New coefficient inequalities for starlike and convex functions

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Abstract

The object of the present paper is to derive new coefficient inequalities for univalent and starlike, and univalent and convex functions defined in the open unit disk $U$. Our results are the improvements of the previous theorems given by J. Clunie and F.R. Keogh ([1]) and by H. Silverman ([2]).

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1 Introduction

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (a_1 = 1)$$
which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). A function \( f(z) \in A \) is said to be univalent and starlike in \( U \) if it satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0
\]
for all \( z \in U \). Also a function \( f(z) \in A \) is said to be univalent and convex in \( U \) if it satisfies
\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0
\]
for all \( z \in U \).

Clunie and Keogh ([1]) (also Silverman ([2])) have proved the following result: If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n|a_n| \leq 1,
\]
then \( f(z) \) is univalent and starlike in \( U \). If \( f(z) \in A \) satisfies
\[
\sum_{n=2}^{\infty} n^2|a_n| \leq 1,
\]
then \( f(z) \) is univalent and convex in \( U \).

In the present paper, we consider new coefficient inequalities for functions \( f(z) \) to be univalent and starlike, and univalent and convex in \( U \).

### 2 Coefficient inequalities

Our main result for the coefficient inequality of \( f(z) \) to be univalent and starlike in \( U \) is contained in

**Theorem 1.** Let \( f(z) \) be in the class \( A \) and
\[
\max_{n \geq 1} |a_n| = p|a_p|.
\]
If $f(z)$ satisfies
\[ \sum_{n=1, n \neq p}^{\infty} (|n - p| + p)|a_n| \leq |p|a_p, \]
then $f(z)$ is univalent and starlike in $U$.

**Proof.** Applying the maximum principle of analytic functions, the following inequality holds true on $|z| = 1$
\[ |zf'(z) - pf(z)| - |pf(z)| = \left| \sum_{n=1}^{\infty} (n - p)a_n z^n \right| - p \left| \sum_{n=1}^{\infty} a_n z^n \right| \leq \sum_{n=1}^{\infty} a_n z^n |n - p||a_n||z^n| - p \left( |a_p||z^p| - \sum_{n=1, n \neq p}^{\infty} |a_n||z^n| \right) = \sum_{n=1, n \neq p}^{\infty} (|n - p| + p)|a_n| - p|a_p| \leq 0. \]

Therefore, it follows that
\[ \left| \frac{zf'(z)}{f(z)} - p \right| < p \]
for all $z \in U$. This shows that $f(z)$ is univalent and starlike in $U$.

**Remark 1.** If
\[ \max_{n \geq 1} |a_n| = |a_1| = 1, \]
then Theorem 1 becomes the result by Clunie and Keogh ([1]) (also by Silverman([2])).

**Corollary 1.** If a function $f(z) \in A$ satisfies
\[ \max_{n \geq 1} n|a_n| = 2|a_2| \]
and
\[ \sum_{n=3}^{\infty} n|a_n| \leq 2|a_2| - 3, \]
then $f(z)$ is univalent and starlike in $U$. 
By means of the definition between starlike functions and convex functions, it follows that \( f(z) \in A \) is univalent and convex in \( U \) if and only if \( zf'(z) \) is univalent starlike in \( U \). Therefore Theorem 1 gives us

**Theorem 2.** Let \( f(z) \) be in the class \( A \) and

\[
\max_{n \geq 1} n^2 |a_n| = p^2 |a_p|.
\]

If \( f(z) \) satisfies

\[
\sum_{n=1, n \neq p}^{\infty} n(|n - p| + p)|a_n| \leq p^2 |a_p|,
\]

then \( f(z) \) is univalent and convex in \( U \).

**Remark 2.** If

\[
\max_{n \geq 1} n^2 |a_n| = |a_1| = 1,
\]

then Theorem 2 becomes the result by Silverman ([2]).

**Corollary 2.** If a function \( f(z) \in A \) satisfies

\[
\max_{n \geq 1} n^2 |a_n| = 4|a_2|
\]

and

\[
\sum_{n=3}^{\infty} n|a_n| \leq 4|a_2| - 3,
\]

then \( f(z) \) is univalent and convex in \( U \).

**References**


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