Subdivision Schemes for Geometric Modelling

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• Sep 5 – Subdivision as a linear process
  • basic concepts, notation, subdivision matrix
• Sep 6 – The Laurent polynomial formalism
  • algebraic approach, polynomial reproduction
• Sep 7 – Smoothness analysis
  • Hölder regularity of limit by spectral radius method
• Sep 8 – Subdivision surfaces
  • overview of most important schemes & properties
The functional setting

- initial data \( f^0 = (f_i^0)_{i \in \mathbb{Z}} \)
- mask \( a = (a_i)_{i \in \mathbb{Z}} \)
- refinement rule \( f_i^{j+1} = \sum_k a_{i-2k} f_k^j \)
- parameter values \( (t_i^j)_{i \in \mathbb{Z}, j \in \mathbb{N}} \)
- piecewise linear functions \( F^j \) with \( F^j(t_i^j) = f_i^j \)
- limit function \( S_\alpha^\infty f^0 = \lim_{j \to \infty} F^j \)
- consider initial data \( \delta^0 = (\delta_i^0)_{i \in \mathbb{Z}} = (\ldots, 0, 1, 0, \ldots) \)
- basic limit function \( \phi_a = S_\alpha^\infty \delta^0 \)
- by linearity of the scheme \( S_\alpha^\infty f^0 = \sum_k \phi_a(\cdot - k) f_k^0 \)
Examples

- polygon subdivision
- Chaikin’s corner cutting
- cubic B-spline scheme
- interpolating 4-point scheme
Convergence

- if the sequence \((F^j)_{j \in \mathbb{N}}\) of piecewise linear functions converges (uniformly) for any initial data, then the scheme \(S_a\) is called \textit{convergent}.
- the limit is necessarily a continuous function.
- necessary conditions for \(S_a\) to be convergent:
  - even/odd coefficients of the mask sum to 1:
    \[ a(z) = (1+z) b(z) \quad \text{and} \quad b(1) = 1 \]
  - 1 is the single dominant eigenvalue of the local subdivision matrix.
Example

- scheme with mask \( a = [-7, 7, 16, 16, 7, -7]/16 \)
- even/odd coefficients sum to 1
- local subdivision matrix
  \[
  S = \begin{pmatrix}
  7 & 16 & -7 & 0 \\
  -7 & 16 & 7 & 0 \\
  0 & 7 & 16 & -7 \\
  0 & -7 & 16 & 7
  \end{pmatrix}
  \]
- eigenvalues: \( 1, \frac{7}{8}, \frac{1}{2} \pm \frac{i}{4}\sqrt{10} \)
- necessary conditions for convergence satisfied
- still, the scheme does not converge
- more analysis needed!
Theorem

the scheme $S_a$ converges, if and only if the scheme $S_b$ is contractive, i.e. $S_b^\infty f^0 = 0$ for any initial data.

- remember: $S_b$ is the scheme for the differences

the scheme $S_b$ is contractive, if

$$\max_{i \in \mathbb{Z}} |f_i^{j+1}| \leq \mu \max_{i \in \mathbb{Z}} |f_i^j|, \quad \mu < 1$$

and that is the case if

$$||b|| = \max\left(\sum_i |b_{2i}|, \sum_i |b_{2i+1}|\right) < 1$$
Examples

- **general primal 3-point** \( a = [ w, \frac{1}{2}, 1 - 2w, \frac{1}{2}, w ] \)
  - difference scheme \( b = [ w, \frac{1}{2} - w, \frac{1}{2} - w, w ] \)
  - \( ||b|| = |w| + |\frac{1}{2} - w| < 1 \) for \( w \in (-\frac{1}{4}, \frac{3}{4}) \)
  - \( S_b \) is contractive, hence the scheme converges

- **scheme with mask** \( a = [ -7, 7, 16, 16, 7, -7 ]/16 \)
  - difference scheme \( b = [ -7, 14, 2, 14, -7 ]/16 \)
  - \( ||b|| = \max (7 + 2 + 7, 14 + 14)/16 = 7/4 > 1 \)
  - \( S_b \) is not contractive, but ...
Example

- scheme with mask $a = [-1, 1, 8, 8, 1, -1] / 8$
  - difference scheme $b = [-1, 2, 6, 2, -1] / 8$
  - $\| b \| = \max (1+6+1, 2+2) / 8 = 1$
  - $S_b$ is not contractive, but ...

- consider 2 steps of the scheme, i.e. the scheme $S_b^2$ with symbol $b(z)b(z^2)$
  - mask $b^2 = [1, -2, -8, 2, 7, 16, 32, 16, 7, 2, -8, -2, 1] / 64$
  - $\| b^2 \| = \max (1+7+7+1, 2+16+2, 8+32+8) / 64 < 1$
  - $S_b^2$ is contractive, hence the scheme converges
Theorem

the scheme $S_a$ converges, if and only if the scheme $S_b$ is contractive

the scheme $S_b$ is contractive, if $\|b^\ell\| < 1$ for some $\ell > 0$, with

$$\|b^\ell\| = \max\left\{\sum_i |b_{k-2\ell i}^\ell| : 0 \leq k < 2^\ell\right\}$$

where $b_{i}^{\ell}$ are the coefficients of the scheme $S_{b}^{\ell}$ with symbol $b^\ell(z) = b(z) b(z^2) \cdots b(z^{2^{\ell-1}})$
Smoothness

- **Theorem**

  if the scheme $S_b$ converges, then the limit curves of the scheme $S_a$ with symbol

  $$a(z) = \left(\frac{1 + z}{2}\right)^m b(z)$$

  are $C^m$-continuous

- $S_b$ is the scheme for the $m$-th divided differences and

  $$\left(S_a^\infty f^0\right)^{(m)} = S_b^\infty (\Delta^m f^0)$$
Example

4-point scheme
- symbol: $a(z) = \frac{1+z}{2}b(z), \quad b(z) = \frac{-1+4z-z^2}{8z^3}(1+z)^3$
- mask of $S_b: b = [-1, 1, 8, 8, 1, -1]/8$
- this scheme converges (see above)
- the limit curves of the 4-point scheme are $C^1$-continuous

To check $C^2$-continuity, consider $a(z) = \frac{(1+z)^3}{4}c(z)$
- but $c = [-1, 3, 3, -1]/4 \Rightarrow \|c\| = 1$
  and $c^2 = [1,-3,-6,10,6,6,10,-6,-3,1]/16 \Rightarrow \|c^2\| = 1$
- likewise for $c^\ell \Rightarrow S_c$ not contractive $\Rightarrow$ no $C^2$-continuity
Definition

A function $\phi$ is called Hölder regular of order $n + \alpha$ $(n \in \mathbb{N}, 0 < \alpha \leq 1)$, if it is $n$ times continuously differentiable and $\phi^{(n)}$ is Lipschitz of order $\alpha$, i.e.

$$|\phi^{(n)}(x + h) - \phi^{(n)}(x)| \leq c |h|^\alpha$$

for all $x$ and $h$ and some constant $c$.

- Remember: A function that is Lipschitz of order 1 is not necessarily differentiable.
- Hölder regularity of order $n + 1$ is weaker than being $n + 1$ times differentiable.
Theorem

the scheme $S_a$ with symbol $a(z) = \left(\frac{1+z}{2}\right)^m b(z)$ generates limit curves with Hölder regularity $r \geq m - \log_2(\|b^{\ell}\|)/\ell$ for any $\ell$

Examples

- 4-point scheme $a(z) = \left(\frac{1+z}{2}\right)^4 \frac{1+4z-z^2}{z^2}$
  
  - $m=4$, $b=[-1,4,-1] \implies r \geq 4 - \log_2(4) = 2$

- cubic B-spline scheme $a(z) = \left(\frac{1+z}{2}\right)^4 \frac{2}{z^2}$
  
  - $m=4$, $b=[2] \implies r \geq 4 - \log_2(2^\ell)/\ell = 3$
Example

- general primal 3-point
  \[ a(z) = \left(\frac{1+z}{2}\right)^2 4w + (2-8w)z + 4wz^2 \]

- \( m=2, \ell=1, b=[4w, 2-8w, 4w] \Rightarrow r \geq 2 - \log_2(||b||) \)

- \( m=2, \ell=2, b^2=[16w^2, 8w(1-4w), 8w(1-2w), 4(1-4w)^2, ...] \Rightarrow r \geq 2 - \log_2(||b^2||)/2 \)

- larger \( \ell, ... \)
suppose the mask \( a \) is supported on \([0, N]\), i.e. \( a_i = 0 \) for \( i < 0 \) and \( i > N \)

- all masks are of this kind after an index shift

refine the initial data \( f^0 = (\ldots, 0, 1, 0, \ldots) \)

- remember: \( f^{i+1}_i = \sum_k a_{i-2k} f^i_k \)
- hence, \( f^1 \) is supported on \([0, N]\)
- likewise, \( f^j \) is supported on \([0, (2^j-1)N]\)

assume primal parameterization \( t^j_i = i/2^j \)
- the support of the basic limit function \( \phi_a \) is \([0, N]\)
for arbitrary initial data \( f^0 \), the limit function is

\[
S^\infty_{\alpha} f^0 = \sum_k \phi_a(\cdot - k) f^0_k
\]

assume the support of \( \phi_a \) is \([0, N]\), then the values of the limit function \( S^\infty_{\alpha} f^0 \)
on \([0,1]\) are determined by the \( N \) control points \( f^0_{-N+1}, f^0_{-N+2}, \ldots, f^0_0 \)
on \([0, \frac{1}{2}]\) are determined by \( f^1_{-N+1}, f^1_{-N+2}, \ldots, f^1_0 \)
on \([\frac{1}{2}, 1]\) are determined by \( f^1_{-N+2}, f^1_{-N+3}, \ldots, f^1_1 \)
Example

- cubic B-spline scheme with $N=4$

\[
\begin{pmatrix}
\vdots \\
{f}_{-3}^1 \\
{f}_{-2}^1 \\
{f}_{-1}^1 \\
{f}_0^1 \\
\vdots \\
\end{pmatrix}
= \begin{pmatrix}
\vdots \\
1 & 1 \\
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\
\frac{1}{16} & \frac{1}{16} \\
\vdots \\
\end{pmatrix}
\begin{pmatrix}
A_0 \\
A_1 \\
\vdots \\
\end{pmatrix}
\begin{pmatrix}
{u}_0^0 \\
{f}_{-3}^0 \\
{f}_{-2}^0 \\
{f}_{-1}^0 \\
{f}_0^0 \\
\vdots \\
\end{pmatrix}
\]

- $u_0^0 = (f_{-3}^0, \ldots, f_0^0)$ determines $S^\infty_\alpha f^0$ on $[0,1]$
- $A_0u^0$ determines the values on $[0,\frac{1}{2}]$
- $A_1u^0$ determines the values on $[\frac{1}{2},1]$
Another local limit analysis

- suppose the mask $a$ of a convergent scheme is *supported* on $[0,N]$
- consider the local $N \times N$ subdivision matrices $A_0, A_1$
- take any $x \in [0,1]$ with binary representation
  \[ x = 0.i_1i_2i_3i_4... \text{ with } i_k \in \{0,1\} \]
- then the limit value $S_\infty a f^0(x)$ is given ($N$ times) by
  \[ \cdots A_{i_4}A_{i_3}A_{i_2}A_{i_1}u^0 \]
  with $u^0 = (f^0_{-N+1}, \ldots, f^0_0)$
Joint spectral radius

**Definition**

The *joint spectral radius* of two matrices $A_0, A_1$ is

$$\rho(A_0, A_1) = \limsup_{k \to \infty} \left( \max \left\{ \|A_{i_k} \cdots A_{i_2} A_{i_1}\|_\infty^{1/k} : i_k \in \{0, 1\} \right\} \right)$$

- is bounded by the spectral radii and the norms of $A_0$ and $A_1$
  $$\max\{\rho(A_0), \rho(A_1)\} \leq \rho(A_0, A_1) \leq \max\{\|A_0\|_\infty, \|A_1\|_\infty\}$$
- does not depend on the chosen matrix norm
- is usually very hard to determine exactly
Theorem

the scheme $S_a$ with symbol $a(z) = \left(\frac{1+z}{2}\right)^m b(z)$ generates limit curves with Hölder regularity $r = m - \log_2(\mu)$, where $\mu$ is the joint spectral radius of the local matrices $B_0, B_1$ from the scheme $S_b$

- in practice, the lower and upper bounds on $\mu$ are used to get upper and lower bounds on $r$
- the lower bound then is the same as before, because $\|b^k\| = \max \left\{ \|B_{i_k} \cdots B_{i_2} B_{i_1}\|_\infty : i_k \in \{0, 1\} \right\}$
Hölder regularity

Example

- cubic B-spline scheme \( a(z) = \left(\frac{1+z}{2}\right)^4 \frac{2}{z^2} \)
  - \( b = [2] \) \( \Rightarrow \) \( B_0 = B_1 = (2) \) \( \Rightarrow \) \( \mu = 2 \) \( \Rightarrow \) \( r = 4 - \log_2(2) = 3 \)
  - scheme gives \( C^2 \) limit curves, whose second derivatives are Lipschitz of order 1; sometimes called \( C^{3-\epsilon} \)

- 4-point scheme \( a(z) = \left(\frac{1+z}{2}\right)^4 \frac{-1+4z-z^2}{z^2} \)
  - \( b = [-1, 4, -1] \) \( \Rightarrow \) \( B_0 = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}, \) \( B_1 = \begin{pmatrix} -1 & -1 \\ 4 & \end{pmatrix} \)
  - \( \|B_0\| = \|B_1\| = \rho(B_0) = \rho(B_2) = 4 = \mu \) \( \Rightarrow \) \( r = 4 - \log_2(4) = 2 \)
  - scheme gives \( C^{3-\epsilon} \) limit curves
Hölder regularity

**Example**

- dual 4-point scheme
  - evaluate local cubic interpolant in a dual fashion
  - \( a = [-5, -7, 35, 105, 105, 35, -7, -5] / 128 \)
  - divide \( m = 5 \) times by \( (1 + z)/2 \) \( \Rightarrow \) \( b = [-5, 18, -5]/4 \)

\[
B_0 = \begin{pmatrix}
\frac{9}{2} \\
-\frac{5}{4} \\
-\frac{5}{4}
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
-\frac{5}{4} \\
-\frac{5}{4} \\
\frac{9}{2}
\end{pmatrix}
\]

- \( \|B_0\| = \|B_1\| = \rho(B_0) = \rho(B_1) = 4.5 = \mu \) \( \Rightarrow \) \( r = 5 - \log_2(4.5) \)
- scheme gives \( C^{2.83} \) limit curves
- basic limit function \( \phi_a \) and support size
  - if all but \( N+1 \) consecutive mask coefficients are zero, then \( N \) is the support size of the mask and the basic limit function

- a scheme \( S_a \) converges if the difference scheme \( S_b \) is contractive
  - the norm of \( b \) or the \( \ell \)-iterated scheme \( b^\ell \) is less than one

- a scheme is \( C^m \)-continuous, if the scheme for the \( m \)-th divided differences converges
the norm of $b^\ell$ leads to a lower bound on the Hölder regularity of the limit functions

lower and upper bound are given by joint spectral radius analysis

- given a scheme $S_a$, divide $a(z)$ by as many factors $(1+z)/2$ as possible, say $m$ such factors
- for the remaining scheme $S_b$ with support size $N$, consider the local $N \times N$ subdivision matrices $B_0, B_1$
- determine the joint spectral radius $\mu = \rho(B_0, B_1)$
- Hölder regularity of limit curves is $r = m - \log_2(\mu)$