A Formula for Tetranacci-Like Sequence

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Abstract
Many papers are concerning a variety of generalizations of the Fibonacci sequence. In this paper, we define a Tetranacci-Like sequence in terms of first four terms and then present the general formula for $n^{th}$ term of the Tetranacci-Like sequence with derivation.

Keywords: Tetranacci sequence, Tetranacci-Like sequence, Tetranacci numbers.

1 Introduction
Many sequences have been studied for many years now. Arithmetic, Geometric, Harmonic, Fibonacci and Lucas sequences have been very well-defined in Mathematical Journals. On the other hand, Fibonacci-Like sequence, Tribonacci-Like sequence received little more attention from mathematicians.
Fibonacci sequence is a sequence obtained by adding two preceding terms with the initial conditions 0 and 1. Similarly, Tribonacci sequence is obtained by adding three preceding terms starting with 0, 0 and 1. Moreover, Fibonacci-Like sequence and Tribonacci-Like sequence defined by the same pattern but the sequences start with two and three arbitrary terms respectively.

Various properties of Fibonacci-Like sequence have been presented in the paper of B. Singh [2]. In [3], Natividad derived a formula in solving a Fibonacci-like sequence using the Binet’s formula and Bueno [1] gives the formula for the $k^{th}$ term of Natividad’s Fibonacci-Like sequence. Also, Natividad [4] established a formula in solving the $n^{th}$ term of the Tribonacci-Like sequence.

In this paper, we will derive a general formula to finding the $n^{th}$ term of the Tetranacci-Like sequence using its first four terms and tetranacci numbers.

The Tetranacci sequence $\{M_n\}$ [5], [6] defined by the recurrence relation

\[ M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} \quad \text{for } n \geq 4, \]

where $M_0 = M_1 = 0$, $M_2 = M_3 = 1$.

First few terms of the Tetranacci sequence are as:

<table>
<thead>
<tr>
<th>$n^{th}$ Term</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetranacci Numbers</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>15</td>
<td>29</td>
<td>56</td>
<td>108</td>
<td>208</td>
<td>401</td>
<td>773</td>
<td>1490</td>
</tr>
</tbody>
</table>

When the first four terms of the Tetranacci sequence become arbitrary, it is known as Tetranacci-Like sequence.

### 2 Main Results

The Tetranacci-Like sequence is a sequence with the arbitrary initial terms or we can say that Tetranacci-Like sequence start at any desired numbers.

Let the first four terms of Tetranacci-Like sequence be $Q_1$, $Q_2$, $Q_3$, and $Q_4$. Then we shall derive a general formula for $Q_n$ given the first four terms.

The sequence $Q_1$, $Q_2$, $Q_3$, $Q_4$, ..., $Q_n$ is known as generalized Tetranacci sequence (or Tetranacci-Like sequence), if
\[ Q_n = Q_{n-4} + Q_{n-3} + Q_{n-2} + Q_{n-1} \]  

(1.2)

To find the general formula for \( n^{th} \) term of the Tetranacci-Like sequence, we follow a specific pattern.

From (1.2), we derive some of the equations as

\[
\begin{align*}
Q_5 &= Q_1 + Q_2 + Q_3 + Q_4 \\
Q_6 &= Q_1 + 2Q_2 + 2Q_3 + 2Q_4 \\
Q_7 &= 2Q_1 + 3Q_2 + 4Q_3 + 4Q_4 \\
Q_8 &= 4Q_1 + 6Q_2 + 7Q_3 + 8Q_4 \\
Q_9 &= 8Q_1 + 12Q_2 + 14Q_3 + 15Q_4 \\
Q_{10} &= 15Q_1 + 23Q_2 + 27Q_3 + 29Q_4 \\
Q_{11} &= 29Q_1 + 44Q_2 + 52Q_3 + 56Q_4
\end{align*}
\]

Now we write all the numerical coefficients of \( Q_1, Q_2, Q_3 \) and \( Q_4 \) in tabular form that were shown in Table 2.

**Table 2**: Coefficients of \( Q_1, Q_2, Q_3 \) and \( Q_4 \) of \( n^{th} \) term of Tetranacci-Like sequence

<table>
<thead>
<tr>
<th>Number of terms</th>
<th>( n^{th} ) term of Tetranacci-Like sequence</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Q_5 )</td>
<td>1, 1</td>
</tr>
<tr>
<td>2</td>
<td>( Q_6 )</td>
<td>1, 2</td>
</tr>
<tr>
<td>3</td>
<td>( Q_7 )</td>
<td>2, 3</td>
</tr>
<tr>
<td>4</td>
<td>( Q_8 )</td>
<td>4, 6</td>
</tr>
<tr>
<td>5</td>
<td>( Q_9 )</td>
<td>8, 12</td>
</tr>
<tr>
<td>6</td>
<td>( Q_{10} )</td>
<td>15, 23</td>
</tr>
<tr>
<td>7</td>
<td>( Q_{11} )</td>
<td>29, 44</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( n )</td>
<td>( Q_n )</td>
<td>( (n-2) )</td>
</tr>
</tbody>
</table>

\[ Q_{n-4} + (n-2) + (n-3) \]

\[ (n-2) + (n-3) + (n-4) + (n-1) \]
After a keen observation of Table 1 and Table 2, we state the following theorem.

**Theorem 1:** For any real numbers $Q_1$, $Q_2$, $Q_3$ and $Q_4$, the formula for finding the $n$th term of the Tetranacci-Like sequence is

$$Q_n = M_{n-2} Q_1 + (M_{n-2} + M_{n-3}) Q_2 + (M_{n-2} + M_{n-3} + M_{n-4}) Q_3 + M_{n-1} Q_4,$$

(1.3)

where

- $Q_1$ = first term
- $Q_2$ = second term
- $Q_3$ = third term
- $Q_4$ = fourth term
- $M_{n-1}, M_{n-2}, M_{n-3}, M_{n-4}$ = corresponding tetranacci numbers.

**Proof:** We shall prove above theorem by the Principle of Mathematical Induction method for $n \geq 5$.

First we take $n = 5$, then we get

$$Q_5 = M_3 Q_1 + (M_3 + M_2) Q_2 + (M_3 + M_2 + M_1) Q_3 + M_4 Q_4$$

$$Q_5 = (1) Q_1 + (1 + 0) Q_2 + (1 + 0 + 0) Q_3 + (1) Q_4$$

which is true. (by definition of Tetranacci-Like sequence)

Now, we assume that the theorem is true for some integer $k (>5)$, i.e.

$$P(k) : Q_k = M_{k-2} Q_1 + (M_{k-2} + M_{k-3}) Q_2 + (M_{k-2} + M_{k-3} + M_{k-4}) Q_3 + M_{k-1} Q_4$$

(1.4)

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true, i.e.

$$P(k+1) : Q_{k+1} = M_{k-1} Q_1 + (M_{k-1} + M_{k-2}) Q_2 + (M_{k-1} + M_{k-2} + M_{k-3}) Q_3 + M_k Q_4$$

(1.5)
To verify above equation, we shall add $Q_{k-1}$, $Q_{k-2}$ and $Q_{k-3}$ on both side of $P(k)$, then eq.(1.4) becomes

$$Q_k + Q_{k-1} + Q_{k-2} + Q_{k-3} = M_{k-2}Q_1 + (M_{k-2} + M_{k-3})Q_2 + (M_{k-2} + M_{k-3} + M_{k-4})Q_3 + M_{k-1}Q_4 + Q_{k-1} + Q_{k-2} + Q_{k-3}$$

(1.6)

By equation (1.4), we have

$$Q_{k-1} = M_{k-3}Q_1 + (M_{k-3} + M_{k-4})Q_2 + (M_{k-3} + M_{k-4} + M_{k-5})Q_3 + M_{k-2}Q_4$$

$$Q_{k-2} = M_{k-4}Q_1 + (M_{k-4} + M_{k-5})Q_2 + (M_{k-4} + M_{k-5} + M_{k-6})Q_3 + M_{k-3}Q_4$$

$$Q_{k-3} = M_{k-5}Q_1 + (M_{k-5} + M_{k-6})Q_2 + (M_{k-5} + M_{k-6} + M_{k-7})Q_3 + M_{k-4}Q_4$$

Use above in eq. (1.6), we obtain

$$Q_k = M_{k-2}Q_1 + (M_{k-2} + M_{k-3})Q_2 + (M_{k-2} + M_{k-3} + M_{k-4})Q_3 + M_{k-1}Q_4$$

$$M_{k-3}Q_1 + (M_{k-3} + M_{k-4})Q_2 + (M_{k-3} + M_{k-4} + M_{k-5})Q_3 + M_{k-2}Q_4$$

$$M_{k-4}Q_1 + (M_{k-4} + M_{k-5})Q_2 + (M_{k-4} + M_{k-5} + M_{k-6})Q_3 + M_{k-3}Q_4$$

$$M_{k-5}Q_1 + (M_{k-5} + M_{k-6})Q_2 + (M_{k-5} + M_{k-6} + M_{k-7})Q_3 + M_{k-4}Q_4$$

(1.7)

Now by the definition of Tetranacci sequence eq. (1.7) becomes

$$Q_{k+1} = M_{k-1}Q_1 + [M_{k-1} + M_{k-2}]Q_2 + [M_{k-1} + M_{k-2} + M_{k-3}]Q_3 + M_{k}Q_4$$

Thus by the Mathematical Induction P(k+1) is true, whenever P(k) is true. Hence the theorem is verified.
3 Conclusion

In this paper, we have introduced Tetranacci-Like sequence using its first four terms and Tetranacci numbers and derived the general formula of \( n \)th term of the Tetranacci-Like sequence. The method of Mathematical Induction has been used.

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References


