Adaptive Wavelet Methods for the Efficient Approximation of Images

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Adaptive Wavelet Methods for Image Approximation

Outline

• Introduction: Adaptive wavelet transforms
  • Generalized lifting schemes
  • Geometric approaches with adaptivity costs
• Description of the EPWT algorithm
• Examples and experiments
• A hybrid method using the EPWT
• Numerical experiments
• Denoising of scattered data
• References
Introduction

Idea  Design adaptive approximation schemes respecting the local geometric regularity of two-dimensional functions

Basic adaptive wavelet approaches

a) Apply a generalized lifting scheme to the data using (nonlinear) data-dependent prediction and update operators

b) Adaptive approximation schemes using geometric image information, usually with extra adaptivity costs
Basic adaptive wavelet approaches

a) Apply a generalized lifting scheme to the data using (nonlinear) data-dependent prediction and update operators

Literature (incomplete)

• discrete MRA and generalized wavelets (Harten ’93)
• second generation wavelets (Sweldens ’97)
• edge adapted multiscale transform (Cohen & Matei ’01)
• Nonlinear wavelet transforms (Claypoole et al. ’03)
• adaptive lifting schemes (Heijmans et al. ’06)
• adaptive directional lifting based wavelet transf. (Ding et al. ’06)
• edge-adapted nonlinear MRA (ENO-EA) (Arandiga et al. ’08)
• meshless multiscale decompositions (Baraniuk et al. ’08)
• nonlinear locally adaptive filter banks (Plonka & Tenorth ’09)
The general lifting scheme consists of three steps.

1. **Split** Split the given data \( a = (a(i, j))_{i,j=0}^{N-1} \) into two sets \( a^e \) and \( a^o \)

2. **Predict** Find a good approximation \( \tilde{a}^o \) of \( a^o \) of the form
   \[
   \tilde{a}^o = P_1 a^o + P_2 a^e
   \]
   Put
   \[
   d^o := \tilde{a}^o - a^o.
   \]
   Assume that \((a^e, a^o) \mapsto (a^e, d^o)\) is invertible, i.e., \( I - P_1 \) is invertible.

3. **Update** Find a “smoothed” approximation of \( a^e \)
   (a low-pass filtered subsampled version of \( a \))
   \[
   \tilde{a}^e := U_1(d^o) + U_2(a^e)
   \]
   Assume that \((a^e, d^o) \mapsto (\tilde{a}^e, d^o)\) is invertible, i.e., that \( U_2 \) is invertible.
How to choose the prediction and update operators?

**Prediction operator** local approximation of $a^o$ by an adaptively weighted average of “neighboring” data

**Example 1.**
- Fix a stencil at a neighborhood of $a^o(i, j)$ (adaptively)
- Compute a polynomial $p$ by interpolating/approximating the data on the stencil
- Choose $p(i, j)$ to approximate $a^o(i, j)$.

**Example 2.** Use nonlinear diffusion filters to determine the prediction operator

**Update operator** usually linear, non-adaptive
Basic adaptive wavelet approaches

b) Adaptive wavelet approximation schemes using geometric image information, usually with extra adaptivity costs

Literature (incomplete)

- **wedgelets** (Donoho ’99)
- **bandelets** (Le Pennec & Mallat ’05)
- **geometric wavelets** (Dekel & Leviatan ’05)
- **geometrical grouplets** (Mallat ’09)
- **EPWT** (Plonka et al. 09)
- **tetrolets** (Krommweh ’10)
- **generalized tree-based wavelet transform** (Ram, Elad et al. ’11)
Basic adaptive wavelet approaches

**wedgelets** (Donoho ’99)
approximation of images using an adaptively chosen domain decomposition

**bandelets** (Le Pennec & Mallat ’05)
wavelet filter bank followed by adaptive geometric orthogonal filters

**geometric wavelets** (Dekel & Leviatan ’05)
binary space partition and polynomial approximations in subdomains

**geometrical grouplets** (Mallat ’09)
association fields that group points, generalized Haar wavelets

**EPWT** (Plonka et al. 09)

**tetrolets** (Krommweh ’10)
generalized Haar wavelets on adaptively chosen tetrolet partitions
Comparison of basic adaptive wavelet approaches

a) Generalized lifting scheme with nonlinear prediction

**Advantages** invertible transform, no side information necessary
usually a justifiable computational effort

**Drawbacks** bad stability of the reconstruction scheme
only slightly better approximation results compared with
linear (nonadaptive) transforms

b) Adaptive wavelet approximation using geometric image information

**Advantages** very good approximation results

**Drawbacks** adaptivity costs for encoding
usually high computational effort
Description of the EPWT

**Problem**  Given a matrix of data points (image values), how to compress the data by a wavelet transform thereby exploiting the local correlations efficiently?

**Idea**

1. Find a (one-dimensional) path through all data points such that there is a strong correlation between neighboring data points.
2. Apply a one-dimensional wavelet transform along the path.
3. Apply the idea repeatedly to the low-pass filtered array of data.
Toy Example

\[
f = \begin{bmatrix}
115 & 108 & 109 & 112 \\
106 & 116 & 107 & 109 \\
112 & 110 & 108 & 108 \\
108 & 109 & 103 & 106 \\
\end{bmatrix}
\]

array of data.

\[
p^4 = ((0, 5, 8, 9, 13, 12), (1, 6, 11, 10, 7, 2, 3), (4), (15, 14)),
\]

\[
f^3 = (115.5, 111, 108.5, 107.5, 108, 109, 109, 104.5),
\]

\[
p^3 = ((0, 1, 6, 5, 4, 3), (2, 7)), \quad p^2 = (0, 1, 2, 3).
\]
The relaxed EPWT

Idea: Change the direction of the path only if the difference of data values is greater than a predetermined value $\theta$.

rigorous EPWT ($\theta = 0$)  
Entropy 2.08 bit per pixel

relaxed EPWT ($\theta = 0.14$)  
Entropy 0.39 bit per pixel
Numerical results

Test: door lock image (128 × 128)

<table>
<thead>
<tr>
<th>WT</th>
<th>( \theta_1 )</th>
<th>levels</th>
<th>nonzero coeff</th>
<th>PSNR</th>
<th>entropy of ( \tilde{p}^{14} )</th>
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<tbody>
<tr>
<td>tensor prod. Haar</td>
<td>-</td>
<td>7</td>
<td>512</td>
<td>22.16</td>
<td>-</td>
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<td>-</td>
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<td>1.11</td>
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<td>512</td>
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<td>0.32</td>
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<td>10</td>
<td>512</td>
<td>28.35</td>
<td>2.22</td>
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<td>10</td>
<td>512</td>
<td>28.99</td>
<td>1.11</td>
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<td>10</td>
<td>512</td>
<td>28.38</td>
<td>0.55</td>
</tr>
</tbody>
</table>
original image

\[
\text{D4, 512 coeff.} \\
\text{PSNR} = 22.94 \text{ dB}
\]

\[
\text{EPWT, } \theta_1 = 0 \\
\text{PSNR} = 28.63 \text{ dB}
\]

\[
\text{EPWT, } \theta_1 = 0.05 \\
\text{PSNR} = 29.23 \text{ dB}
\]

\[
\text{EPWT, } \theta_1 = 0.1 \\
\text{PSNR} = 28.67 \text{ dB}
\]

\[
\text{EPWT, } \theta_1 = 0.15 \\
\text{PSNR} = 27.65 \text{ dB}
\]
Results for N-term approximation

Theorem 1 (Plonka, Tenorth, Iske (2011))
The EPWT (with the Haar wavelet transform) leads for suitable path vectors to an $N$-term approximation of the form

$$\|f - f_N\|_2^2 \leq C N^{-\alpha}$$

for piecewise Hölder continuous functions of order $\alpha$ (with $0 < \alpha \leq 1$) possessing discontinuities along curves of finite length.

Theorem 2 (Plonka, Iske, Tenorth (2013))
The application of the EPWT leads for suitably chosen path vectors to an $N$-term approximation of the form

$$\|f - f_N\|_2^2 \leq C N^{-\alpha}$$

for piecewise Hölder smooth functions of order $\alpha > 0$ possessing discontinuities along curves of finite length.
The hybrid method using the EPWT

Idea

1. Apply an image separation into a smooth image part and a remainder part containing edges and texture

\[ u = u^{sm} + u^r \]

using e.g. a suitable smoothing filter.

2. Apply a tensor product wavelet transform to the smooth image part \( u^{sm} \) to get an \( N \)-term approximation \( u_{N}^{sm} \).

3. Apply the EPWT to the (shrinked) remainder \( u^r \) to get an \( M \)-term approximation \( u_{M}^r \).

4. Add \( u_{N}^{sm} \) and \( u_{M}^r \) to find a good approximation of \( u \).
A sketch of the hybrid method

We use the tensor-product wavelet transform for the smoothed image and the EPWT for the (shrunken) difference image.
Example

Original image

smoothed image $u^{sm}$

wavelet approximation $u^{sm}_{1200}$

difference image $u^r$

shrunken difference $u^r_{1/4}$

EPWT approximation $u^r_{800}$
Example continued


(a) $u_{1200+800}$ using the new hybrid method

(b) $u_{2000}$ using the 9/7 wavelet transform with 2000 non-zero elements
Numerical results for the hybrid method

<table>
<thead>
<tr>
<th>image</th>
<th>nzc</th>
<th>9/7 PSNR</th>
<th>Hybrid PSNR</th>
<th>entropy</th>
</tr>
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<tbody>
<tr>
<td>barbara</td>
<td>500</td>
<td>23.33</td>
<td>27.28</td>
<td>1.0070</td>
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<tr>
<td>cameraman</td>
<td>500</td>
<td>22.54</td>
<td>27.49</td>
<td>0.9893</td>
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<td>clock</td>
<td>500</td>
<td>24.61</td>
<td>30.87</td>
<td>0.8742</td>
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<td>goldhill</td>
<td>500</td>
<td>24.18</td>
<td>28.19</td>
<td>0.8408</td>
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<tr>
<td>lena</td>
<td>500</td>
<td>23.21</td>
<td>27.91</td>
<td>0.9022</td>
</tr>
<tr>
<td>pepper</td>
<td>500</td>
<td>23.41</td>
<td>28.03</td>
<td>0.8795</td>
</tr>
<tr>
<td>sails</td>
<td>500</td>
<td>21.32</td>
<td>25.42</td>
<td>0.9190</td>
</tr>
</tbody>
</table>

Hybrid: Search for suitable path vectors in each level
Original image

7/9, 500 coeff.
PSNR = 23.21

Hybrid, 500 coeff.
PSNR = 27.91

Original image

7/9, 500 coeff.
PSNR = 23.41

Hybrid, 500 coeff.
PSNR = 28.03
Denoising of scattered data using the EPWT approach

Given

a set of $d$-dimensional points $\Gamma = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^d$

noisy function values $\tilde{f}(x_j) = f(x_j) + z_j$, $j = 1, \ldots, N$

where

$f : \mathbb{R}^d \to \mathbb{R}$ piecewise smooth

$z_j$ independent and $\mathcal{N}(0, \sigma_j^2)$ distributed (Gaussian noise)

Wanted  denoised function values $f(x_j)$

Classical wavelet shrinkage

wavelet decomposition

shrinkage: set small high-pass coefficients to zero

wavelet reconstruction

Analogon of cycle shift:

average [ shift $\to$ wavelet shrinkage $\to$ un-shift ]
Denoising scheme (wavelet decomposition and shrinkage)

- find path through all points

- find path through all points
Denoising scheme (wavelet decomposition and shrinkage)

- find path through all points

![Diagram of path through points](image-url)
Denoising scheme (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform along the path
  low pass coefficients \((3, 10, 8, 1)\)
  high pass coefficients \((1, 2, 2, 1)\)
Denoising scheme (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform along the path
  low pass coefficients (3, 10, 8, 1)
  high pass coefficients (1, 2, 2, 1)
- update point set
- apply shrinkage to wavelet coefficients
Denoising scheme (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform along the path
  low pass coefficients (3, 10, 8, 1)
  high pass coefficients (1, 2, 2, 1)
- update point set
- apply shrinkage to wavelet coefficients
- relate low pass coefficients to the updated point set
Denoising scheme (wavelet decomposition and shrinkage)

- find path through all points
- apply 1D wavelet transform along the path
  low pass coefficients (3, 10, 8, 1)
  high pass coefficients (1, 2, 2, 1)
- update point set
- apply shrinkage to wavelet coefficients
- relate low pass coefficients to the updated point set
- continue at the next level
Adaptive path reconstruction

- Choose first path index \( p(1) \) randomly from \( \Gamma := \{1, \ldots, N\} \).
- For \( k = 1, \ldots, N - 1 \) choose \( p(k + 1) \) such that

\[
x_{p(k+1)} = \arg\max_{x \in N_{C,\theta}(x_{p(k)})} \frac{\langle x_{p(k)} - x_{p(k-1)}, x - x_{p(k)} \rangle}{\|x_{p(k)} - x_{p(k-1)}\| \cdot \|x - x_{p(k)}\|}
\]

where \( N_{C,\theta}(x_{p(k)}) \) contains all points \( x_r \in \Gamma \) fulfilling:

1. \( r \notin \{p(1), \ldots, p(k)\} \)
2. \( \|x_r - x_{p(k)}\|_2 \leq C \)
3. \( |f(x_r) - f(x_{p(k)})| \leq \theta. \)

If \( N_{C,\theta}(x_{p(k)}) = \emptyset \), randomly choose \( p(k+1) \) among the indices fulfilling 1 & 2 or only 1.
Example: Adaptive path reconstruction
Original image

noisy image
PSNR = 19.97

adaptive path constr. 
PSNR = 29.01

random path constr. 
PSNR = 27.96

σ = 0.1
Original image
PSNR = 16.45

noisy image

adaptive path constr.
PSNR = 26.44

random path constr.
PSNR = 25.69

\( \sigma = 0.15 \)
Comparison of denoising results

<table>
<thead>
<tr>
<th>Method</th>
<th>peppers noisy image</th>
<th>peppers noisy image</th>
<th>cameraman noisy image</th>
<th>cameraman noisy image</th>
</tr>
</thead>
<tbody>
<tr>
<td>tensor product wavelet shrinkage</td>
<td>24.91</td>
<td>23.20</td>
<td>24.74</td>
<td>22.86</td>
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<tr>
<td>with cycle spinning</td>
<td>28.11</td>
<td>25.86</td>
<td>27.19</td>
<td>25.14</td>
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<tr>
<td>4-pixel scheme</td>
<td>28.26</td>
<td>26.13</td>
<td>27.64</td>
<td>25.73</td>
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<td>curvelet shrinkage</td>
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<td>23.95</td>
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<td>23.73</td>
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<td>shearlet shrinkage</td>
<td>26.82</td>
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<td>26.07</td>
<td>24.23</td>
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<td>deterministic path</td>
<td>29.01</td>
<td>26.44</td>
<td>28.28</td>
<td>26.15</td>
</tr>
<tr>
<td>random path</td>
<td>27.96</td>
<td>25.69</td>
<td>27.44</td>
<td>24.85</td>
</tr>
</tbody>
</table>
Denoising of non-rectangular domains

Original image

noisy image
PSNR = 19.97

denoised image
PSNR = 27.77
Original image

noisy image

PSNR = 19.98

PSNR = 26.31

denoised image

Original image

noisy image

PSNR = 19.96

PSNR = 28.71

denoised image
Our publications

- Gerlind Plonka.  
  *The easy path wavelet transform: A new adaptive wavelet transform for sparse representation of two-dimensional data.*  
  SIAM Multiscale Modeling and Simulation 7(3) (2009), 1474-1496.

- Gerlind Plonka, Daniela Roșca.  
  *Easy Path Wavelet Transform on triangulations of the sphere.*  
  Mathematical Geosciences 42(7) (2010), 839-855.

- Jianwei Ma, Gerlind Plonka, Hervé Chauris.  
  *A new sparse representation of seismic data using adaptive easy-path wavelet transform.*  

- Gerlind Plonka, Stefanie Tenorth, Daniela Roșca.  
  *A hybrid method for image approximation using the easy path wavelet transform.*  
• Gerlind Plonka, Stefanie Tenorth, Armin Iske.  
*Optimally sparse image representation by the easy path wavelet transform.*

• Dennis Heinen, Gerlind Plonka.  
*Wavelet shrinkage on paths for denoising of scattered data.*
Results in Mathematics 62(3) (2012), 337-354.

• Gerlind Plonka, Armin Iske, Stefanie Tenorth.  
*Optimal representation of piecewise Hölder smooth bivariate functions by the easy path wavelet transform.*
\thankyou