A certain class of quadratures

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

Our aim is to investigate a quadrature of form:

\[ \int_0^1 f(x)dx = c_1f(x_1)+c_2f(x_2)+c_3f(x_3)+c_4f(x_4)+c_5f(x_5)+R(f) \] (1)

where \( f : [0, 1] \to \mathbb{R} \) is integrable, \( R(f) \) is the remainder-term and the distinct knots \( x_j \) are supposed to be symmetric distributed in \([0, 1]\). Under the additional hypothesis that all \( x_j \) are of rational type (see(4)), we are interested to find maximum degree of exactness of such quadrature.

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1 Introduction

Let $\prod_m$ be the linear space of all real polynomials of degree $\leq m$ and denote $e_j(t) = t^j, j \in \mathbb{N}$. A quadrature of form

$$\int_0^1 f(x)dx = \sum_{k=0}^n c_k f(x_k) + R(f)$$

has degrees (of exactness) $m$ if $R(h) = 0$ for any polynomial $h \in \prod_m$. If $R(h) = 0$ for all $h \in \prod_m$ and moreover $R(e_{m+1}) \neq 0$ it is said that (2) has the exact degree $m$. It is known that if (2) has degree $m$, then $m \leq 2n - 1$. Likewise, there exists only one formula (2) having maximum degree $2n - 1$.

The aim of this paper is to study the formulas like (2) for $n = 5$ having some practical properties. Let us note that in this case, the optimal formula having maximum degree $m = 9$ is

$$\int_0^1 f(x)dx = \sum_{k=1}^5 c_k f(x_k) + r(f)$$

$$x_k = \frac{1}{2} \pm \frac{1}{6} \sqrt{5 \pm 2 \sqrt{\frac{10}{7}}}, 1 \leq k \leq 4, x_5 = \frac{1}{2}$$

It is clear that not all knots $x_k$ are rational numbers.

**Definition 1.** Formula (1) is said to be of “practical-type”, if

i) the knots $x_j$ are of form

$$x_1 = r_1, x_2 = r_2, x_3 = \frac{1}{2}, x_4 = 1 - r_2, x_5 = 1 - r_1$$

where $r_1, r_2$ distinct rational numbers from $\left[0, \frac{1}{2}\right]$

ii) all coefficients $c_1, c_2, c_3, c_4, c_5$ are rational numbers with $c_1 = c_5$ and $c_2 = c_4$. 
iii) (1) is of order \( p \), with \( p \geq 1 \). Therefore, in case \( n = 5 \) a practical-type formula has the form

\[
(5) \int_{0}^{1} f(x)dx = A(f(r_1)+f(1-r_1))+B(f(r_2)+f(1-r_2))+C \cdot f\left(\frac{1}{2}\right) + R(f)
\]

\( A, B \) being rational numbers, \( C = 1 - 2(A + B) \), and when \( r_1, r_2 \) are distinct rational numbers from \([0, \frac{1}{2})\).

**Lemma 1.** Let \( s \) be a natural number and suppose in (5) we have \( R(h) = 0 \) for all \( h \in \prod_{2s} \). Then \( R(g) = 0 \) for every \( g \) from \( \prod_{2s+1} \).

**Proof.** Let \( H(x) = \left(x - \frac{1}{2}\right)^{2s+1} \). According to symmetry \( \int_{0}^{1} H(x)dx = 0 \) and also \( R(H) = 0 \). Observe that \( e_{2s+1}(x) \equiv x^{2s+1} = H(x) + h_1(x) \) with \( h_1 \in \prod_{2s} \). Therefore \( R(e_{2s+1}) = 0 \) and supposing \( g \in \prod_{2s+1} \) with \( g(x) = a_0x^{2s+1} + ... \), we have \( R(g) = a_0 \cdot R(e_{2s+1}) + R(h_2), h_2 \in \prod_{2s} \), that is \( R(g) = 0 \).

**Lemma 2.** If in (5) we have \( R(h) = 0 \) for every polynomial of degree \( \leq 4 \), then

\[
(6) \quad A = \frac{10r_2^2 - 10r_2 + 1}{60(1 - 2r_1)^2(r_1 - r_2)(1 - r_1 - r_2)}
\]

\[
B = \frac{10r_2^2 - 10r_1 + 1}{60(1 - 2r_2)^2(r_2 - r_1)(1 - r_1 - r_2)}
\]

\[
C = \frac{8 + 40(r_1^2 + r_2^2) - 40(r_1 - r_2) + 240r_1r_2(1 - r_1 - r_2 + r_1r_2)}{15(1 - 2r_1)^2(1 - 2r_2)^2}
\]

**Proof.** We use standard method, namely by considering polynomials

\[
l_j = \frac{\omega(x)}{(x - x_j)\omega'(x_j)}, \ j \in \{1, 2, 3, 4, 5\}, \ \omega(x) = \prod_{k=1}^{5}(x - x_k)
\]
For instance, taking into account that
\[ \omega'(x) = -\frac{1}{4}(1 - 2r_1)^2(r_1 - r_2), \text{ with } \delta = \frac{1}{2} \]
are found
\[ 0 = R(l_1) = \int_{0}^{1} l_1(x)dx - A l_1(x_1) \]
and we conclude with
\[ A = \frac{1}{\omega'(x_1)} \int_{-\frac{1}{2}}^{\frac{1}{2}} t[t - (1 - 2r_1)h][t^2 - (1 - 2r_2)^2h^2]dt = \]
\[ = \frac{10r_2^2 - 10r_2 + 1}{60(1 - 2r_1)^2(r_1 - r_2)(1 - r_1 - r_2)} \]

In a similar way are found coefficients B and C. Taking into account that (5) is symmetric, we give:

**Corollary 1.** Quadrature formula (5) has order, \( m \geq 5 \), if and only if the coefficients are given by (6).

**Lemma 3.** If (5) has order \( m \), \( m \geq 6 \), then \( r_1, r_2 \) must be distinct rational numbers from \( (0, 1] \) such that

\[ 560r_1^2r_2^2 + 56(r_1^2 + r_2^2) - 56(r_1 + r_2) + 560r_1r_2(1 - r_1 - r_2) + 5 = 0. \]

**Proof.** It is sufficient to impose condition \( R(e_6) = 0, e_6(x) = x^6 \). By considering \([a, b] = [-1, 1]\), are found \( R(e_6) = \frac{1}{7} - 2Ar_1^6 - 2Br_2^6 = 0 \). Using Lemma 2, see (6) we obtain condition (7).
Corollary 2. Suppose that (5) is of practical-type. If \( r_1, r_2 \) are distinct rational numbers from \((0, 1]\) such that equalities (6) and (7) are verified, then (5) has order \( m = 7 \).

Let us remark, that the above proposition implies that
\[
r_1 + r_2 - 2r_1r_2 \geq \frac{2}{7}
\]

Corollary 3. The maximum order of \( m \) of practical-type quadratus formula at 5-knots satisfied \( m \leq 7 \).

Proof. Formulas like (7) having order \( m = 8 \) does not exist. The reason is that by assuming \( m \geq 8 \), then according to Lemma 1 we must have \( m = 9 \). But in this case numbers \( r_1 \) and \( r_2 \) are not rational (see (3)).

Lemma 4. Then does not exist pairs of rational numbers \((r_1, r_2)\) which satisfy
\[
560r_1^2r_2^2 + 56(r_1^2 + r_2^2) - 56(r_1 + r_2) + 560r_1r_2(1 - r_1 - r_2) + 5 = 0.
\]

Proof. The case \((1 - 2r_1)(1 - 2r_2) = 0\) is impossible. Further, consider
\[
(1 - 2r_1)(1 - 2r_2) \neq 0
\]
and let \( 1 - 2r_1 = \frac{p}{z}, 1 - 2r_2 = \frac{x}{y} \). \( p, q, x, y, \in \mathbb{Z} \), \( q > 0, y > 0 \), with \((p, q) = 1\), \((x, y) = 1\).

Because \((1 - 2r_2)^2 = \frac{3[5 - 7(1 - 2r_1)^2]}{7[3 - 5(1 - 2r_1)^2]}\), we obtain
\[
7x^2(3q^2 - 5p^2) = 3y^2(5q^2 - 7p^2).\]
It follows that \( x^2 \equiv 0 \mod{3} \) or \( p^2 \equiv 0 \mod{3} \). Therefore \( x \) or \( p \) is divisible by 3, \( x = 0 \mod{3} \), \( x = 3k \) with \( k \in \mathbb{Z} \). Then after dividing by 3, are finds \( y^2(5q^2 - 7p^2) = 3\cdot 7(3q^2 - 5p^2)\),
which means that \(5q^2 - 7p^2\) must be divisible by 3. From \((x, y) = 1\) it is clear that \(y\) is not divisible by 3. Now
\[
5q^2 - 7p^2 = 6(q^2 - p^2) - (q^2 + p^2) \equiv -(q^2 + p^2) \equiv 0 \pmod{3}
\]
implies \(p^2 + q^2 \equiv 0 \pmod{3}\) which is impossible unless \(p \equiv q \equiv 0 \pmod{3}\), which can’t happen because \((p, q) = 1\).

**Theorem 1.** The practical quadratures at five knots, having maximal degree of exactness \(m = 5\) are those of form
\[
(8) \int_0^1 f(x)dx = A[f(r_1) + f(1-r_1)] + B[f(r_2) + f(1-r_2)] + Cf\left(\frac{1}{2}\right) + R(f)
\]
where \(R(f)\) is remainder, \(r_1, r_2\) are distinct rational numbers from \((0, 1]\) and
\[
A = \frac{10r_2^2 - 10r_2 + 1}{60(1-2r_1)^2(r_2-r_1)(1-r_1-r_2)}
\]
\[
B = \frac{10r_1^2 - 10r_1 + 1}{60(1-2r_2)^2(r_2-r_1)(1-r_1-r_2)}
\]
\[
C = \frac{8 + 40(r_1^2 + r_2^2) - 40(r_1 - r_2) + 240r_1r_2(1 - r_1 - r_2 + r_1r_2)}{15(1-2r_1)^2(1-2r_2)^2}
\]
Let us note that in quadrature formula from (8) we have
\[
R(c_6) = \frac{560r_1^2r_2^2 + 56(r_1^2 + r_2^2) - 56(r_1 + r_2) + 560r_1r_2(1 - r_1 - r_2) + 5 \cdot \frac{1}{2^6}}{105}
\]
If by \([z_0, z_1, ..., z_k; f]\) is denoted the difference of a function \(f : [0, 1] \to \mathbb{R}\) at a system of distinct points \(\{z_0, z_1, ..., z_k\} \subset [0, 1]\), it may be shown that.

**Theorem 2.** Any partial quadratures at five knots, having maximal degree \(m = 5\) may be written as
\[
(9) \int_0^1 f(x)dx = f\left(\frac{1}{2}\right) + \frac{1}{12} \left[ r_1, \frac{1}{2}, 1 - r_1; f \right] + \frac{3 - 5(1 - 2r_1)^2}{240}.
\]
A certain class of quadratures

\[
\cdot \left[ r_1, r_2, \frac{1}{2}, 1 - r_2, 1 - r_1; f \right] + R(f),
\]

where \( r_1, r_2 \) are distinct rational numbers from \((0, 1)\]

2 Examples

In the following of \( R_j(f), j \in \mathbb{N}^* \), we shall denote the remainders terms in certain quadratures formulas.

**Example 1.** The closed formulas like (8) are obtained in case \( r_2 = 1 \), namely

\[
\int_0^1 f(x)dx = A_0[f(0) + f(1)] + C_0 f\left(\frac{1}{2}\right) + B_0[f(r) + f(1-r)] + R_1(f)
\]

where \( r \in \mathbb{Q}, r \in (0, 1), R_1(e_6) = \frac{14(1 - 2r)^6 - 6}{105 \cdot 2^6} \) and

\[
A_0 = \frac{1}{6} - \frac{1}{15(1-2r)^2}; \quad B_0 = \frac{1}{60r(1-2r)^2(1-r)}; \quad C_0 = \frac{3}{2} - \frac{2}{15(1-2r)^2}.
\]

**Example 2.** For instance, when \((r_1, r_2) = \left(1; \frac{1}{2}\right)\) , (10) gives

\[
\int_0^1 f(x)dx = \frac{7}{90} [f(0) + f(1)] + \\
+ \frac{16}{25} \left[ f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + \frac{2}{15} f\left(\frac{1}{2}\right) + R_2(f) \right] + R_2(e_6) = \frac{1}{21 \cdot 2^7}
\]

**Example 3.** In case \((r_1, r_2) = \left(\frac{1}{2}; \frac{1}{4}\right)\) are found

\[
\int_0^1 f(x)dx = \frac{86}{45} \left[ f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) \right] - \frac{224}{45} \left[ f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) \right] + \\
\]
\[ +\frac{107}{15} f \left( \frac{1}{2} \right) + R_3(f) \]

\[ R_3(e_6) = \frac{115}{21 \cdot 2^{12}} \]

### 3 The remainder term

In order to investigate the remainder we use same methods as in [1] – [6].

**Theorem 3.** Let \( m = \frac{1}{2}, h = \frac{1}{2}, x_1 = r_1, x_2 = r_2, x_3 = \frac{1}{2}, x_4 = 1 - r_2, x_5 = 1 - r_1. \)

If \( \Omega(t) = \left[ t^2 - (1 - 2r_1)^2 \cdot \frac{1}{4} \right] \left[ t^2 - (1 - 2r_2)^2 \cdot \frac{1}{4} \right]. \)

\begin{equation}
R(f) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} t^2 \Omega(t) \left[ \frac{1}{2} - t, r_1, r_2, \frac{1}{2}, 1 - r_2, 1 - r_1, \frac{1}{2} + t; f \right] dt
\end{equation}

**Proof.** Let \( \omega(x) = \prod_{j=1}^{5} (x - x_j). \) Because our formula (8) is of interpolatory type, it follows that we have

\[ \int_{0}^{1} f(x) dx = \int_{0}^{1} L_4(x_1, x_2, x_3, x_4, x_5; f) dx + R(f) \]

where \( R(f) = \int_{0}^{1} \omega(x)[x, x_1, x_2, x_3, x_4, x_5; f] dx. \)

But \( \int_{0}^{1} f(1-x) dx = \int_{0}^{1} f(x) dx \) and using the symmetry of knots \( \{x_1, x_2, ..., x_5\} \) we have

\[ L_4 \left( r_1, r_2, \frac{1}{2}, 1 - r_2, 1 - r_1; f | 1 - x \right) = L_4 \left( r_1, r_2, \frac{1}{2}, 1 - r_2, 1 - r_1; f | x \right). \]
Further, the equality $\omega(1 - x) = -\omega(x)$ gives

$$R(f) = -\int_0^1 \omega(x) \left[ 1 - x, r_1, r_2, \frac{1}{2}; 1 - r_2, 1 - r_1; f \right] dx$$

Therefore the remainder from (8) may be written as $R(f) = \frac{1}{2} \int_0^1 \omega(x) D(f; x) dx$

with

$$D(f; x) = \left[ x, r_1, r_2, \frac{1}{2}; 1 - r_2, 1 - r_1; f \right] - \left[ 1 - x; r_1, r_2, \frac{1}{2}; 1 - r_2, 1 - r_1; f \right] =$$

$$= 2 \left( x - \frac{1}{2} \right) \left[ x, r_1, r_2, \frac{1}{2}; 1 - r_2, 1 - r_1; f \right]$$

In this manner

$$R(f) = \int_0^1 \left( x - \frac{1}{2} \right) \omega(x) \left[ x, r_1, r_2, \frac{1}{2}; 1 - r_2, 1 - r_1; f \right] dx$$

which is the same with (13).

Further for $g \in C[0, 1]$ we use the uniform norm $||g|| = \max_{x \in [a, b]} |g(x)|$.

**Corollary 4.** Let us denote

$$\omega(x) = (x-r_1)(x-r_2)(x-1+r_1)(x-1+r_2), J(r_1, r_2) = \int_0^1 \left( x - \frac{1}{2} \right)^2 |\omega(x)| dx$$

If $R(f)$ is the remainder in (8), then for $f \in C^6[0, 1]$

(14) $|R(f)| \leq \frac{1}{46080} J(r_1, r_2) ||f^{(6)}||$. 

References


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