Unique Common Fixed Point Theorem for Three Pairs of Weakly Compatible Mappings in Complete G-metric Space

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Abstract

In this paper a unique common fixed point theorem has been proved for three pairs of weakly compatible mappings in complete G – metric space. This theorem is the extension of many other results existing in the literature. An example has been provided to validate the main result of this paper.

Keywords: Common fixed point, Complete G – metric space, G – Cauchy sequence, Weakly compatible maps.

1 Introduction

The concept of the commutativity has been generalized in several ways. S. Sessa, [11] has introduced the concept of weakly commuting whereas Gerald Jungck [5] initiated the concept of compatibility. It can be easily verified that
When the two mappings are commuting then they are compatible but not conversely.

Compatible mappings are more general than commuting and weakly commuting mappings.

Compatible maps are weakly compatible but not conversely.

Many authors like [3], [4], [1] and [10] worked on compatible mappings in metric space.

Mustafa in collaboration with Sims [14] introduced a new notation of generalized metric space called G-metric space in 2006. He proved many fixed point results for a self mapping in G-metric space under certain conditions.

The main aim of this paper is to prove unique common fixed point theorem for three pairs of weakly compatible maps satisfying a new contractive condition in a complete G-metric space.

Now, we give preliminaries and basic definitions which are used throughout the paper.

**Definition 1.1:** Let $X$ be a non empty set, and let $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

1. $(G_1)$ $G(x, y, z) = 0$ if $x = y = z$
2. $(G_2)$ $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$
3. $(G_3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$
4. $(G_4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x)$ (Symmetry in all three variables)
5. $(G_5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality)

Then the function $G$ is called a generalized metric space, or more specially a G-metric on $X$, and the pair $(X, G)$ is called a G-metric space.

**Definition 1.2:** Let $(X, G)$ be a G-metric space and let $\{x_n\}$ be a sequence of points of $X$, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if

$$\lim_{n,m \to \infty} G(x_n, x_m) = 0,$$

and we say that the sequence $\{x_n\}$ is $G$-convergent to $x$ or $\{x_n\}$ $G$-converges to $x$.

Thus, $x_n \rightarrow x$ in a G-metric space $(X, G)$ if for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $m, n \geq k$. 

**Proposition 1.3:** Let \((X,G)\) be a \(G\)-metric space. Then the following are equivalent:

i) \(\{x_n\} \) is \(G\)-convergent to \(x\)

ii) \(G(x_n, x_n, x) \to 0 \text{ as } n \to +\infty\)

iii) \(G(x_n, x, x) \to 0 \text{ as } n \to +\infty\)

iv) \(G(x_n, x_m, x) \to 0 \text{ as } n, m \to +\infty\)

**Proposition 1.4:** Let \((X,G)\) be a \(G\)-metric space. Then for any \(x, y, z, a\) in \(X\) it follows that

i) If \(G(x, y, z) = 0\) then \(x = y = z\)

ii) \(G(x, y, z) \leq G(x, x, y) + G(x, x, z)\)

iii) \(G(x, y, y) \leq 2G(y, x, x)\)

iv) \(G(x, y, z) \leq G(x, a, z) + G(a, y, z)\)

**Definition 1.5:** Let \((X,G)\) be a \(G\)-metric space. A sequence \(\{x_n\}\) is called a \(G\)-Cauchy sequence if for any \(\epsilon > 0\) there exists \(k \in N\) such that \(G(x_n, x_m, x_l) < \epsilon\) for all \(m, n, l \geq k\), that is \(G(x_n, x_m, x_l) \to 0\) as \(n, m, l \to +\infty\).

**Proposition 1.6:** Let \((X,G)\) be a \(G\)-metric space. Then the following are equivalent:

i) The sequence \(\{x_n\}\) is \(G\)-Cauchy;

ii) For any \(\epsilon > 0\) there exists \(k \in N\) such that \(G(x_n, x_m, x_m) < \epsilon\) for all \(m, n \geq k\)

**Proposition 1.7:** A \(G\)-metric space \((X,G)\) is called \(G\)-complete if every \(G\)-Cauchy sequence is \(G\)-convergent in \((X,G)\).

**Proposition 1.8:** Let \((X,G)\) be a \(G\)-metric space. Then \(f:X \to X\) is \(G\)-continuous at \(x \in X\), if and only if it is \(G\)-sequentially continuous at \(x\), that is, whenever \(\{x_n\}\) is \(G\)-convergent to \(x\), \(\{f(x_n)\}\) is \(G\)-convergent to \(f(x)\).

**Definition 1.9:** Let \(f\) and \(g\) be two self–maps on a set \(X\). Maps \(f\) and \(g\) are said to be commuting if \(fgx =gfx\), for all \(x \in X\).

**Definition 1.10:** Let \(f\) and \(g\) be two self–maps on a set \(X\). If \(fx = gx\), for some \(x \in X\) then \(x\) is called coincidence point of \(f\) and \(g\).
Definition 1.11[6]: Let \( f \) and \( g \) be two self-maps defined on a set \( X \), then \( f \) and \( g \) are said to be weakly compatible if they commute at coincidence points. That is if \( fu = gu \) for some \( u \in X \), then \( fg u = gf u \).

Lemma 1.12 [5]: Let \( f \) and \( g \) be weakly compatible self-mappings of a set \( X \). If \( f \) and \( g \) have a unique point of coincidence, that is, \( w = fx = gx \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

Definition 1.13: A function \( \phi : [0, \infty) \to [0, \infty) \) is said to be special phi function if it satisfies:

i) \( 0 < \phi(t) < t \), for all \( t > 0 \)

ii) The series \( \sum_{n=1}^{\infty} \phi^n(t) \) converges for all \( t > 0 \)

iii) \( \phi \) is an upper semi continuous function.

Definition 1.15: A real valued function \( \phi \) defined on \( X \subseteq R \) is said to be upper semi continuous if \( \limsup_{n \to \infty} \phi(t_n) \leq \phi(t) \), for every sequence \( \{t_n\} \subseteq X \) with \( t_n \to t \) as \( n \to \infty \).

2 Main Result

Theorem 2.1: Let \((X, G)\) be a complete \( G \)-metric space and \( A, B, C, L, M, N : X \to X \) be mappings such that

I) \( N(X) \subseteq A(X), \ L(X) \subseteq B(X), \ M(X) \subseteq C(X) \)

II) \( G(Lx, My, Nz) \leq \phi(\lambda(x, y, z)) \), where \( \phi \) is a special phi function and

\[ \lambda(x, y, z) = \max \{G(Ax, By, Cz), G(Lx, Ax, Cz), G(My, By, Ax), G(Nz, Cz, By)\} \]

III) The pairs \((L, A), (M, B)\) and \((N, C)\) are weakly compatible.

Then \( A, B, C, L, M, N \) have a unique common fixed point in \( X \).

Proof: Let \( x_0 \) be an arbitrary point of \( X \) and define the sequence \( \{x_n\} \) in \( X \) such that

\[ y_n = Lx_n = Bx_{n+1}, \quad y_{n+1} = Mx_{n+1} = Cx_{n+2}, \quad y_{n+2} = Nx_{n+2} = Ax_{n+3} \]

Consider, \( G(y_n, y_{n+1}, y_{n+2}) = G(Lx_n, Mx_{n+1}, Nx_{n+2}) \)

\[ \leq \phi(\lambda(x_n, x_{n+1}, x_{n+2})) \]
where

\[ \lambda(x_n, x_{n+1}, x_{n+2}) = \max \left\{ G(Ax_n, Bx_{n+1}, Cx_{n+2}), \right. \]
\[ \left. G(Lx_n, Ax_n, Cx_{n+2}), G(Mx_{n+1}, Bx_{n+1}, Ax_n), G(Nx_{n+2}, Cx_{n+2}, Bx_{n+1}) \right\} \]
\[ = \max \left\{ G(Nx_{n-1}, Lx_n, Mx_{n+1}), G(Lx_n, Nx_{n-1}, Mx_{n+1}), \right. \]
\[ \left. G(Mx_{n+1}, Lx_n, Nx_{n-1}), G(Nx_{n+2}, Mx_{n+1}, Lx_n) \right\} \]

i.e. \( \lambda(x_n, x_{n+1}, x_{n+2}) = \max \{ G(y_{n-1}, y_n, y_{n+1}), G(y_n, y_{n-1}, y_{n+1}), G(y_{n+1}, y_n, y_{n-1}), G(y_{n+2}, y_{n+1}, y_n) \} \)

Since \( \phi \) is a phi function,

Therefore \( \lambda(x_n, x_{n+1}, x_{n+2}) = G(y_n, y_{n+1}, y_{n+2}) \) is not possible.

Therefore \( G(y_n, y_{n+1}, y_{n+2}) \leq \phi(G(y_{n-1}, y_n, y_{n+1})) \)

---------- (2.1.1)

Since \( \phi \) is an upper semi continuous, special phi function, so equation (2.1.1) implies that the sequence \( \{y_n\} \) is monotonic decreasing and continuous.

Hence there exists a real number say \( r \geq 0 \), such that \( \lim_{n \to \infty} G(y_n, y_{n+1}, y_{n+2}) = r \)

As \( n \to \infty \), equation (2.1.1) implies that \( r \leq \phi(r) \), which is possible only if \( r = 0 \), because \( \phi \) is a special phi function.

Therefore \( \lim_{n \to \infty} G(y_n, y_{n+1}, y_{n+2}) = 0 \)

---------- (2.1.2)

Now we show that \( \{y_n\} \) is a Cauchy sequence.

We have,

\[ G(y_n, y_{n+1}, y_{n+2}) \leq \phi(G(y_{n-1}, y_n, y_{n+1})) \]
\[ \leq \phi(\phi(G(y_{n-2}, y_{n-1}, y_n))) \]
\[ = \phi^2(G(y_{n-2}, y_{n-1}, y_n)) \]
\[ \leq \phi^3(G(y_{n-2}, y_{n-1}, y_n)) \]
\[ \leq \phi^n(G(y_0, y_1, y_2)) \]

By using \( (G_3), (G_4), (G_5) \) and condition (2.1.1) for any \( k \in \mathbb{N} \), we write
\[ G(y_n, y_{n+k}, y_{n+k}) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) \]
\[ + \ldots + G(y_{n+k-2}, y_{n+k-1}, y_{n+k-1}) + G(y_{n+k-1}, y_{n+k}, y_{n+k}) \]
\[ \leq G(y_n, y_{n+1}, y_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+3}) + G(y_{n+2}, y_{n+3}, y_{n+4}) \]
\[ + \ldots + G(y_{n+k-2}, y_{n+k-1}, y_{n+k}) + G(y_{n+k-1}, y_{n+k}, y_{n+k+1}) \]
\[ \leq \phi^k(G(y_0, y_1, y_2)) + \phi^{n+1}(G(y_0, y_1, y_2)) + \phi^{n+2}(G(y_0, y_1, y_2)) \]
\[ + \ldots + \phi^{n+k}(G(y_0, y_1, y_2)) \]
\[ = \sum_{i=0}^{n+k} \phi^i(G(y_0, y_1, y_2)) \]
i.e. \[ G(y_n, y_{n+k}, y_{n+k}) \leq \sum_{i=0}^{\infty} \phi^i(G(y_0, y_1, y_2)) \] \[ \text{----------- (2.1.3)} \]

By definition of function \( \phi \), we have \[ \sum_{i=0}^{\infty} \phi^i(G(y_0, y_1, y_2)) \text{ tends to 0 as } n \to \infty \]

Therefore \[ \lim_{n \to \infty} G(y_n, y_{n+k}, y_{n+k}) = 0 \quad \text{for all } k \in N \] \[ \text{----------- (2.1.4)} \]

This means that \( \{y_n\} \) is a Cauchy sequence and since \( X \) is complete, therefore there exists a point \( u \in X \), such that \[ \lim_{n \to \infty} y_n = u \]

Therefore \[ \lim_{n \to \infty} Lx_n = \lim_{n \to \infty} Bx_{n+1} = u \], \[ \lim_{n \to \infty} Mx_{n+1} = \lim_{n \to \infty} Cx_{n+2} = u \]

and \[ \lim_{n \to \infty} Nx_{n+2} = \lim_{n \to \infty} Ax_{n+3} = u \]

Since \( N(X) \subseteq A(X) \), there exists a point \( v \in X \) such that \( u = Av \)

Therefore by (II) we have,
\[ G(Lv, u, u) \leq G(Lv, Mx_{n+1}, u) + G(Mx_{n+1}, u, u) \leq G(Lv, Mx_{n+1}, Nx_{n+2}) + G(Nx_{n+2}, u, Mx_{n+1}) + G(Mx_{n+1}, u, u) \]
\[ \leq \phi(\lambda(v, x_{n+1}, x_{n+2})) + G(Nx_{n+2}, u, Mx_{n+1}) + G(Mx_{n+1}, u, u) \] \[ \text{----- (2.1.5)} \]

Where,
\[ \lambda(v, x_{n+1}, x_{n+2}) = \max \{ \begin{align*} &G(Av, Bx_{n+1}, Cx_{n+2}), G(Lv, Av, Cx_{n+2}), \\ &G(Mx_{n+1}, Bx_{n+1}, Av), G(Nx_{n+2}, Cx_{n+2}, Bx_{n+2}) \end{align*} \}
\[ = \max \{ G(u, Lx_n, Mx_{n+1}), G(Lv, u, Mx_{n+1}), G(Mx_{n+1}, Lx_n, u), G(Nx_{n+2}, Mx_{n+1}, Lx_n) \} \]
Taking limit as \( n \to \infty \) in the above relation, we get

\[
\lambda(v, x_{n+1}, x_{n+2}) = \max \{G(u, u, u), G(Lv, u, u), G(u, u, u), G(u, u, u)\}
\]

Therefore \( \lambda(v, x_{n+1}, x_{n+2}) = G(Lv, u, u) \)

Thus as \( n \to \infty \), we get from (2.1.5)

\[
G(Lv, u, u) \leq \phi(G(Lv, u, u)) + G(u, u, u) + G(u, u, u)
\]

i.e. \( G(Lv, u, u) \leq \phi(G(Lv, u, u)) \)

--------- (2.1.6)

If \( Lv \neq u \), then \( G(Lv, u, u) > 0 \), and hence as \( \phi \) is a special phi function

\[
\phi(G(Lv, u, u)) < G(Lv, u, u)
\]

Therefore from (2.1.6) we have \( G(Lv, u, u) < G(Lv, u, u) \), which is a contradiction

\[
\therefore \text{ we must have } \ Lv = u \ . \text{ So we have } \ Av = Lv = u \ .
\]

i.e. \( v \) is a coincidence point of \( L \) and \( A \).

Since the pair of maps \( L \) and \( A \) are weakly compatible,

\[
\therefore \ LAv = ALv \quad \text{i.e.} \quad Lu = Au
\]

Again, since \( L(X) \subseteq B(X) \), there exists a point \( w \in X \) such that \( u = Bw \)

Therefore by (II) we have,

\[
G(u, u, Mw) = G(Lv, Lv, Mw) \quad (\because G(x, x, y) \leq G(x, y, z))
\leq G(Lv, Mw, Nx_{n+2})
\leq \phi(\lambda(v, w, x_{n+2}))
\]

--------- (2.1.7)

Where \( \lambda(v, w, x_{n+2}) = \max \{G(Av, Bw, Cx_{n+2}), G(Lv, Av, Cx_{n+2}), G(Mw, Bw, Ax_{n+2}), G(Nx_{n+2}, Cx_{n+2}, Bw)\} \)

\[
= \max \{G(u, u, Mx_{n+1}), G(u, u, Mx_{n+1}), G(Mw, u, Nx_{n+1}), G(Nx_{n+2}, Mx_{n+1}, u)\}
\]

Taking limit as \( n \to \infty \), we get
\[ \lambda(v, w, x_{n+2}) = \max \{G(u, u, u), G(u, u, u), G(Mw, u, u), G(u, u, u) \} \]

Therefore \[ \lambda(v, w, x_{n+2}) = G(Mw, u, u) = G(u, u, Mw) \]

Therefore from (2.1.7), we get \[ G(u, u, Mw) \leq \phi(G(u, u, Mw)) \] --------- (2.1.8)

If \( Mw \neq u \), then \( G(u, u, Mw) > 0 \) and hence as \( \phi \) is a special phi function,

\[ \phi(G(u, u, Mw)) < G(u, u, Mw) \]

Therefore by using (2.1.8), we get, \( G(u, u, Mw) < G(u, u, Mw) \), which is a contradiction.

Hence we have \( Mw = u \). Thus we have \( Mw = Bw = u \) i.e. \( w \) is a coincidence point of \( M \) and \( B \).

Since the pair of maps \( M \) and \( B \) are weakly compatible,

\[ \therefore MBw = BMw \text{ i.e. } Mu = Bu \]

Now again, since \( M(X) \subseteq C(X) \), there exists a point \( p \in X \), such that \( u = Cp \)

Therefore by (II), we have,

\[ G(u, u, Np) = G(Lv, Mw, Np) \]
\[ \leq \phi(\lambda(v, w, p)) \] --------- (2.1.9)

Where

\[ \lambda(v, w, p) = \max \{G(Av, Bw, Cp), G(Lv, Av, Cp), G(Mw, Bw, Av), G(Np, Cp, Bw) \} \]
\[ = \max \{G(u, u, u), G(u, u, u), G(u, u, u), G(Np, u, u) \} \]

Therefore \[ \lambda(v, w, p) = G(Np, u, u) = G(u, u, Np) \]

Therefore from (2.1.9), we have \[ G(u, u, Np) \leq \phi(G(u, u, Np)) \] ------- (2.1.10)

If \( Np \neq u \), then \( G(u, u, Np) > 0 \) and hence as \( \phi \) is a special phi function,

\[ \phi(G(u, u, Np)) < G(u, u, Np) \]

Therefore from (2.1.10) we get, \( G(u, u, Np) < G(u, u, Np) \), which is a contradiction.

Hence we must have \( Np = u \). Thus we have \( Np = Cp = u \) i.e. \( p \) is a coincidence point of \( N \) and \( C \). Since the pair of maps \( N \) and \( C \) are weakly compatible,
\[ \therefore \ NCP = Cnp \quad \text{i.e.} \quad Nu = Cu \]

Now we show that ‘u’ is a fixed point of \( L \).

By (II), we have
\[ G(Lu, u, u) = G(Lu, Mw, Np) \leq \phi(\lambda(u, w, p)) \]
\[ \quad \text{--------- (2.1.11)} \]

Where
\[ \lambda(u, w, p) = \max \{G(Au, Bw, Cp), G(Lu, Au, Cp), G(Mw, Bw, Au), G(Np, Cp, Bw)\} \]
\[ = \max \{G(Lu, u, u), G(Lu, Lu, u), G(u, u, Lu), G(u, u, u)\} \]
\[ = G(Lu, u, u) \quad \text{--------- by (iv) of Proposition 1.4} \]

Therefore from (2.1.11), we have,
\[ G(Lu, u, u) \leq \phi(G(Lu, u, u)) \quad \text{--------- (2.1.12)} \]

If \( Lu \neq u \), then \( G(Lu, u, u) > 0 \) and hence as \( \phi \) is a special phi function,
\[ \therefore \phi(G(Lu, u, u)) < G(Lu, u, u) \]

Therefore (2.1.12) implies that \( G(Lu, u, u) < G(Lu, u, u) \), which is a contradiction.

Hence we have \( Lu = u \). So we get \( Lu = Au = u \).

Now, we show that \( u \) is a fixed point of \( M \).

Therefore by (II) we have,
\[ G(u, u, Mu) = G(Lu, Np, Mu) \]
\[ = G(Lu, Mu, Np) \]
\[ \leq \phi(\lambda(u, u, p)) \]
\[ \text{--------- (2.1.13)} \]

Where
\[ \lambda(u, u, p) = \max \{G(Au, Bu, Cp), G(Lu, Au, Cp), G(Mu, Bu, Au), G(Np, Cp, Bu)\} \]
\[ = \max \{G(Lu, Mu, u), G(Lu, Lu, u), G(Mu, Mu, Mu), G(u, u, Mu)\} \]
\[ = \max \{G(u, Mu, u), G(u, u, u), G(Mu, Mu, u), G(u, u, Mu)\} \]
\[ = G(u, u, Mu) \quad \text{--------- by (iv) of Proposition 1.4} \]

So from (2.1.13) we get,
\[ G(u, u, Mu) \leq \phi(G(u, u, Mu)) \]
\[ \text{--------- (2.1.14)} \]

If \( Mu \neq u \), then \( G(u, u, Mu) > 0 \) and hence as \( \phi \) is a special phi function,
\[ \phi(G(u, u, Mu)) < G(u, u, Mu) \]

Thus from (2.1.14) we get, \( G(u, u, Mu) < G(u, u, Mu) \), which is a contradiction.

Therefore \( Mu = u \). Hence \( Mu = Bu = u \).
Now we show that $u$ is a fixed point of $N$.
Therefore from (II) we have, $G(u, u, Nu) = G(Lu, Mu, Nu) \leq \phi(\lambda(u, u, u)) \quad \text{(2.1.15)}$

Where

$\lambda(u, u, u) = \max \{G(Au, Bu, Cu), G(Lu, Au, Cu), G(Mu, Bu, Au), G(Nu, Cu, Bu)\}$
$= \max \{G(u, u, Nu), G(u, Lu, Nu), G(u, Mu, Lu), G(Nu, Nu, Mu)\}$
$= \max \{G(u, u, Nu), G(u, u, Nu), G(u, u, u), G(Nu, Nu, u)\}$
$= G(u, u, Nu) \quad \text{by (iv) of Proposition 1.4}$

Thus from (2.1.15) we have, $G(u, u, Nu) \leq \phi(G(u, u, Nu)) \quad \text{(2.1.16)}$

If $Nu \neq u$, then $G(u, u, Nu) > 0$ and hence as $\phi$ is a special phi function,

$\phi(G(u, u, hu)) < G(u, u, hu)$

Thus by using (2.1.16) we get, $G(u, u, Nu) < G(u, u, Nu)$, which is a contradiction.

Hence $Nu = u$. Thus we have $Nu = Cu = u$

Therefore $Lu = Au = Mu = Bu = Nu = Cu = u$ i.e. $u$ is a common fixed point of $L, A, M, B, N$ and $C$.

Now we show that ‘$u$’ is unique common fixed point of $L, A, M, B, N$ and $C$.

If possible, let us assume that ‘$m$’ is another common fixed point of $L, A, M, B, N$ and $C$.

By using (II) we have, $G(u, u, m) = G(Lu, Mu, Nm) \leq \phi(\lambda(u, u, m)) \quad \text{(2.1.17)}$

Where

$\lambda(u, u, m) = \max \{G(Au, Bu, Cm), G(Lu, Au, Cm), G(Mu, Bu, Au), G(Nm, Cm, Bu)\}$
$= \max \{G(u, u, m), G(u, u, m), G(u, u, u), G(m, m, u)\}$
$= G(u, u, m) \quad \text{by (iv) of Proposition 1.4}$

Thus from (2.1.17) we have, $G(u, u, m) \leq \phi(G(u, u, m)) \quad \text{(2.1.18)}$

If $u \neq m$, then $G(u, u, m) > 0$ and hence as $\phi$ is a special phi function,

$\phi(G(u, u, m)) < G(u, u, m)$

Hence from (2.1.18) we get, $G(u, u, m) < G(u, u, m)$, which is a contradiction.
Hence we have $u = m$.
Thus ‘$u$’ is the unique common fixed point of $L, A, M, B, N$ and $C$.

**Example 2.2:** Let $X = [0, \infty)$ and $G$ be a mapping defined on $X$ as

$$G(x, y, z) = |x - y| + |y - z| + |z - x|,$$

for all $x, y, z \in X$.

Then $G$ is a complete $G$-metric on $X$ and $(X, G)$ is a complete $G$-metric space.

Let $A, B, C, L, M, N : X \to X$ be defined as $Ax = \frac{x}{3}$, $Tx = \frac{x}{6}$, $Cx = \frac{x}{9}$,

$Lx = \frac{x}{24}$, $Mx = \frac{x}{36}$

and $Nx = \frac{x}{12}$ then (i) $N(X) \subseteq A(X)$, $L(X) \subseteq B(X)$, $M(X) \subseteq C(X)$

(ii) The pairs $(L, A)$, $(M, B)$ and $(N, C)$ are weakly compatible.

(iii) Also $G(Lx, My, Nz) \leq \phi(\lambda(x, y, z))$

Where

$$\lambda(x, y, z) = \max \{G(Ax, By, Cz), G(Lx, Ax, Cz), G(My, By, Ax), G(Nz, Cz, By)\}$$

Then ‘0’ is unique common fixed point of $L, A, M, B, N$ and $C$ in $X$.

**Corollary 2.3:** Let $(X, G)$ be a complete $G$-metric space and $A, L, M, N : X \to X$ be mappings such that

I) $N(X) \subseteq A(X)$, $L(X) \subseteq A(X)$, $M(X) \subseteq A(X)$

II) $G(Lx, My, Nz) \leq \phi(\lambda(x, y, z))$, where $\phi$ is a special phi function and

$$\lambda(x, y, z) = \max \{G(Ax, Ay, Az), G(Lx, Ax, Az), G(My, Ay, Ax), G(Nz, Az, Ay)\}$$

III) The pairs $(L, A)$, $(M, A)$ and $(N, A)$ are weakly compatible.

Then $A, L, M$, and $N$ have a unique common fixed point in $X$.

**Proof:** By taking $A = B = C$ in Theorem 2.1 we get the proof.

**Corollary 2.4:** Let $(X, G)$ be a complete $G$-metric space and $A, L : X \to X$ be mappings such that

I) $L(X) \subseteq A(X)$
II) \[ G(Lx, Ly, Lz) \leq \phi(\lambda(x, y, z)), \] where \( \phi \) is a special phi function and \( \lambda(x, y, z) = \max \{G(Ax, Ay, Az), G(Lx, Ax, Az), G(Ly, Ay, Ax), G(Lz, Az, Ay)\} \)

III) The pair \((L, A)\) is weakly compatible.

Then \( A, L \) have a unique common fixed point in \( X \).

**Proof:** By taking \( A = B = C \) \& \( L = M = N \) in Theorem 2.1 we get the proof.

**References**


