Subdivision Schemes for Geometric Modelling

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Sep 5 – Subdivision as a linear process
- basic concepts, notation, subdivision matrix

Sep 6 – The Laurent polynomial formalism
- algebraic approach, polynomial reproduction

Sep 7 – Smoothness analysis
- Hölder regularity of limit by spectral radius method

Sep 8 – Subdivision surfaces
- overview of most important schemes & properties
Schemes considered so far

- polygon subdivision
  - even stencil \([1]\)  odd stencil \([1,1]/2\)
  - interpolatory \(C^0\) limit curve (piecewise linear)

- cubic B-spline subdivision
  - even stencil \([1,6,1]/8\)  odd stencil \([1,1]/2\)
  - approximating \(C^2\) limit curve (piecewise cubic)

- 4-point scheme
  - even stencil \([1]\)  odd stencil \([-1,9,9,-1]/16\)
  - interpolatory \(C^1\) limit curve (non-polynomial)
Primal and dual schemes

- primal schemes
  - even and odd stencil are both symmetric
  - even stencil: modify old points
  - odd stencil: insert new points (between old points)
  - point → point, edge → point

- dual schemes
  - insert two new points (between old points)
  - discard old points
  - point → edge, edge → edge
Chaikin’s corner cutting

- **Example**

  \[ p^{j+1}_{2i} = \frac{3}{4} p^j_i + \frac{1}{4} p^j_{i+1}, \quad p^{j+1}_{2i+1} = \frac{1}{4} p^j_i + \frac{3}{4} p^j_{i+1} \]

- even stencil \([3,1]/4\)  
  odd stencil \([1,3]/4\)

- invariant neighbourhood size: 2

- local subdivision matrix \( S = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \)

- eigenvalues: 1, \(1/2\)

- limit stencil \([1,1]/2\)

- note: even/odd stencil are symmetric to each other
The subdivision mask

- refinement rules
  - even stencil \([..., \alpha_2, \alpha_1, \alpha_0, \alpha_{-1}, \alpha_{-2}, ...]\)
  - odd stencil \([..., \beta_2, \beta_1, \beta_0, \beta_{-1}, \beta_{-2}, ...]\)
  - rules
    \[p_{2i}^{j+1} = \sum_{k} \alpha_k p_{i-k}^j, \quad p_{2i+1}^{j+1} = \sum_{k} \beta_k p_{i-k}^j\]
  - combine stencils into subdivision mask
    \[a = [..., \alpha_2, \beta_1, \alpha_1, \beta_0, \alpha_0, \beta_{-1}, \alpha_{-1}, \beta_{-2}, \alpha_{-2}, ...]\]
    \[= [..., a_4, a_3, a_2, a_1, a_0, a_{-1}, a_{-2}, a_{-3}, a_{-4}, ...]\]
  - one single refinement rule
    \[p_{i}^{j+1} = \sum_{k} a_{i-2k} p_{k}^j\]
Masks of the schemes so far

- polygon subdivision
  \[ a = \left[ 1, 2, 1 \right]/2 \]

- Chaikin’s corner cutting
  \[ a = \left[ 1, 3, 3, 1 \right]/4 \]

- cubic B-spline scheme
  \[ a = \left[ 1, 4, 6, 4, 1 \right]/8 \]

- 4-point scheme
  \[ a = \left[ -1, 0, 9, 16, 9, 0, -1 \right]/16 \]
Definition

given a sequence \( c = \{ c_i : i \in \mathbb{Z} \} \), we call

\[
c(z) = \sum_{i \in \mathbb{Z}} c_i z^i
\]

the \( z \)-transform of \( c \)

- if \( c \) is finitely supported, then \( c(z) \) is a \textit{Laurent polynomial}

- for a subdivision scheme with mask \( a \), we call the Laurent polynomial \( a(z) \) the \textit{symbol} of the scheme
Symbols of the schemes so far

- polygon subdivision
  \[ a = \left[ 1, 2, 1 \right] / 2 \]
  \[ a(z) = \frac{1}{2z}(1 + z)^2 \]

- Chaikin’s corner cutting
  \[ a = \left[ 1, 3, 3, 1 \right] / 4 \]
  \[ a(z) = \frac{1}{4z^2}(1 + z)^3 \]

- cubic B-spline scheme
  \[ a = \left[ 1, 4, 6, 4, 1 \right] / 8 \]
  \[ a(z) = \frac{1}{8z^2}(1 + z)^4 \]

- 4-point scheme
  \[ a = \left[ -1, 0, 9, 16, 9, 0, -1 \right] / 16 \]
  \[ a(z) = \frac{-1 + 4z - z^2}{16z^3} (1 + z)^4 \]
**necessary condition** for convergence

- coefficients of even/odd stencil sum to 1

\[
\sum_{i \in \mathbb{Z}} \alpha_i = \sum_{i \in \mathbb{Z}} a_{2i} = 1, \quad \sum_{i \in \mathbb{Z}} \beta_i = \sum_{i \in \mathbb{Z}} a_{2i+1} = 1
\]

- equivalent to

\[
a(-1) = 0, \quad a(1) = 2
\]

- implies

\[
a(z) = (1 + z)b(z), \quad b(1) = 1
\]
Subdivision in terms of symbols

- consider the $z$-transform $p^j(z)$ of the data $\{p_i^j\}$ at level $j$, then the refinement rule can be written as

$$p_i^{j+1} = \sum_k a_{i-2k} p_k^j$$

can be written as

$$p^{j+1}(z) = a(z)p^j(z^2)$$

- very neat and compact way of writing the rule
- note: we are not interested in the polynomials as such, but rather in their coefficients
Subdivision of differences

- let $\Delta$ denote the (finite) *difference operator* on sequences

$$\Delta c = \{ c_i - c_{i-1} : i \in \mathbb{Z} \}$$

- if $a(z) = (1+z)$ $b(z)$ is the symbol of a convergent subdivision scheme, then $b(z)$ is the symbol of the scheme for the differences

$$\Delta p_{i}^{j+1} = \sum_{k} b_{i-2k} \Delta p_{k}^{j}$$
Subdivision of differences

Example

- cubic B-spline scheme

\[ a(z) = \frac{1}{8z^2}(1+z)^4 = (1+z)b(z), \quad b(z) = \frac{1}{8z^2}(1+z)^3 \]

- corresponding scheme for the differences
  - mask \( b = [1, 3, 3, 1]/8 \)
  - local subdivision matrix \( S = \begin{pmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{pmatrix} \)
  - eigenvalues: \( 1/2, 1/4 \)
  - maps differences (edge vectors) to 0
  - can we conclude that the scheme converges?
Symbols and masks

- multiplying a symbol by \((1+z)\)
  - write down the mask \([1, 2, 3, -1]\)
  - write it again, shifted to the left \([1, 2, 3, -1]\)
  - add both rows \([1, 3, 5, 2, -1]\)

- dividing a symbol by \((1+z)\)
  - check, if sum of odd/even coefficients is the same
  - write down the mask \([1, 4, 6, 4, 1]/8\)
  - copy first coefficient, then take differences \([1, 3, 3, 1]/8\)
Polynomial sequences

- **Definition**
  
  A sequence \( c = \{ c_i : i \in \mathbb{Z} \} \) is called *polynomial of degree \( d \)*, if there exists some polynomial \( \pi \) of degree \( d \) such that \( c_i = \pi(i) \) for all \( i \in \mathbb{Z} \).

- **Examples**

  - (..., 3, 3, 3, 3, ...) is of degree 0
  - (..., −2, 1, 4, 7, ...) is of degree 1

- If \( c \) is polynomial of degree \( d \), then
  
  \[ \Delta^{d+1} c = 0 \iff (1 - z)^{d+1} c(z) = 0 \]
Symbols and masks

- multiplying a symbol by \((1-\ z)\)
  - write down the mask \([1, -2, 3, -1]\)
  - write it negated, shifted to left \([-1, 2, -3, 1]\)
  - add both rows \([-1, 3, -5, 4, -1]\)

- dividing a symbol by \((1-\ z)\)
  - check, if coefficients add to zero
  - write down the mask \([1, -4, 6, -4, 1]/8\)
  - copy first coefficient negated, then take differences \([-1, 3, -3, 1]/8\)
• suppose the initial data $p^0$ is polynomial of degree $d$ and the symbol of the scheme is

$$a(z) = (1 + z)^{d+1} b(z)$$

then the refined data $p^j$ at any level $j$ is polynomial of degree $d$, and so is the limit curve

• in fact, condition $(\star)$ is necessary and sufficient for the scheme being able to *generate polynomials of degree $d$*
Example

- cubic B-spline scheme $a = [1, 4, 6, 4, 1]/8$
- symbol $a(z) = \frac{1}{8z^2}(1 + z)^4$
- generates polynomials up to degree 3
- initial data $p^0 = (..., 9, 4, 1, 0, 1, 4, 9, ...)$
- refined data $p^1 = (..., 10, 5, 2, 1, 2, 5, 10, ...) / 4$
  $p^2 = (..., 14, 9, 6, 5, 6, 9, 14, ...) / 16$
- limit curve is a quadratic polynomial, but not the one from which the initial data was sampled
- no polynomial reproduction
The functional setting

- initial data \( f^0 = (f^0_i)_{i \in \mathbb{Z}} \)
- mask \( a = (a_i)_{i \in \mathbb{Z}} \)
- refinement rule
  \[ f^{j+1}_i = \sum_k a_{i-2k} f^j_k \]
- parameter values \( (t^j_i)_{i \in \mathbb{Z}, j \in \mathbb{N}} \)
- piecewise linear functions \( F^j \) with \( F^j(t^j_i) = f^j_i \)
- limit function
  \[ \lim_{j \to \infty} F^j = S^\infty f^0 = S^\infty_a f^0 \]
Parameterizations

- **primal** parameterization
  \[ t_i^j = i / 2^j \]
  for **primal** schemes with **odd** symmetry
  \[ a_{-i} = a_i \]

- **dual** parameterization
  \[ t_i^j = \frac{1}{2} + (i - \frac{1}{2}) / 2^j \]
  for **dual** schemes with **even** symmetry
  \[ a_{-i} = a_{i-1} \]
Example

- Chaikin’s corner cutting: \( a = \left[ 1, 3, 3, 1 \right]/4 \)
- Initial data: \( f_i^0 = \pi(t_i^0) \), for \( \pi(x) = x \)
- Piecewise linear functions \( F^j \) with \( F^j(t_i^j) = f_i^j \)
- Does the scheme reproduce \( \pi \), i.e. \( \lim_{j \to \infty} F^j = \pi ? \)

- Primal parameterization
  \[
  t_i^j = i/2^j \quad \Rightarrow \quad \left( \lim_{j \to \infty} F^j \right)(x) = x + \frac{1}{2}
  \]

- Dual parameterization
  \[
  t_i^j = \frac{1}{2} + \left( i - \frac{1}{2} \right)/2^j \quad \Rightarrow \quad \left( \lim_{j \to \infty} F^j \right)(x) = x
  \]
for any scheme that \textit{generates} linear functions

with symbol \( a(z) = (1+z)^2 b(z) \)

let \( \tau = a'(1)/2 \)

attach data \( f^j_i \) to parameter \( t^j_i = -\tau + (i + \tau)/2^j \)

then the scheme also \textit{reproduces} linear functions

\section*{Examples}

- cubic B-spline scheme \( a'(1) = 0 \) \( \Rightarrow \) \( t^j_i = i/2^j \)

- Chaikin’s corner cutting

\[ a'(1) = -1 \quad \Rightarrow \quad t^j_i = \frac{1}{2} + (i - \frac{1}{2})/2^j \]
Polynomial reproduction

- reproduction of degree $d$ requires generation of degree $d$

- generation of degree $d$ is equivalent to
  - $a(z) = (1+z)^{d+1}b(z) \iff $ zero of order $d+1$ at $z=-1$

- **Theorem**
  the subdivision scheme with symbol $a(z)$ reproduces polynomials of degree $d$ w.r.t. the parameterization with $\tau = a'(1)/2$, if and only if
  
  $$a^{(k)}(-1) = 0, \quad a^{(k)}(1) = 2 \prod_{j=0}^{k-1} (\tau-j), \quad k = 0, \ldots, d$$
Polynomial reproduction

- **Examples**
  - **polygon subdivision** \( a = \left[ 1, 2, 1 \right]/2 \)
    - linear reproduction w.r.t. primal parameterization
  - **Chaikin’s corner cutting** \( a = \left[ 1, 3, 3, 1 \right]/4 \)
    - linear reproduction w.r.t. dual parameterization
  - **general primal 3-point** \( a = \left[ w, \frac{1}{2}, 1 - 2w, \frac{1}{2}, w \right] \)
    - linear reproduction w.r.t. primal parameterization
  - **an unsymmetric scheme** \( a = \left[ -1, 0, 6, 8, 3 \right]/8 \)
    - quadratic reproduction w.r.t. primal parameterization
    - note: this is an interpolating scheme!
A scheme that reproduces polynomials of degree $d$ has approximation order $d+1$

- Given a sufficiently smooth function $F$
- Take the initial data $f^0_i = F(ih)$
- Then

$$\| F - S^\infty_a f^0 \| \leq C h^{d+1}$$

- The constant $C$ does not depend on $h$
- combining even/odd stencils into the mask
  - one common subdivision rule
- use $\mathcal{Z}$-transform to turn mask into the symbol
  - formally, the symbol is a Laurent polynomial
  - transform data in the same way
  - yields an algebraic way to describe subdivision
  - necessary convergence condition: $a(-1)=0, a(1)=2$
- hands-on rules for multiplying and dividing masks by $(1+\mathcal{Z})$ and by $(1-\mathcal{Z})$
- polynomial generation of degree $d$
  \[ a(z) = (1+z)^{d+1}b(z) \]
  \[ \Leftrightarrow \quad a^{(k)}(-1) = 0 \quad \text{for} \quad k = 0, \ldots, d \]

- polynomial reproduction of degree $d$
  - requires polynomial generation of degree $d$
  - depends on the correct parameterization
    \[ t_i^j = -\tau + (i + \tau)/2^j \quad \text{with} \quad \tau = a'(1)/2 \]
  - correct values of the $d$ derivatives of $a$ at $z=1$
    \[ a^{(k)}(1) = 2 \prod_{j=0}^{k-1} (\tau - j) \quad \text{for} \quad k = 0, \ldots, d \]