3 [4]:** Let $f : X \rightarrow X$ and $\alpha, \eta : X \times X \rightarrow [0, \infty)$ be two function, $f$ is said to be $\alpha$-admissible function with respect $\eta$ if

$$\alpha(x,y) \geq \eta(x,y) \implies \alpha(f(x),f(y)) \geq \eta(f(x),f(y)) \text{ for all } x,y \in X.$$  

Note that if we take $\eta(x,y) = 1$, then this definition reduces to definition (2.1).

**Definition 2.4:** Let $(X, M, *)$ be a fuzzy metric space and

$$\alpha, \eta : X \times X \rightarrow [0, \infty), \text{ } X \text{ is said to be } \alpha$-$\eta$-complete iff every Cauchy sequence \{x_n\} with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converge in $X$.

Note: $X$ is said to be $\alpha$-complete if $\eta(x,y) = 1$ for all $x,y \in X$.

**Example 2.5:** Let $X = [0,1]$ and $M(x, y, t) = \frac{t}{t + |x - y|}$, if $t > 0$ with $a * b = \min \{a, b\}$ for every $a, b \in [0,1]$. Define $\alpha, \eta : X \times X \rightarrow [0, \infty)$

$\alpha(x, y) = \begin{cases} 
(x + y)^2 & \text{if } x, y \in X \\
0 & \text{if } o.w
\end{cases}$. 

By $\alpha(x, y)$
\[ \eta(x,y) = 2xy \]

Then \((X, M, \ast)\) is \(\alpha\)-\(\eta\)-complete fuzzy metric space.

**Definition 2.6:** Let \((X, M, \ast)\) be a fuzzy metric space and let \(\alpha, \eta: X \times X \to [0, \infty)\) and \(f: X \to X\), \(f\) is said to be \(\alpha\)-\(\eta\)-continuous function on \(X\) if for \(x \in X\) and \(\{x_n\}\) be a sequence in \(X\) with \(x_n \to x\) as \(n \to \infty\), \(\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\) for all \(n \in \mathbb{N}\) implies \(f(x_n) \to f(x)\).

**Definition 2.7** [9]:

1. (1) Let \(f\) be a function of a fuzzy metric space \((X, M, \ast)\) into itself. \((X, M, \ast)\) is said to be \(\ast\)-orbitally complete if and only if every Cauchy sequence which is contained in \(\{x, f(x), f^2(x), f^3(x), \ldots\}\) for some \(x \in X\) converges in \(X\).

A \(\ast\)-orbitally complete fuzzy metric space may not be complete.

2. (2) A function \(f: X \to X\) is called orbitally continuous at \(x \in X\) if \(\lim_{n \to \infty} f^n(x) = x\) implies \(\lim_{n \to \infty} f^n(x) = f(x)\)

The function \(f\) is orbitally continuous on \(X\) if \(f\) is orbitally continuous for all \(x \in X\).

**Remark 2.8:**

1. (1) Let \((X, M, \ast)\) be a fuzzy metric space and \(f: X \to X\) be a function and let \(X\) be an orbitally \(f\)-complete. Define \(\alpha, \eta: X \times X \to [0, \infty)\) by

\[
\alpha(x,y) = \begin{cases} 3 & \text{if } x,y \in O(w) \\ 1 & \text{otherwise} \end{cases}, \quad \eta(x,y) = 1
\]

Where \(O(w)\) is an orbit of a point \(w \in X\). \((X, M, \ast)\) is an \(\alpha\)-\(\eta\)-complete.

If \(\{x_n\}\) be a Cauchy sequence with \(\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\) for all \(n \in \mathbb{N}\) then \(\{x_n\} \subseteq O(w)\)

Now, since \(X\) is an orbitally \(f\)-complete fuzzy metric space, then \(\{x_n\}\) converge in \(X\). We can say that \(X\) is \(\alpha\)-\(\eta\)-complete.

2. (2) Let \((X, M, \ast)\) and \(\alpha, \eta: X \times X \to [0, \infty)\) is in (1), let \(f: X \to X\) be an orbitally continuous function on \((X, M, \ast)\). Then \(f\) is \(\alpha\)-\(\eta\)-continuous function. Indeed if \(x_n \to x\) as \(n \to \infty\) and \(\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\) for all \(n \in \mathbb{N}\), so \(x_n \in O(w)\) for all \(n \in \mathbb{N}\), then there exist sequence \((k_i)_{i \in \mathbb{N}}\) of positive integer such that \(x_n \to f^{k_i w} \to x\) as \(i \to \infty\). Now since \(f\) is an orbitally continuous on \((X, M, \ast)\), then \(f(x_n) = f(f^{k_i w}) \to f(x)\) as \(i \to \infty\).
Denote with \( \phi \) the set of all the function \( \varphi: [0,1] \rightarrow [0,1] \) with the following properties:

(1) \( \varphi \) is nondecreasing and continuous
(2) \( \varphi(\tau) = 0 \) if \( \tau = 1 \)

**Definition 2.9:** Let \((X, M, \ast)\) be a fuzzy metric space and \( d: X \rightarrow X \), let \( \mathbb{M}(x,y) = \min \{ M(x,y, t), M(x, f(x), t), M(y, f(y), t), M(x, f(y), t) \ast M(y, f(x), t) \} \) 

We say

(1) \( f \) is an \( \alpha \)-\( \eta \)-\( \varphi \)-contraction function if 
\[
\alpha(x, y) \geq \eta(x, y) \Rightarrow \mathbb{M}(f(x), f(y), t) \geq \varphi(\mathbb{M}(x, y)).
\]

(2) \( f \) is an \( \alpha \)-\( \varphi \)-contraction function if \( \eta(x, y) = 1 \) for all \( x, y \in X \).

**Theorem 2.10:** Let \((X, M, \ast)\) be a fuzzy metric space and let \( f: X \rightarrow X \), suppose that \( \alpha, \eta: X \times X \rightarrow [0, \infty) \) are two function. Assume that the following assertions hold

(i) \((X, M, \ast)\) is an \( \alpha \)-\( \eta \)-complete fuzzy metric space.
(ii) \( f \) is an \( \alpha \)-admissible function with respect to \( \eta \).
(iii) \( f \) is an \( \alpha \)-\( \eta \)-\( \varphi \)-contraction function on \( X \).
(iv) \( f \) is an \( \alpha \)-\( \eta \)-continuous function.
(v) There exist \( x_0 \in X \) such that \( \alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0)) \).

Then \( f \) has a fixed point in \( X \).

**Proof:** Let \( x_0 \in X \) such that \( \alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0)) \)

Define a sequence \( \{x_n\} \) such that \( x_n = f(x_{n-1}) \) for all \( n \in \mathbb{N} \)

If \( x_n = x_{n+1} \) for some \( n \), then \( x = x_n \) is a fixed point of \( f \).

Suppose \( x_n \neq x_{n+1} \)

Since \( f \) is \( \alpha \)-admissible function with respect \( \eta \) and 

\( \alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0)) \)

Then \( \alpha(x_1, x_2) = \alpha(f(x_0), f(x_1)) \geq \eta(f(x_0), f(x_1)) = \eta(x_1, x_2) \)

By continuing this process, we get 

\( \alpha(x_n, f(x_n)) = \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) = \eta(x_n, f(x_n)) \)
By (1) in definition (2.9)

$$M(x_n, x_{n+1}, t) = M(f(x_{n-1}), f(x_n), t) \geq \alpha(M(x_{n-1}, x_n))$$

Where

$$M(x_{n-1}, x_n) = \min \{M(x_{n-1}, x_n), M(x_{n-1}, f(x_n), t), M(x_n, f(x_{n-1}), t), M(x_{n-1}, f(x_n), t) \cdot M(x_n, f(x_{n-1}), t)\}$$

$$\geq \min \{M(x_{n-1}, x_n), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t) \cdot M(x_n, x_n, t)\}$$

$$= \min \{M(x_{n-1}, x_n), M(x_n, x_{n+1}, t)\}$$

Since \(\phi\) is nondecreasing and continuous, we have

$$M(x_n, x_{n+1}, t) \geq \phi(\min\{M(x_{n-1}, x_n), M(x_n, x_{n+1})\})$$

If \(\min\{M(x_{n-1}, x_n), M(x_n, x_{n+1})\} = M(x_n, x_{n+1}, t)\)

Then

$$M(x_n, x_{n+1}, t) \geq \phi(\min\{M(x_{n-1}, x_n), M(x_n, x_{n+1})\}) \geq \phi(M(x_n, x_{n+1}, t)) > M(x_n, x_{n+1}, t)$$

Which is contradiction.

Therefore \(\min\{M(x_{n-1}, x_n), M(x_n, x_{n+1})\} = M(x_{n-1}, x_n, t)\)

Hence for all \(n \in \mathbb{N}\) we have

$$M(x_n, x_{n+1}, t) \geq \phi^2(M(x_{n-1}, x_{n-1}, t)) \geq \cdots \geq \phi^n(M(x_0, x_1, t))$$

Let \(n, m \in \mathbb{N}\) with \(n > m\), then

$$M(x_n, x_m, t) \geq \phi(M(x_{n-1}, x_{n-1}, t)) \geq \cdots \geq \phi^n(M(x_0, x_1, t))$$

Therefore \(\lim_{n \to \infty} M(x_n, x_m, t) = 1\)

Hence \(\{x_n\}\) is a Cauchy sequence

Since \(X\) is an \(\alpha\)-\(\eta\)-complete fuzzy metric space there is \(x \in X\) such that \(x_n \to x\) as \(n \to \infty\).

Since \(f\) is an \(\alpha\)-\(\eta\)-continuous function so \(x_{n+1} = f(x_n) \to f(x)\) as \(n \to \infty\).
Hence \( f(x) = x \)

Suppose \( y \) is a fixed point of \( f \) such that \( f(y) = y \)

\[ M(x, y, t) = M(f(x), f(y), t) \geq \varphi(M(x, y)) \]

Thus \( x = y \)

**Example 2.11:** Let \( X = [0,3] \) be equipped with the ordinary metric

\[ d(x, y) = |x - y|, \quad \varphi(\tau) = \sqrt{\tau} \text{ for all } \tau \in [0,1]. \]

Define \( M(x, y, t) = e^{-\frac{2|x-y|}{t}} \) for all \( x, y \in X \) and \( t > 0 \), and \( \alpha, \eta: X \times X \to [0, \infty) \)

By \( \alpha(x, y) = \begin{cases} (x + y)^2 & \text{if } x, y \in X \\ 0 & \text{if } \text{otherwise} \end{cases} \)

\( \eta(x, y) = 2xy, \)

Clearly, \((X, M, \ast)\) is an \( \alpha \)-\( \eta \)-complete fuzzy metric space with respect to \( t \)-norm \( a \ast b = ab \). (by [8])

Let \( f: X \to X \) be defined as

\[ f(x) = \begin{cases} 1 & x \in [0,1] \\ \frac{3-x}{2} & x \in (1,3] \end{cases} \]

\[ d(f(x), f(y)) = |f(x) - f(y)| = \frac{1}{2}|x - y| \leq |x - y| = d(x, y) \]

It follows that \( M(f(x), f(y), t) = e^{-\frac{2|f(x)-f(y)|}{t}} \geq e^{-\frac{|x-y|}{t}} = \varphi(M(x, y)) \)

Thus \( f \) is \( \alpha \)-\( \eta \)-\( \varphi \)-contraction function in fuzzy metric space \((X, M, \ast)\).

**Corollary 2.12:** Let \((X, M, \ast)\) be a fuzzy metric space and let \( f: X \to X \), suppose that \( \alpha, \eta: X \times X \to [0, \infty) \) are two function. Assume that the following assertions hold:

(i) \((X, M, \ast)\) is an \( \alpha \)-complete fuzzy metric space.
(ii) \( f \) is an \( \alpha \)-admissible function.
(iii) \( f \) is \( \alpha \)-\( \varphi \)-contraction function on \( X \).
(iv) \( f \) is an \( \alpha \)-continuous function.
(v) There exist \( x_0 \in X \) such that \( \alpha(x_0, f(x_0)) \geq 1 \).

Then \( f \) has a fixed point in \( X \).
Theorem 2.13: Let \((X, M, \ast)\) be a fuzzy metric space and let \(f : X \rightarrow X\), suppose that \(\alpha, \eta : X \times X \rightarrow [0, \infty)\) are two function. Assume that the following assertions hold:

(i) \((X, M, \ast)\) is an \(\alpha\)-complete fuzzy metric space
(ii) \(f\) is an \(\alpha\)-admissible function with respect to \(\eta\).
(iii) \(f\) is a \(\alpha\)-\(\eta\)-\(\varphi\)-contraction function on \(X\).
(iv) There exist \(x_0 \in X\) such that \(\alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0))\).
(v) If \(\{x_n\}\) is a sequence in \(X\) \(\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\)

With \(x_n \rightarrow x\) as \(n \rightarrow \infty\), then

Either \(\alpha(f(x_n), x) \geq \eta(f(x_n), f^2(x_n))\)

Or \(\alpha(f^2(x_n), x) \geq \eta(f^2(x_n), f^3(x_n))\) for all \(n \in \mathbb{N}\)

Then \(f\) has a fixed point in \(X\).

Proof: Let \(x_0 \in X\) such that \(\alpha(x_0, f(x_0)) \geq \eta(x_0, f(x_0))\)

Define a sequence \(\{x_n\}\) in \(X\) by \(x_n = f^n(x_0) = f(x_{n-1})\) for all \(n \in \mathbb{N}\).

Now is in the proof of theorem (2.10) we have \(\alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n)\)

There exist \(x \in X\) such that \(x_n \rightarrow x\) as \(n \rightarrow \infty\)

Let \(M(x, f(x), t) \neq 1\), from (v)

Either \(\alpha(f(x_{n-1}), x) \geq \eta(f(x_{n-1}), f^2(x_{n-1}))\)

Or \(\alpha(f^2(x_{n-1}), x) \geq \eta(f^2(x_{n-1}), f^3(x_{n-1}))\)

Then either \(\alpha(x_n, x) \geq \eta(x_n, x_{n+1})\)

Or \(\alpha(x_{n+1}, x) \geq \eta(x_{n+1}, x_{n+2})\)

Let \(\alpha(x_n, x) \geq \eta(x_n, x_{n+1})\) from definition (2.9) condition (1), we get

\[M(x_{n+1}, f(x), t) = M(f(x_n), f(x), t) \geq \varphi(M(x_n, x))\]

Where

\[M(x_n, x) = \min \{M(x_n, x, t), M(x_n, f(x_n), t), M(x, f(x), t), \]

\[M(x_n, f(x), t) * M(x, f(x_n), t)\}\]

\[= \varphi(\min \{M(x_n, x, t), M(x_n, x_{n+1} t), M(x, f(x), t), \]

\[M(x_n, f(x), t) * M(x, x_{n+1}, t)\})\]
By taking \( n \to \infty \) in the above we get
\[
M(x, f(x), t) = \varphi(\min \{1, M(x, f(x), t)\}) > M(x, f(x), t)
\]
Which is contradiction.

Hence \( M(x, f(x), t) = 1 \implies f(x) = x \).

**Corollary 2.14:** Let \((X, M, *)\) be a fuzzy metric space and let \( f: X \to X \), suppose that \( \alpha, \eta: X \times X \to [0, \infty) \) are two function. Assume that the following assertions hold:

(i) \((X, M, *)\) is an \( \alpha \)-complete fuzzy metric space.
(ii) \( f \) is an \( \alpha \)-admissible function.
(iii) \( f \) is \( \alpha \)-\( \varphi \)-contraction function on \( X \).
(iv) There exist \( x_0 \in X \) such that \( \alpha(x_0, f(x_0)) \geq 1 \).
(v) If \( \{x_n\} \) is a sequence in \( X \) \( \alpha(x_n, x_{n+1}) \geq 1 \)

With \( x_n \to x \) as \( n \to \infty \), then

Either \( \alpha(f(x_n), x) \geq 1 \)

Or \( \alpha(f^2(x_n), x) \geq 1 \) for all \( n \in \mathbb{N} \)

Then \( f \) has a fixed point in \( X \).

**Corollary 2.15:** Let \((X, M, *)\) be a complete fuzzy metric space and let \( f: X \to X \) be a continuous function such that \( f \) is contraction function that is
\[
M(f(x), f(y), t) \geq \varphi(\mathbb{M}(x, y)) \text{ for all } x, y \in X
\]
Then \( f \) has a fixed point in \( X \).

### 2.2 Fixed Point in Orbitally \( f \)-Complete Fuzzy Metric Space

**Theorem 2.2.16:** Let \((X, M, *)\) be a fuzzy metric space and \( f: X \to X \) such that the following assertion hold:

(i) \((X, M, *)\) is an orbitally \( f \)-complete fuzzy metric space.
(ii) there exist \( \varphi \) be a function such that \( M(f(x), f(y), t) \geq \varphi(\mathbb{M}(x, y)) \) for all \( x, y \in O(w) \) for some \( w \in X \).
(iii) If \( \{x_n\} \) be a sequence .such that \( \{x_n\} \subseteq O(w) \) with \( x_n \to x \) as \( n \to \infty \), \( x \in O(w) \).

Then \( f \) has fixed point in \( X \).
Proof: Define $\alpha: X \times X \to [0, \infty)$ from remark (2.8), $(X, M, \ast)$ is an $\alpha$-complete fuzzy metric space and $f$ is an $\alpha$-admissible function.

Let $\alpha(x, y) \geq 1$ for all $x, y \in O(w)$

Then from (ii)

$$M(f(x), f(y), t) \geq \varphi(M(x, y))$$

That is $f$ is an $\alpha$-$\varphi$-contraction function

Let $\{x_n\}$ sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x$

So $\{x_n\} \subseteq O(w)$ from (iii) we have $x \in O(w)$

That is $\alpha(x_n, x) \geq 1$ by corollary (2.13)

Then $f$ has unique fixed point in $X$.

Corollary 2.2.17: Let $(X, M, \ast)$ be a fuzzy metric space and $f: X \to X$ such that the following assertion hold:

(i) $(X, M, \ast)$ is an orbitally $f$-complete fuzzy metric space.  
(ii) there exist $k \in (0,1)$ such that $M(f(x), f(y), t) \geq kM(x, y)$ for all $x, y \in O(w)$ for some $w \in X$.  
(iii) If $\{x_n\}$ be a sequence such that $\{x_n\} \subseteq O(w)$ with $x_n \to x$ as $n \to \infty$. Then $x \in O(w)$. Then $f$ has fixed point in $X$.

2.3 Suzuki Type Fixed Point Result

From Theorem (2.10) we deduce the following Suzuki type fixed point result

**Theorem 2.3.18:** Let $(X, M, \ast)$ be a complete fuzzy metric space and let $f: X \to X$ continuous function. Assume that there exist $k \in (0,1)$ such that

$$M(x, f(x), t) \geq M(x, y, t) \Rightarrow M(f(x), f(y), t) \geq kM(x, y) \quad (1)$$

For all $x, y \in X$, where

$$M(x, y) = \min \{M(x, y, t), M(x, f(x), t), M(y, f(y), t), M(x, f(y), t) + M(y, f(x), t)\}$$

Then $f$ has a fixed point in $X$.

Proof: Define $\alpha, \eta: X \times X \to [0, \infty)$ and $\varphi: [0,1] \to [0,1]$ by
\( \alpha(x, y) = M(x, f(x), t), \quad \eta(x, y) = M(x, y, t) \)

Such that \( \alpha(x, y) \geq \eta(x, y) \) for all \( x, y \in X \)

And let \( \varphi(\tau) = k\tau, \tau \in [0,1] \)

By condition (i) –(v) of theorem (2.10)

Let \( \alpha(x, f(x)) \geq \eta(x, y) \)

Then \( M(x, f(x), t) \geq M(x, y, t) \)

From (1) we have \( M(f(x), f(y), t) \geq kM(x, y) = \varphi(M(x, y)) \)

Then \( f \) is \( \alpha\cdot\eta\cdot\varphi \)-contraction function by condition of theorem (2.10) and \( f \) has fixed point, then the unique fixed point follows from (1).

**Corollary 2.3.19:** Let \((X, M, *)\) be a complete fuzzy metric space and let \( f : X \to X \) a continuous function. Assume that there exist \( k \in (0, 1) \) such that

\[
M(x, f(x), t) \geq M(x, y, t) \Rightarrow M(f(x), f(y), t) \geq kM(x, y, t)
\]

For all \( x, y \in X \),

Then \( f \) has a fixed point in \( X \).

### 2.4 An \( \alpha \cdot \psi \cdot \phi \)-Contraction Function

**Definition 2.4.20:** Let \((X, M, *)\) be a complete fuzzy metric space. Let \( f : X \to X \) be an \( \alpha \)-admissible which satisfies the following

\[
\psi(M(f(x), f(y), t)) \geq \psi(M(x, y)) - \phi(M(x, y))
\]

Such that

(i) \( \psi \) is continuous and decreasing with \( \psi(\tau) = 0 \) iff \( \tau = 1 \).

(ii) \( \phi \) is continuous with \( \phi(\tau) = 0 \) iff \( \tau = 1 \)

\( f \) is called \( \alpha \cdot \psi \cdot \phi \)-contraction function.

**Theorem 2.4.21:** Let \((X, M, *)\) be a complete fuzzy metric space. Let \( f : X \to X \) be an \( \alpha \)-admissible which satisfies the following

\[
\psi(M(f(x), f(y), t)) \geq \psi(M(x, y)) - \phi(M(x, y))
\]
Such that

(i) \((X, M, *)\) is an \(\alpha\)-complete.

(ii) \(f\) is \(\alpha\)-continuous function

(iv) There exist \(x_0 \in X\) such that \(\alpha(x_0, f(x_0)) \geq 1\).

**Proof:** Let \(x_0 \in X\) such that \(\alpha(x_0, f(x_0)) \geq 1\)

Define a sequence \(\{x_n\}\) such that \(x_n = f(x_{n-1})\) for all \(n \in \mathbb{N}\)

If \(x_n = x_{n+1}\) for some \(n\), then \(x = x_n\) is a fixed point of \(f\).

Suppose \(x_n \neq x_{n+1}\)

Since \(f\) is \(\alpha\)-admissible function with respect \(\eta\) and

\[\alpha(x_0, f(x_0)) = \alpha(x_0, x_1) \geq 1\]

Then \(\alpha(x_1, x_2) = \alpha(f(x_0), f(x_1)) \geq \eta(f(x_0), f(x_1)) = \eta(x_1, x_2)\)

By continuing this process, we get

\[\alpha(x_n, f(x_n)) = \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}\]

\[\psi(M(x_n, x_{n+1}, t)) = \psi(M(f(x_{n-1}), f(x_n), t)) \geq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n))\]

Where

\[M(x_{n-1}, x_n) = \min \{M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_n), t), M(x_n, f(x_{n-1}), t)\} \times M(x_{n-1}, f(x_n), t) \times M(x_n, f(x_{n-1}), t)\]

\[\geq \min \{M(x_{n-1}, x_n), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_{n+1}, t) \times M(x_n, x_n, t)\}\]

Since \(\psi\) decreasing, then \(M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t)\) that is \(\{M(x_n, x_{n+1}, t)\}\) is an increasing sequence thus there exist \(l(t) \in (0,1]\) such that \(\lim_{n \to x} M(x_n, x_{n+1}, t) = l(t) < 1\)

Now taking \(n \to \infty\) we obtain for all \(t > 0\)

\[\psi(l(t)) \geq \psi(l(t)) - \phi(l(t))\] which is contradiction \(l(t) = 1\)
Thus we conclude for all \( t > 0 \)

\[
\lim_{n \to \infty} M(x_n, x_{n+1}, t) = 1
\]

\( \{x_n\} \) is Cauchy sequence. Since \( X \) is \( \alpha \)-complete, then \( x_n \to x \)

\[
\psi(M(x_{n+1}, f(x), t)) = \psi(M(f(x_n), f(x), t)) \\
\geq \psi(M(x_n, x)) - \phi(M(x_n, x))
\]

Letting \( n \to \infty \) in the above inequality using properties \( \psi, \phi \)

\[
\psi(\lim_{n \to \infty} M(x_n, f(x), t)) \geq \psi(1) - \phi(1) = 0
\]

Thus \( \lim_{n \to \infty} M(x_n, f(x), t) = 1 \)

Hence \( x_n \to f(x) \implies f(x) = x \).

Now we assume \( y \) is a fixed point of \( f \) such that \( f(y) = y \)

\[
\psi(M(x, y, t)) = \psi(M(f(x), f(y), t)) \geq \psi(M(x, y)) - \phi(M(x, y)) = 0 \\
\implies M(x, y, t) = 1 \implies x = y.
\]

### 2.5 Application On \( \alpha\eta\phi \)-Contraction

**Definition 2.5.22:** Let \( (X, M, *) \) be a fuzzy metric space and \( f: X \to X \) be an \((\alpha, \beta)\)-admissible function, \( f \) is said to be

(a) \((\alpha, \beta)\)-contraction function of type (I), if

\[
\alpha(x, y) \beta(x, y) M(f(x), f(y), t) \geq \phi(M(x, y)).
\]

(b) \((\alpha, \beta)\)-contraction function of type (II), if there exist \( 0 < \ell \leq 1 \) such that

\[
(\alpha(x, y) \beta(x, y) + \ell) M(f(x), f(y), t) \geq (1 + \ell) \phi(M(x, y))
\]

**Theorem 2.5.23:** Let \( (X, M, *) \) be a complete fuzzy metric space and let \( f: X \to X \) be an \( \alpha \)-continuous and \((\alpha, \beta)\)-contraction function of type (I), (II), if there exist \( \alpha(x_0, f(x_0)) \geq 1 \) and \( \beta(x_0, f(x_0)) \geq 1 \), then \( f \) has a unique fixed point in \( X \).

**Proof:** Let \( x_0 \in X \) such that \( \alpha(x_0, f(x_0)) \geq 1 \) and \( \beta(x_0, f(x_0)) \geq 1 \)

Define a sequence \( \{x_n\} \) such that \( x_n = f(x_{n-1}) \) for all \( n \in \mathbb{N} \)

Since \( f \) is \((\alpha, \beta)\)-admissible function and \( \alpha(x_0, f(x_0)) \geq 1 \)

Then \( \alpha(x_1, x_2) = \alpha(f(x_0), f(x_1)) \geq 1 \)

By continuing this process, we get
\[ \alpha(x_n, x_{n+1}) = \alpha(x_n, f(x_n)) \geq 1 \]

Similarly we have \[ \beta(x_n, x_{n+1}) = \beta(x_n, f(x_n)) \geq 1 \]

If \( x_n = x_{n+1} \) for some \( n \), then \( x = x_n \) is a fixed point of \( f \).

Suppose \( x_n \neq x_{n+1} \)

(a) \[ \alpha(x_n, x_{n+1}) \beta(x_n, x_{n+1}) M(x_n, x_{n+1}, t) \]
\[ = \alpha(x_n, x_{n+1}) \beta(x_n, x_{n+1}) M(f(x_{n-1}), f(x_n), t) \]
\[ \geq \varphi(\mathbb{M}(x_{n-1}, x_n)) \]

Where \[ \mathbb{M}(x_{n-1}, x_n) = \min \{ M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_n), t), \]
\[ M(x_{n-1}, f(x_n), t) \} \]
\[ = \min \{ M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t) \} \]
\[ \geq \min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t) \} \]
\[ = \min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \} \]

If \( \min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \} = M(x_n, x_{n+1}, t) \)

Then \[ M(x_n, x_{n+1}, t) \geq \varphi(\min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \}) \geq \varphi(\varphi(M(x_n, x_{n+1}, t))) \]
\[ > M(x_n, x_{n+1}, t) \]

Which is contradiction

Therefore \( \min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \} = M(x_{n-1}, x_n, t) \)

Hence for all \( n \in \mathbb{N} \) we have

\[ M(x_n, x_{n+1}, t) \geq \varphi(\varphi(M(x_n, x_{n+1}, t))) \geq \varphi^2(M(x_{n-1}, x_{n-1}, t)) \]
\[ \geq \cdots \geq \varphi^n(M(x_0, x_1, t)) \]

Let \( n, m \in \mathbb{N} \) with \( n > m \), then

\[ M(x_n, x_m, t) = \alpha(x_n, x_{n+1}) \beta(x_n, x_{n+1}) M(f(x_{n-1}), f(x_n), t) \geq \varphi(\mathbb{M}(x_{n-1}, x_n)) \]
\[ \geq \cdots \geq \varphi^n(M(x_0, x_1, t)) \]

Therefore \( \lim_{n \to \infty} M(x_n, x_m, t) = 1 \)
Hence \{x_n\} is a Cauchy sequence.

Since \( X \) is an \( \alpha, \eta \)-complete fuzzy metric space there is \( x \in X \) such that \( x_n \to x \) as \( n \to \infty \).

\[
M(x_n, f(x), t) = \alpha(x_n, x) \beta(x_n, x) M(f(x_n), f(x), t) \geq \varphi(\mathcal{M}(x_n, x))
\]

Hence \( f(x) = x \).

Suppose \( y \) is a fixed point of \( f \) such that \( f(y) = y \).

\[
M(x, y, t) = \alpha(x, y) \beta(x, y) M(f(x), f(y), t) \geq \varphi(\mathcal{M}(x, y)).
\]

Hence \( x = y \).

(b)

\[
(\alpha(x_n, x_{n+1}) \beta(x_n, x_{n+1}) + \ell)M(x_{n+1}, x) = (\alpha(x_n, x_{n+1}) \beta(x_n, x_{n+1}) + \ell)M(f(x_n), f(x_{n+1}), t) \\
\geq (1 + \ell)\varphi(\mathcal{M}(x_n, x_{n+1}))
\]

Where

\[
\mathcal{M}(x_{n-1}, x_n) = \min \{ M(x_{n-1}, x_n, t), M(x_{n-1}, f(x_{n-1}), t), M(x_n, f(x_n), t), \\
M(x_{n-1}, f(x_{n}), t) * M(x_n, f(x_{n-1}), t) \}
\]

\[
= \min \{ M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), \\
M(x_{n-1}, x_{n+1}, t) * M(x_n, x_n, t) \}
\]

\[
\geq \min \{ M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) * M(x_n, x_{n+1}, t) \}
\]

\[
= \min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \}
\]

If \( \min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \} = M(x_n, x_{n+1}, t) \)

Then

\[
M(x_n, x_{n+1}, t) \geq \varphi(\min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \}) \geq \varphi(\mathcal{M}(x_n, x_{n+1}, t)) \\
> M(x_n, x_{n+1}, t)
\]

Which is contradiction.

Therefore \( \min \{ M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t) \} = M(x_{n-1}, x_n, t) \)

Hence for all \( n \in \mathbb{N} \) we have

\[
M(x_n, x_{n+1}, t) \geq \varphi(M(x_{n-1}, x_n, t)) \geq \varphi^2(M(x_{n-2}, x_{n-1}, t)) \\
\geq \cdots \geq \varphi^n(M(x_0, x_1, t))
\]

Let \( n, m \in \mathbb{N} \) with \( n > m \), then
\[ M(x_n, x_m, t) = (\alpha(x_{n-1}, x_1) \beta(x_{n-1}, x_{n+1}) + 1)^{M(f(x_{n-1}), f(x_n), t)} \geq (1 + \ell)^{\varphi^n(M(x_0, x_1, t))} \]

Therefore \( \lim_{n \to \infty} M(x_n, x_m, t) = 1 \)

Hence \( \{x_n\} \) is a Cauchy sequence

Since \( X \) is an \( \alpha-\eta \)-complete fuzzy metric space there is \( x \in X \) such that \( x_n \to x \) as \( n \to \infty \).

\[ (\alpha(x_n, x) \beta(x_n, x) + \ell)^{M(x_n, f(x), t)} = (\alpha(x_n, x) \beta(x_n, x) + \ell)^{M(f(x_n), f(x), t)} \geq (1 + \ell)^{\varphi^n(M(x_n, x))} \]

Hence \( f(x) = x \)

Suppose \( y \) is a fixed point of \( f \) such that \( f(y) = y \)

\[ (\alpha(x, y) \beta(x, y) + \ell)^{M(f(x), f(y), t)} \geq (1 + \ell)^{\varphi^n(M(x, y))} \]

Hence \( x = y \).

References

[6] P. Salimia, C. Vetro and P. Vetro, Fixed point theorems for twisted \( (\alpha, \beta) \)-\( \psi \)-contractive type mappings and applications, *Faculty of Sciences and Mathematics*, University of Niš, Serbia, 24(4) (2013), 605-615.