On a subclass of functions with negative coefficients

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Dedicated to Professor dr. Gheorghe Micula on his 60th birthday

Abstract

We determine conditions for a function to be n-close to convex of order $\alpha$, $\alpha \in [0,1)$, $n \in \mathbb{N}$, with negative coefficients.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$,

$$A = \{ f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0 \}$$

and $S = \{ f \in A : f \text{ is univalent in } U \}$.

In ([4]) the subfamily $T$ of $S$ consisting of functions $f$ of the form

$$(1) \quad f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geq 0, j = 2, 3, ..., \quad z \in U$$
was introduced.

The purpose of this paper is to give a condition for $f \in T$ to be n-close to convex of order $\alpha$, $\alpha \in [0, 1)$, $n \in \mathbb{N}$, and to determine some properties of this class.

## 2 Preliminary results

Let $D^n$ be the Sălăgean differential operator (see [2]) $D^n : A \rightarrow A, n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^n f(z) = D(D^{n-1}f(z))$$

**Remark 2.1.** If $f \in T, f(z) = \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \ldots, z \in U$ then

$$D^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j.$$

**Theorem 2.1.[2].** If $f(z) = \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \ldots, z \in U$ then the next assertions are equivalent:

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, where $T^* = T \cap S^*$ and $S^*$ is the well-known class of starlike functions.

**Definition 2.1.[2].** Let $\alpha \in [0, 1)$ and $n \in \mathbb{N}$, then

$$S_n(\alpha) = \left\{ f \in A : \text{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, z \in U \right\}$$
is the set of n-starlike functions of order $\alpha$.

**Remark 2.2.** If $f \in S_n(\alpha)$ according to the definition of the Sălăgean differential operator we can write that

$$\text{Re} \left( \frac{z(D^n f(z))'}{D^n f(z)} \right) > \alpha$$

and thus the function $F(z) = D^n f(z) \in S(\alpha)$, $\alpha \in [0, 1)$, where

$$S(\alpha) = \left\{ h \in A : \text{Re} \left( \frac{z h'(z)}{h(z)} \right) > \alpha, \ z \in U \right\}.$$

**Definition 2.2.** $T_n(\alpha) = T \cap S_n(\alpha)$.

**Definition 2.3.** Let $\alpha \in [0, 1), \beta \in (0, 1]$ and let $n \in \mathbb{N}$; we define the class $T_n(\alpha, \beta)$ of n-starlike functions of order $\alpha$ and type $\beta$ with negative coefficients by

$$T_n(\alpha, \beta) = \{ f \in A : |J_n(f, \alpha; z)| < \beta, \ z \in U \},$$

where

$$J_n(f, \alpha; z) = \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{\frac{D^{n+1} f(z)}{D^n f(z)} + 1 - 2\alpha}, \ z \in U.$$

**Remark 3.2.** The class $T_0(\alpha, 1)$ is the class of starlike functions of order $\alpha$ with negative coefficients; $T_1(\alpha, 1)$ is the well-known class of convex functions of order $\alpha$ with negative coefficients; $T_n(\alpha, 1)$ is the class of n-starlike functions of order $\alpha$ with negative coefficients i.e. $T_n(\alpha, 1) = T \cap S_n(\alpha)$. We also note that the functions in $T_n(\alpha, \beta)$ are univalent because $T_n(\alpha, \beta) \subset T_n(\alpha, 1)$, $\beta \in (0, 1)$ and $T_n(\alpha_1, \beta) \subset T_n(\alpha, \beta)$ with $1 > \alpha_1 > \alpha \geq 0$, $\beta \in (0, 1]$. 
Theorem 2.2.[3]. Let $\alpha \in [0, 1), \beta \in (0, 1]$ and $n \in \mathbb{N}$. The function $f$ of the form (1) is in $T_n(\alpha, \beta)$ if and only if

$$\sum_{j=2}^{\infty} j^n[j - 1 + \beta(j + 1 - 2\alpha)]a_j \leq 2\beta(1 - \alpha)$$

The result is sharp and the extremal functions are:

$$f_j(z) = z - \frac{2\beta(1 - \alpha)}{j^n[j - 1 + \beta(j + 1 - 2\alpha)]}z^j, j = 2, 3, ...$$

From this result we have $T_{n+1}(\alpha, \beta) \subset T_n(\alpha, \beta), n \in \mathbb{N}$.

Definition 2.4.[3]. Let $I_c : A \to A$ be the integral operator defined by $f = I_c(F)$, where $c \in (-1, \infty)$, $F \in A$ and

$$f(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1}F(t)dt. \quad (2)$$

We note if $F \in A$ is a function of the form (1), then

$$f(z) = I_c(F(z) = z - \sum_{j=2}^{\infty} \frac{c + 1}{c + j}a_jz^j. \quad (3)$$

Remark 2.4. In [3] is showed that if $F \in T_n(\alpha, \beta)$ then $f = I_c(F) \in T_n(\alpha, \beta)$.

Definition 2.5.[1]. Let $f \in A$. We say that $f$ is $n$-close to convex of order $\alpha$ with respect to a half-plane, and denote by $CC_n(\alpha)$ the set of these functions, if there exists $g \in S_n(0) = S_n$ so that

$$Re\frac{D^{n+1}f(z)}{D^n g(z)} > \alpha, \ z \in U,$$

where $n \in \mathbb{N}, \alpha \in [0, 1)$. 
Remark 2.5. $CC_0(\alpha) = CC(\alpha)$, where $CC(\alpha)$ is the well-known class of close to convex functions of order $\alpha$.

Remark 2.6. In [1] the author show that if $n \in \mathbb{N}$ and $\alpha \in [0,1)$ then $CC_{n+1}(\alpha) \subset CC_n(\alpha)$ and thus the functions from $CC_n(\alpha)$ are univalent.

Remark 2.7. From Remark 2.3 and Theorem 2.2 we have for $f$ of the form (1) with $f \in T_n(\alpha, 1) = T_n(\alpha)$:

$$\sum_{j=2}^{\infty} j^n (j - \alpha) a_j \leq 1 - \alpha, \text{ where } \alpha \in [0,1)$$

3 Main results

Definition 3.1. Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \ldots$, $z \in U$.

We say that $f$ is in the class $CCT_n(\alpha)$, $\alpha \in [0,1)$, $n \in \mathbb{N}$, with respect to the function $g \in T_n(0)$, if:

$$\text{Re} \frac{D^{n+1} f(z)}{D^n g(z)} > \alpha, \ z \in U.$$

Theorem 3.1. Let $\alpha \in [0,1)$ and $n \in \mathbb{N}$. The function $f \in T$ of the form (1) is in $CCT_n(\alpha)$, with respect to the function $g \in T_n(0)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, \ldots$, if and only if

$$\sum_{j=2}^{\infty} j^n [ja_j + (2 - \alpha)b_j] < 1 - \alpha \quad (4)$$

Proof. Let $f \in CCT_n(\alpha)$, with $\alpha \in [0,1)$. We have

$$\text{Re} \frac{D^{n+1} f(z)}{D^n g(z)} > \alpha.$$
If we take $z \in [0, 1)$, we have (see Remark 2.1):

\[
1 - \sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1} \quad > \alpha
\]

(5)

\[
1 - \sum_{j=2}^{\infty} j^{n} b_j z^{j-1}
\]

From $g \in T_n(0) = T_n(0, 1)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, \ldots$, we have (see Remark 2.7):

\[
\sum_{j=2}^{\infty} j^{n+1} b_j \leq 1.
\]

(6)

We have: $\sum_{j=2}^{\infty} j^{n} b_j z^{j-1} \leq \sum_{j=2}^{\infty} j^{n+1} b_j z^{j-1} < \sum_{j=2}^{\infty} j^{n+1} b_j$.

From (6) we obtain: $\sum_{j=2}^{\infty} j^{n} b_j z^{j-1} < 1$ and thus $1 - \sum_{j=2}^{\infty} j^{n} b_j z^{j-1} > 0$.

In this condition from (5) we obtain:

\[
1 - \sum_{j=2}^{\infty} j^{n+1} a_j z^{j-1} > \alpha \left[ 1 - \sum_{j=2}^{\infty} j^{n} b_j z^{j-1} \right]
\]

Letting $z \to 1^-$ along the real axis we have:

\[
1 - \sum_{j=2}^{\infty} j^{n+1} a_j > \alpha - \sum_{j=2}^{\infty} j^{n} \alpha b_j,
\]

and thus:

\[
\sum_{j=2}^{\infty} j^{n}[ja_j - \alpha b_j] < 1 - \alpha.
\]

From $\sum_{j=2}^{\infty} j^{n}[ja_j + (2 - \alpha)b_j] > \sum_{j=2}^{\infty} j^{n}[ja_j - \alpha b_j]$ we have that from

(7)

\[
\sum_{j=2}^{\infty} j^{n}[ja_j + (2 - \alpha)b_j] < 1 - \alpha
\]
we obtain $\text{Re} \frac{D^{n+1}f(z)}{D^n g(z)} > \alpha$.

Now let take $f \in T$ and $g \in T_n(0)$ for which the relation (4) hold.

The condition $\text{Re} \frac{D^{n+1}f(z)}{D^n f(z)} > \alpha$ is equivalent with

(8) \[ \alpha - \text{Re} \left( \frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right) < 1 \]

We have

\[
\alpha - \text{Re} \left( \frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right) \leq \alpha + \left| \frac{D^{n+1}f(z)}{D^n g(z)} - 1 \right| = \\
= \alpha + \left| \frac{1 - \sum_{j=2}^{\infty} j^n a_j z^{j-1}}{1 - \sum_{j=2}^{\infty} j^n b_j z^{j-1}} - 1 \right| \leq \sum_{j=2}^{\infty} j^n |b_j - ja_j| \cdot |z|^{j-1} \\
\leq \sum_{j=2}^{\infty} j^n |b_j - ja_j| \leq \sum_{j=2}^{\infty} j^n (b_j + ja_j) \\
= \alpha + \sum_{j=2}^{\infty} j^n [ja_j + (1 - \alpha) b_j] \\
= \frac{\alpha + \sum_{j=2}^{\infty} j^n [ja_j + (1 - \alpha) b_j]}{1 - \sum_{j=2}^{\infty} j^n b_j} \\
\leq \sum_{j=2}^{\infty} j^n [ja_j + (2 - \alpha) b_j] < 1
\]

that is the condition (4).
Remark 3.1. If we take $f \equiv g$ we obtain from Theorem 3.1

$$\sum_{j=2}^{\infty} j^na_j[ja_j + 2 - \alpha] < 1 - \alpha$$

From $\sum_{j=2}^{\infty} j^na_j[j + 2 - \alpha] > \sum_{j=2}^{\infty} j a_j(j - \alpha)$ we obtain:

$$\sum_{j=2}^{\infty} j a_j(j - \alpha) < 1 - \alpha$$

Thus we obtain the result from Remark 2.7.

Remark 3.2. From the proof of the Theorem 3.1 we obtain a necessary condition for a function $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_jz^j$ to be in the class $CCT_n(\alpha)$, $\alpha \in [0,1)$, $n \in \mathbb{N}$, with respect to the function $g \in T_n(0)$, $g(z) = z - \sum_{j=2}^{\infty} b_jz^j$:

$$\sum_{j=2}^{\infty} j^na_j - \alpha b_j < 1 - \alpha.$$  

Theorem 3.2. If $F \in CCT_n(\alpha), \alpha \in [0,1), n \in \mathbb{N}$, with respect to the function $G \in T_n(0)$ and $f = I_c(F)$, $g = I_c(F)$ where $I_c$ is defined by (2), then $f \in CCT_n(\alpha)$ with respect to the function $g \in T_n(0)$ (see Remark 2.4)

Proof. From $F(z) = z - \sum_{j=2}^{\infty} a_jz^j$, $a_j \geq 0, j = 2, 3, ...$ and $f(z) = I_c(F)(z)$ we have (see (3)):

$$f(z) = z - \sum_{j=2}^{\infty} \alpha_jz^j$$

where $\alpha_j = \frac{c+1}{c+j}a_j$, $j = 2, 3, ...$
From $G(z) = z - \sum_{j=2}^{\infty} b_j z^j, b_j \geq 0, j = 2, 3, ...$ and $g(z) = I_e(G)(z)$ we have:

$$g(z) = z - \sum_{j=2}^{\infty} \beta_j z^j,$$

where $\beta_j = \frac{c+1}{c+j} b_j, \ j = 2, 3, ...$

From $F \in CCT_n(\alpha)$ with respect to the function $G \in T_n(0)$ we have (see Theorem 3.1):

$$\sum_{j=2}^{\infty} j^n[ja_j + (2 - \alpha) b_j] < 1 - \alpha. \quad (9)$$

From Theorem 3.1 we need only to show that:

$$\sum_{j=2}^{\infty} j^n[ja_j + (2 - \alpha) \beta_j] < 1 - \alpha.$$  

We have for $c \in (-1, \infty)$ and $j = 2, 3, ...$:

$$\sum_{j=2}^{\infty} j^n[ja_j + (2 - \alpha) \beta_j] =$$

$$= \sum_{j=2}^{\infty} \frac{c+1}{c+j} j^n[ja_j + (2 - \alpha) b_j] < \sum_{j=2}^{\infty} j^n[ja_j + (2 - \alpha) b_j]$$

From (9) we have:

$$\sum_{j=2}^{\infty} j^n[ja_j + (2 - \alpha) \beta_j] < 1 - \alpha.$$  

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