SHILOV PARABOLIC SYSTEMS

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Abstract. We apply semigroups of growth order $r > 0$ in the study of Shilov parabolic systems, and improve results obtained by the use of $C$-regularized semigroups. In the final part of paper, we consider some concrete examples and directly compute matrix exponentials with a view to strengthening the estimates obtained by abstract methods.

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1. Introduction and Preliminaries

The main purpose of the present paper is to apply (analytic) semigroups of growth order $r > 0$ in the study of Shilov parabolic systems of abstract differential equations. In the case of $L^2$-type spaces, the pioneering results in this direction were obtained by S. G. Krein [19, Chapter 1, §8], T. Ushijima

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[29, Section 10] and J. Wiener, L. Debnath [33]. It is also worth noting that H. Sunouchi [27] analyzed Shilov parabolic systems within the frame of \((0, A)\)-semigroups ([24]). Concerning the applications of \(C\)-regularized semigroups to systems of abstract differential equations, the references [5]-[6], [14] and [36]-[38] are crucially important. In this paper, we refine results on the well-posedness of abstract Shilov parabolic systems obtained by Q. Zheng, Y. Li [36] and Q. Zheng [38]. In order to do that, we use the theory of \(C\)-regularized semigroups as an auxiliary tool.

Henceforth \((X, \| \cdot \|)\) denotes a non-trivial complex Banach space and \(L(X)\) denotes the Banach algebra of all linear continuous mappings from \(X\) into \(X\). The range of a closed, linear operator \(B\) acting on \(X\) is denoted by \(R(B)\), and by \(K\) is denoted an absolute positive constant appearing in our estimations and formulae. In the sequel, we basically follow the notation given in [5]. Given \(\gamma \in (0, \pi]\) and \(s \in \mathbb{R}\), set \(\Sigma_\gamma := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \gamma\}\) and \([s] := \inf\{k \in \mathbb{Z} : s \leq k\}\).

We need the following definitions.

**Definition 1.1.** ([4]) Let \(r > 0\). An operator family \((T(t))_{t > 0}\) in \(L(X)\) is said to be a semigroup of growth order \(r > 0\) iff the following conditions hold:

(i) \(T(t + s) = T(t)T(s), t, s > 0,\)

(ii) for every \(x \in X\), the mapping \(t \mapsto T(t)x, t > 0\) is continuous,

(iii) \(||t^rT(t)|| = O(1), t \to 0^+,\)

(iv) \(T(t)x = 0\) for all \(t > 0\) implies \(x = 0\), and

(v) \(X_0 = \bigcup_{t > 0} R(T(t))\) is dense in \(X\).

The infinitesimal generator of \((T(t))_{t > 0}\) is defined by

\[
G = \left\{(x, y) \in X \times X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} = y\right\}.
\]

The infinitesimal generator \(G\) of \((T(t))_{t > 0}\) is a closable linear operator and its closure \(\overline{G}\) is called the complete infinitesimal generator, in short c.i.g., of \((T(t))_{t > 0}\).

**Definition 1.2.** ([28]) Let \((T(t))_{t > 0}\) be a semigroup of growth order \(r > 0\), let \(\gamma \in (0, \frac{\pi}{2}]\) and let \((T(t))_{t > 0}\) possess an analytic extension to the sector \(\Sigma_\gamma\), denoted by the same symbol. Suppose, further, that there
exists $\omega \in \mathbb{R}$ such that, for every $\delta \in (0, \gamma)$, there exists $M_\delta > 0$ satisfying $\|z^r T(z)\| \leq M_\delta e^{\omega R z}$, $z \in \Sigma$. Then the operator family $(T(t))_{t \in \Sigma}$ is called an analytic semigroup of growth order $r$.

We refer the reader to [22] and [32] for the notions of a $C$-regularized semigroup (cosine function) and its subgenerator (integral generator).

**Definition 1.3.** ([18]) Let $(S(t))_{t \in [0, \tau]}$ be a (local) $C$-regularized semigroup having $A$ as a subgenerator and let the mapping $t \mapsto S(t)$, $t \in (0, \tau)$ be infinitely differentiable in the uniform operator topology. Then it is said that $(S(t))_{t \in [0, \tau]}$ is $\rho$-hypoanalytic, $1 \leq \rho < \infty$, iff for every compact set $L \subseteq (0, \tau)$ there exists $h_L > 0$ such that

$$
\sup_{t \in L, i \in \mathbb{N}_0} \left\| \frac{d^i}{dt^i} S(t) \right\| < \infty.
$$

Let $n \in \mathbb{N}$ and let $i A_j$, $1 \leq j \leq n$ be commuting generators of bounded $C_0$-groups on $X$. Put $|A|^2 := \sum_{j=1}^n A_j^2$ and

$$
(1 + |A|^2)^{-r} := (1 + |x|^2)^{-r}(A), \ r > 0,
$$

where the right hand side of (1) is defined by means of the functional calculus for commuting generators of bounded $C_0$-groups (cf. [2] and [12, Section XII] for further information).

Let $P(x) = [p_{ij}(x)]_{m \times m}$, $x \in \mathbb{R}^n$ be an $m \times m$ polynomial matrix and let $\lambda_j(x)$, $1 \leq j \leq m$ be the eigenvalues of $P(x)$, $x \in \mathbb{R}^n$; see [36] for the definition of a closable operator $P(A)$. Set $k := 1 + \left\lceil \frac{n}{2} \right\rceil$, $\Lambda := \sup_{1 \leq j \leq m} \| \lambda_j(x) \|$, $x \in \mathbb{R}^n$, $N := \max(\text{deg}(p_{ij}(x)))$ and assume that $r \in (0, N]$. Then it is said that $P(x)$ is Shilov $r$-parabolic [12] if there exist $\omega > 0$ and $\omega' \in \mathbb{R}$ such that $\Lambda(P(x)) \leq -\omega |x|^r + \omega'$, $x \in \mathbb{R}^n$; in the case $r = N$, it is also said that $P(x)$ is Petrovskii parabolic. Define $\pi_1(r) := \min_{1 \leq j \leq m, |x|=r} |\lambda_j(x)|$, $r \geq 0$, $\pi_2(r) := \max_{1 \leq j \leq m, |x|=r} |\lambda_j(x)|$, $r \geq 0$ and $S(P) := \{ \lambda_j(x) : x \in \mathbb{R}^n, 1 \leq j \leq m \}$. Let $P(x)$ be Shilov $r$-parabolic for some $r \in (0, N]$. By a corollary of Seidenberg-Tarski theorem (cf. [13] and [29, Lemma 10.2]), we know that there exist real numbers $a_1$, $a_2$, $\alpha_1$ and $\alpha_2$ such that $\pi_1(r) = a_1 r^{\alpha_1}(1 + o(1))$ as $r \to \infty$ and $\pi_2(r) = a_2 r^{\alpha_2}(1 + o(1))$ as $r \to \infty$. Obviously, $r \leq \alpha_1 \leq \alpha_2 \leq N$ and, by the proof of [29, Proposition 10.4], there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\{ \lambda \in \mathbb{C} : \Re \lambda \geq -\alpha |3\lambda|^{\alpha_2 + \beta} \} \cap S(P) = \emptyset$. 
2. Shilov Parabolic Systems

We start this section with the following theorem. Notice that the proof of this theorem strongly depends on the estimate (4), which can be trivially shown only in the case \( m = 1 \).

**Theorem 2.1.** Let \( P(x) \) be Shilov \( r \)-parabolic for some \( r \in (0, N) \). Put \( \kappa := \frac{(N-r)(m-1+N)}{r} \).

(a) Then the operator \( \overline{P(A)} \) is the c.i.g. of a semigroup \( (T_0(t))_{t>0} \) of growth order \( \kappa \) which additionally satisfies that the mapping \( t \mapsto T_0(t) \), \( t > 0 \) is infinitely differentiable in the uniform operator topology and that:

\[
\left\| \frac{d^l}{dt^l} T_0(t) \right\| \leq K^{l+1} (1 + t)^{m-1+Nl} e^{\omega t l^N/r} (1 + t^{-\kappa - Nl}), \quad t > 0, \quad l \in \mathbb{N}_0, (2)
\]

where \( \omega \equiv \sup_{x \in \mathbb{R}^n} \Lambda(P(x)) \).

(b) Suppose that there exist \( \alpha \in (0, \frac{N}{2}] \) and \( \omega \in \mathbb{R} \) such that \( \sigma(P(x)) \subseteq \omega + (\mathbb{C} \setminus \Sigma_{\frac{N}{2}+\alpha}), \ x \in \mathbb{R}^n \). Then the operator \( \overline{P(A)} \) is the c.i.g. of an analytic semigroup \( (T_0(t))_{t \in \Sigma_\alpha} \) of growth order \( \kappa \).

**Proof.** By the well known result of A. Friedman [11, p. 171] and the definition of Shilov parabolic systems, it follows that for each \( \omega_0 < \omega \) there exists \( \omega' > 0 \) such that:

\[
\left\| e^{tP(x)} \right\| \leq K (1 + t + t|x|^N)^{m-1} e^{t(-\omega'|x|+\omega_0)}, \quad t \geq 0, \quad x \in \mathbb{R}^n. (3)
\]

Using (3), [36, (3.2), p. 190] and the product rule, we have that there exists \( \omega'' > 0 \) such that, for every \( t \geq 0, \ x \in \mathbb{R}^n, \ s \in \mathbb{R}, \ l \in \mathbb{N}_0 \) and \( \alpha \in \mathbb{N}_0^0 \) :

\[
\left\| D^\alpha_{x} (P(x)^t e^{tP(x)}(1 + |x|^2)^s) \right\| \leq K^{l+1} (1 + t)^{|\alpha|+m-1} \times (1 + |x|)^{(N-r-1)|\alpha|+(N-r)(m-1)+2s+Nl} e^{t(-\omega''|x|+\omega)}. (4)
\]

This simply implies that, for every \( s \in \mathbb{R}, \ t > 0, \ l \in \mathbb{N}_0 \) and \( p \in \mathbb{N}, \ P(x)^te^{tP(x)}(1 + |x|^2)^s \in BCP(\mathbb{R}^n) \). Using [5, Proposition 12.3], we get that, for every \( s \in \mathbb{R}, \ t > 0 \) and \( p \in \mathbb{N}, \ e^{tP(x)}(1 + |x|^2)^s \in \mathcal{A} \equiv \{ f \in C_0(\mathbb{R}^n) : \mathcal{F}f \in L^1(\mathbb{R}^n) \} \), where \( \mathcal{F} \) denotes the Fourier transform. Set \( T_{r'}(t) := (e^{tP(x)}(1 + |x|^2)^{-r'})(A), \ t > 0, \ r' \geq 0 \) and notice that [9, Lemma 2.3(c)] implies that, for every \( r' > 0, \ T_{r'}(0) := (1 + |A|^2)^{-r'} \in L(X) \). Put \( T_0(0) := I \).
By [5, (12.2)], it follows that \( T_\nu(t + s)(1 + |A|^2)^{-r'} = T_\nu(t)T_\nu(s), t \geq 0, s \geq 0, r' > 0, T_\nu(t)(1 + |A|^2)^{-s} = T_{\nu+s}(t), t \geq 0, s > 0, r' \geq 0 \) and \( T_0(t + s) = T_0(t)T_0(s), t \geq 0, s \geq 0 \). Keeping in mind (4), one can simply prove that the mapping \( t \mapsto T_\nu(t), t > 0 \) is infinitely differentiable in the uniform operator topology for all \( r' \geq 0 \), and that

\[
\frac{d}{dt} T_\nu(t) = (P(x)^{t}e^{tP(x)}(1 + |x|^2)^{-r'})(A), t > 0, r' \geq 0, l \in \mathbb{N}_0. \tag{5}
\]

Let \( \epsilon > 0 \) and \( s > n/2 \) be fixed. Then Stirling’s formula, (4) and the inequality

\[
\xi^\sigma e^{-\epsilon \xi} \leq \frac{(\sigma)}{l^\alpha}, t > 0, \xi > 0, \sigma > 0,
\]

(6)
taken together imply that, for every \( t > 0, l \in \mathbb{N}_0, x \in \mathbb{R}^n, r' \geq 0 \), and for every \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \):

\[
\left\| D_x^{\alpha} (P(x)^{t}e^{tP(x)}(1 + |x|^2)^{-r'}) \right\| \\
\leq K^{l+1}(1 + t)^{m-1+|\alpha|}(1 + |x|)^{(N-r)|\alpha|+(N-r)(m-1)+Nl-2r'} e^{t(-\omega|x|^r+\omega}(1 + |x|)^{-|\alpha|} \\
\leq K^{l+1}(1 + t)^{m-1+|\alpha|} t^{N/r} e^{t\omega}(1 + t^{-\frac{(N-r)}{m-1+r}+Nl-2r'}) \\
(1 + t^{-\frac{N-r}{m-1+r}})^{|\alpha|}(1 + |x|)^{(\epsilon-1)|\alpha|},
\]

with \( K \) being independent of \( t, x, \epsilon \) and \( r' \). The preceding estimate combined with [9, Theorem 2.7(2)] yields that:

\[
\left\| (P(x)^{t}e^{tP(x)}(1 + |x|^2)^{-r'} - s)(A) \right\| \leq K^{l+1}(1 + t)^{m-1+\frac{n}{2}} e^{t\omega} t^{N/r} \\
(1 + t^{-\frac{(N-r)}{m-1+r}+Nl-2r'}), \tag{7}
\]

provided \( t > 0, l \in \mathbb{N}_0 \) and \( r' \geq 0 \). Invoking the concrete representation of \( D_x^{\alpha} e^{tP(x)} \) and an elementary argumentation, we infer that \( \lim_{t \to 0^+} (P(x)^{t}e^{tP(x)}(1 + |x|^2)^{-r'} \in H^k(\mathbb{R}^n) (t > 0, l \in \mathbb{N}_0, r' \geq 0) \), which implies by Bernstein’s lemma \([9, \text{Lemma 2.3}(d)]\) and (7) that:

\[
\left\| (P(x)^{t}e^{tP(x)}(1 + |x|^2)^{-r'})(A) \right\| \leq K^{l+1}(1 + t)^{m-1+\frac{n}{2}} e^{t\omega} t^{N/r} \\
(1 + t^{-\frac{(N-r)}{m-1+r}+Nl-2r'}). \tag{8}
\]
Hence, \((T_0(t))_{t > 0}\) is a semigroup of growth order \(\kappa\). Thanks to (5) and (8), we have that for every \(t > 0\), \(l \in \mathbb{N}_0\) and \(r' \geq 0\):

\[
\left\| \frac{d^l}{dt^l} T_{r'}(t) \right\| \leq K^{l+1}(1 + t)^{m-1+\frac{n}{2}} e^{\omega t}(1 + t^{-\frac{(N-r)(m-1+\frac{n}{2})+Nl-2r'}{r'}}),
\]

and that (2) holds. Since \(T_{r'}(t)(1 + |A|^2)^{-s} = T_{r'+s}(t), \ t \geq 0, \ s > 0\) and \((T_{r'}(t))_{t \geq 0}\) is a global exponentially bounded \((1 + |A|^2)^{-r'}\)-regularized semigroup generated by \(\overline{P}(A)\) for all \(r' > \frac{1}{2}(N-r)(m-1+\frac{n}{2})\) (cf. [36, Theorem 3.1]), we immediately obtain that the c.i.g. \(\mathcal{G}\) of \((T_0(t))_{t > 0}\) is contained in \(\overline{P}(A)\). To prove that \(\overline{P}(A) \subseteq \mathcal{G}\), one can use [36, Lemma 3.1(a)] and the inclusions \(D(P(A)_{|E^n}) \subseteq (1 + |A|^2)^{-r'}(D(P(A))) \subseteq D(\mathcal{G})\), where \(E \equiv \{ \phi(A)x : \phi \in \mathcal{S}, \ x \in X\}\) and \(\mathcal{S}\) denotes the Schwartz space of rapidly decreasing functions on \(\mathbb{R}^n\). This completes the proof of (a). The proof of (b) follows from (a) and the arguments given in the proof of [38, Theorem 1.1]. □

**Remark 2.2.**

(a) Let \(r' \geq 0\). Then the proof of Theorem 2.1 implies that \(L(X) \supseteq (T_{r'}(t))_{t > 0}\) satisfies:

(a1) \(T_{r'}(t+s)(1 + |A|^2)^{-r'} = T_{r'}(t)T_{r'}(s), \ t, s > 0,\)

(a2) for every \(x \in E\), the mapping \(t \mapsto T_{r'}(t)x, \ t > 0\) is continuous,

(a3) \(T_{r'}(t)x = 0\) for all \(t > 0\) implies \(x = 0,\)

(a4) \(X_{0,r'} = \bigcup_{t > 0} R(T_{r'}(t))\) is dense in \(X,\) and

(a5) \(\left\| t^{(N-r)(m-1+\frac{n}{2})-2r'}/r' \right\| T_{r'}(t) = O(1), \ t \in (0,1].\)

The properties (a1)-(a5) betoken that \((T_{r'}(t))_{t > 0}\) is a \((1 + |A|^2)^{-r'}\)-regularized semigroup of growth order \(\frac{(N-r)(m-1+\frac{n}{2})-2r'}{r'},\) provided \(r' \in (0, \frac{1}{2}(N-r)(m-1+\frac{n}{2}));\) the results established in this paper do not represent the real basis for further theoretical research of \(C\)-regularized semigroups of growth order \(\xi > 0.\) In our concrete situation, we have additionally that the mapping \(t \mapsto T_{r'}(t), \ t > 0\) is infinitely differentiable in the uniform operator topology.

(b) By the proof of Theorem 2.1, the inequality

\[
\left\| \frac{d^l}{dt^l} T_{r'}(t) \right\| \leq K^{l+1}(1 + t)^{m-1+\frac{n}{2}} e^{\omega t}, \ t > 0, \ l \in \mathbb{N}_0,
\]

is obvious.
holds provided \((N - r)(m - 1) + Nl - 2r' \leq -(N - r)k\), and the inequality
\[
\left\| \frac{d^l}{dt^l} T_{r'}(t) \right\| \leq \begin{cases} 
K^{l+1}l!N/r t^{-(N-r)(m-1+rac{n}{2})+N-2r'}, & t \in (0, 1], \ l \in \mathbb{N}_0, \\
K^{l+1}l!N/r (1+t)^{m-1+rac{n}{2}} e^{\omega t}, & t > 1, \ l \in \mathbb{N}_0, 
\end{cases}
\]
holds provided \((N - r)(m - 1) + Nl - 2r' \in (- (N - r)k, 0)\).

(c) Let \(l \in \mathbb{N}\). Then, for every \(r' > \frac{1}{2}(N-r)(m-1+\frac{n}{2})\), the estimate (9), in regard to the decay rate of derivatives of \((T_{r'}(t))_{t > 0}\) in a neighborhood of zero, sharpens the corresponding one given in the formulation of [36, Theorem 3.1(a)].

(d) Now we will explain how one can improve \(\rho\)-hypoanalyticity of \(C\)-regularized semigroups constructed in [36, Theorem 3.1(a)], provided that \(X = L^2(\mathbb{R}^n)\). By [29, Proposition 10.5] (cf. also [29, Remark 2, p. 124] and [36, Lemma 1.2]), we easily infer that there exist numbers \(\xi > 0, \ a, b, c > 0\) and \(\varpi \in \mathbb{R}\) such that:
\[
\left\| \frac{d^l}{dt^l}(t^{\xi} T_0(t)) \right\| \leq ab^l l! \frac{\omega^2}{2} e^{\varpi t} t^{-\frac{n}{2}(l+c)}, \ l \in \mathbb{N}_0, \ t > 0. \quad (10)
\]

By the product rule, it follows, that for every compact set \(L \subseteq (0, \infty)\), there exists \(h_L > 0\) such that \(\sup_{t \in L, \ l \in \mathbb{N}_0} \left\| \frac{h^l}{l!} \frac{d^l}{dt^l} T_0(t) \right\| < \infty\). It could be of interest to know whether the estimate (10) holds in the case of a general space \(X\).

Now we pay our attention to the numerical range of \(P(x)\), defined by \(\text{n.r.}(P(x)) := \{(P(x)y, y) : y \in \mathbb{R}^n, \ ||y|| = 1\}, \ x \in \mathbb{R}^n\), where \((\cdot, \cdot)\) denotes the inner product in \(\mathbb{C}^n\) and \(||y|| := (y, y)^{1/2}\). Set \(\Lambda(P(x)) := \sup \{\Re z : z \in \text{n.r.}(P(x))\}, \ x \in \mathbb{R}^n\).

**Theorem 2.3.** Let \(r \in (0, N), \ \omega' > 0\) and \(\omega'' > 0\). Put \(\kappa_{n.r.} := \frac{n(N-r)}{2r}\).

(a) Suppose
\[
\Lambda(P(x)) \leq -\omega'|x|^r + \omega'', \ x \in \mathbb{R}^n. \quad (11)
\]
Then the operator \(\hat{P}(A)\) is the c.i.g. of a semigroup \((T_0(t))_{t > 0}\) of growth order \(\kappa_{n.r.}\), which additionally satisfies that the mapping \(t \mapsto
\( T_0(t), t > 0 \) is infinitely differentiable in the uniform operator topology and that:
\[
\left\| \frac{d^l}{dt^l}T_0(t) \right\| \leq K^{l+1}(1 + t)^{\frac{k}{2}}e^{\omega t}t^{N/r}(1 + t^{-\frac{(N-r)}{2}r}), \quad t > 0, \quad l \in \mathbb{N}_0,
\]
where \( \omega \equiv \sup_{x \in \mathbb{R}^n} \tilde{\Lambda}(P(x)). \)

(b) Let \( \alpha \in (0, \frac{\pi}{2}], \omega \in \mathbb{R}, \ n.r. (P(x)) \subseteq \omega + (\mathcal{C} \setminus \sum_{\frac{\pi}{2} + \alpha}) \), \( x \in \mathbb{R}^n \) and let \( P(x) \) be Shilov \( r \)-parabolic. Then the operator \( \overline{P(A)} \) is the c.i.g. of an analytic semigroup \( (T_0(t))_{t \in \Sigma_n} \) of growth order \( \frac{2N}{2r} \).

(c) Let \( \alpha \in (0, \frac{\pi}{2}], \omega \in \mathbb{R}, \ n.r. (P(x)) \subseteq \omega + (\mathcal{C} \setminus \sum_{\frac{\pi}{2} + \alpha}), \ x \in \mathbb{R}^n \) and let (11) hold. Then the operator \( \overline{P(A)} \) is the c.i.g. of an analytic semigroup \( (T_0(t))_{t \in \Sigma_n} \) of growth order \( \kappa_{n.r.} \).

**Proof.** Suppose that (11) holds. Then the Lumer-Phillips theorem implies that there exist \( \omega_1 > 0 \) and \( L \geq 1 \) such that \( \|e^{tP(x)}\| \leq Ke^{-2\omega_1|x|^2}e^{\omega t}, \quad t \geq 0, \quad |x| \geq L \) and \( \|e^{tP(x)}\| \leq K^{\omega t}, \quad t \geq 0, \quad |x| \leq L. \)

Making use of this fact, we obtain inductively from [15, (8.10)] that there exists \( a \in (0, 2\omega_1) \) such that, for every \( t \geq 0, \ x \in \mathbb{R}^n, \ r' \geq 0, \ l \in \mathbb{N}_0 \), and for every multi-index \( \alpha \in \mathbb{N}_0^n \) such that \( |\alpha| \leq k \):
\[
\|D^\alpha e^{tP(x)}\| \leq Ke^{\omega t}(1 + t)^{|\alpha|}(1 + |x|)^{(N-r)|k|}e^{-a|\alpha|x^2}e^{(-\frac{N-r}{2}r)t}, \quad t > 0. \ (13)
\]

By (13) and the product rule, we reveal that, for every \( t \geq 0, \ x \in \mathbb{R}^n, \ r' \geq 0, \ l \in \mathbb{N}_0 \), and for every \( \alpha \in \mathbb{N}_0^n \) such that \( |\alpha| \leq k \):
\[
\|D^\alpha (P(x)^le^{tP(x)}(1 + |x|^2)^{-r'})\|
\leq K^{l+1}e^{\omega t}(1 + t)^{|\alpha|}(1 + |x|)^{(N-r)|\alpha|+Nl-2r'}e^{-a|\alpha|x^2}e^{(-\frac{N-r}{2}r)t}, \quad t > 0. \ (14)
\]

In the case \( t > 0 \), Stirling’s formula combined with (6) implies that the latter does not exceed
\[
K^{l+1}e^{\omega t}(1 + t)^{|\alpha|}t^{N/r}(1 + t^{-\frac{Nl-2r'}{2r'}})(1 + t^{-\frac{(N-r)n}{2r'}}), \quad t > 0, \quad r' \geq 0, \ l \in \mathbb{N}_0. \ (15)
\]
The decay rate of derivatives of \((T_r(t))_{t \geq 0}\) in a neighborhood of zero (cf. (15)) improves the corresponding one given in the formulation of \([36, \text{Theorem 3.2}].\) Further on, observe that the \((1 + |A|^2)^{-r'}\)-regularized semigroup \((T_r(t))_{t \geq 0}\) constructed in the proof of \([36, \text{Theorem 3.1}],\) resp. \([36, \text{Theorem 3.2}],\) is norm continuous provided \(r' > \frac{1}{2}(N-r)(m-1+\frac{1}{2}),\) resp. \(r' > \frac{1}{2}n(N-r).\) Now we will prove that, in the situation in which the assumptions of \([36, \text{Theorem 3.1}],\) are fulfilled, the operator \(\overline{P}(A)\) generates an exponentially bounded \((1 + |A|^2)^{-r'}\)-regularized semigroup \((T_r(t))_{t \geq 0}\) for \(r' = \frac{1}{2}(N-r)(m-1+\frac{1}{2}).\) By Remark 2.2(a), the only non-trivial thing that should be explained is the strong continuity of \((T_r(t))_{t \geq 0}\) at \(t = 0.\) Using the denseness of \(E\) in \(X,\) and the boundedness of \(\|T_r(t)\|\) for \(0 \leq t \leq 1,\) it suffices to show that the mapping \(t \mapsto T_r(t)\bar{x},\) \(t \geq 0\) is continuous at \(t = 0\) for every fixed \(\bar{x} \in E^m.\) Let \(L \geq 1\) be sufficiently large.

\[
\|e^{tP(x)} - I\| \leq t\|P(x)\| \sum_{n=0}^{\infty} \frac{t^n\|P(x)\|^n}{(n+1)!},
\]

\[
\leq t\|P(x)\| \sum_{n=0}^{\infty} \frac{t^n(K(1 + L)^N)^n}{(n+1)!}, \quad t \geq 0, \quad |x| \leq L. \tag{16}
\]

Given \(x \in \mathbb{R}^n,\) denote by \(H(\lambda_1(x), \cdots, \lambda_m(x))\) the convex hull of the set \(\{\lambda_1(x), \cdots, \lambda_m(x)\}.\) Then, for every \(z \in H(\lambda_1(x), \cdots, \lambda_m(x)),\) there exist non-negative scalars \(\alpha_1, \cdots, \alpha_m\) such that \(\alpha_1 + \cdots + \alpha_m = 1\) and \(z = \alpha_1\lambda_1(x) + \cdots + \alpha_m\lambda_m(x),\) which implies that there exists \(\omega^m > 0\) with \(Rz \leq -\omega^m|x|^r + \omega \leq -|\omega|, \quad |x| \geq L, \quad z \in H(\lambda_1(x), \cdots, \lambda_m(x)).\) By \([11, \text{Theorem 2, p. 169}],\) we obtain that for every \(t \geq 0\) and \(|x| \geq L,\)

\[
\|e^{tP(x)} - I\| \leq t\|P(x)\| \sum_{n=0}^{\infty} \frac{t^n\|P(x)\|^n}{(n+1)!},
\]

\[
\leq Kt(1 + |x|)^{mN} \sup_{z \in H(\lambda_1(x), \cdots, \lambda_m(x))} \|[f_t(z)] + \cdots + [f^{(m-1)}_t(z)]\|, \tag{17}
\]

where \(f_t(z) = \sum_{n=0}^{\infty} \frac{t^n\|z^n\|(n+1)!}, \quad t \geq 0, \quad z \in \mathbb{C}.\) It is clear that \([f_t(z)] = \|z^{\frac{n}{t-1}}\| \leq t|z|e^{\delta R^2}, \quad t > 0, \quad z \in \mathbb{C} \setminus \{0\}.\) This yields the existence of a number \(\sigma > 0\) such that \([f_t(z)] + \cdots + [f^{(m-1)}_t(z)] \leq K(1+t)^\sigma e^{\delta R^2} \leq \)
$$K(1 + t)^\sigma e^{-|\omega|t}, \quad t > 0, \quad |x| \geq L, \quad z \in H(\lambda_1(x), \ldots, \lambda_m(x)).$$

In view of (17), the above implies:

$$\left\| e^{tP(x)} - I \right\| \leq Kt(1 + |x|)^m N(1 + t)^\sigma e^{-|\omega|t}, \quad t \geq 0, \quad |x| \geq L. \quad (18)$$

On the other hand, the concrete representation of $D^\alpha e^{tP(x)}$ for $0 < |\alpha| \leq k$ combined with the estimate (3) indicates that there exists $\sigma_1 > 0$ such that:

$$\left\| D^\alpha (e^{tP(x)} - I) \right\| \leq Kt(1 + t)^{\sigma_1} (1 + |x|)^{\sigma_1} e^{\omega t}, \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad 0 < |\alpha| \leq k. \quad (19)$$

Now the claimed assertion follows from (16), (18)-(19) and Bernstein’s lemma. Notice finally that, in the situation of [36, Theorem 3.2], the operator $\overline{P(A)}$ generates an exponentially bounded $(1 + |A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t \geq 0}$ for $r' = \frac{1}{2} n(N - r)$, and that the above assertions can be simply reformulated for exponentially bounded, analytic $(1 + |A|^2)^{-r'}$-regularized semigroups.

(b) Suppose $1 < p < \infty$, $X = L^p(\mathbb{R}^n)$ and set $n_X := n\left(\frac{1}{2} - \frac{1}{p}\right)$. Then the growth order of $(T_0(t))_{t > 0}$ and the final estimates for the derivatives of $(T_{r'}(t))_{t \geq 0}$ ($r' \geq 0$), in Theorem 2.1 and Theorem 2.3, can be slightly refined (cf. [9, p. 134], [14, p. 375] and [36, p. 193]):

Theorem 2.1 and its proof: One can replace $\kappa$, resp. the term $\frac{n}{2}$, appearing in (9), with $\frac{(N-r)(m-1+n_X)}{r}$, resp. $n_X$.

Theorem 2.3 and its proof: In the definitions of $\kappa_{n,r}$ and the growth order of the semigroup appearing in the formulation of (b) of this theorem, one can replace the term $\frac{n}{2}$ with $n_X$. The same comment can be applied to the derivatives of $(T_{r'}(t))_{t \geq 0}$ ($r' \geq 0$).

(c) The definition of an (analytic) semigroup of growth order $r > 0$ has been recently reconsidered in [26] and [18] by removing the density assumptions. Having this notion in mind, one can prove (cf. [21]), with some obvious technical modifications, that the assertions of Theorem 2.1 and Theorem 2.3 remain true in the case $X = C_b(\mathbb{R}^n)$ or $X = L^\infty(\mathbb{R}^n)$.

(d) Suppose that $P(x)$ is Shilov $r$-parabolic for some $r \in (0, N)$, and denote by $\Omega(T_0)$ the continuity set of the semigroup $(T_0(t))_{t \geq 0}$ given in Theorem 2.1, resp. Theorem 2.3; that is $\Omega(T_0) := \{ \bar{x} \in X^m :$
\[ \lim_{t \to 0^+} T_0(t) \vec{x} = \vec{x} \}. \] Then \( \Omega(T_0) \) contains \( R((1 + |A|^2)^{-r'}) \) for all \( r' \geq \frac{1}{2}(N-r)(m-1+\frac{n}{2}) \), resp. \( r' \geq \frac{1}{4}n(N-r) \), and the abstract Cauchy problem

\[
(ACP) : \begin{cases}
\vec{u} \in C([0, \infty) : X^m) \cap C^\infty((0, \infty) : X^m),
\frac{d}{dt} \vec{u}(t) = \mathcal{P}(A)\vec{u}(t), \ t > 0,
\vec{u}(0) = \vec{x},
\end{cases}
\]

has a unique solution for all \( \vec{x} \in \Omega(T_0) \), improving the corresponding result of Q. Zheng and Y. Li (cf. [36, Lemma 1.2(b)]). As Remark 2.7(c) shows, \( R((1 + |A|^2)^{-r'}) \) can be strictly contained in \( \Omega(T_0) \).

(e) Semigroups of growth order \( r > 0 \) can be also applied in the analysis of time-dependent Shilov parabolic systems ([20], [36]). Suppose that \( T, \mathcal{P}(\cdot, \cdot), T_\Delta \) and \( T_\Delta \) possess the same meaning as in [36, Section 4]. Then a two-parameter operator family \( (W(t, s))_{(t,s) \in T_\Delta} \) is called an evolution system of growth order \( r > 0 \) iff the following holds:

\begin{enumerate}
\item[(e1)] \( W(t, r)W(r, s) = W(t, s), \ 0 \leq s \leq r \leq t \leq T, \)
\item[(e2)] \( W(t, t) = I, \ 0 \leq t \leq T, \)
\item[(e3)] the mapping \((t, s) \mapsto W(t, s)x, \ (t,s) \in T_\Delta \) is continuous for every fixed \( x \in X \), and
\item[(e4)] \( ||(t-s)^rW(t, s)|| = O(1), \ (t,s) \in T_\Delta. \)
\end{enumerate}

Let \( \Lambda(P(t,x)) \) (resp. \( \tilde{\Lambda}(P(t,x)) \)) \( \leq -\delta|x|^r + \omega, \ t \in [0, T], \ x \in \mathbb{R}^n, \) for some \( \delta > 0 \) and \( \omega \in \mathbb{R} \). Using the proofs of [36, Theorem 4.1-Theorem 4.3], Theorem 2.1 and Theorem 2.3, we are in a position to conclude that there exists a unique evolution system of growth order \( \kappa \) (resp. \( \kappa_{n,r} \)) satisfying the conditions (a) and (b) of [36, Theorem 4.1]. In such a way, we improve results on the well-posedness of the abstract Cauchy problem [36, (4.4), p. 197]; the growth order of evolution system can be additionally refined in the case of \( L^p \) type spaces (\( 1 < p < \infty \)).

Now we will consider some concrete examples of polynomial operator matrices. The one-dimensional equation describing sound propagation in a viscous gas ([5], [12]) has the form

\[ u_{tt} = 2u_{txx} + u_{xx} \]
and after the usual matrix reduction becomes

\[
\frac{d}{dt} \vec{u}(t) = P(D)\vec{u}(t), \quad t \geq 0,
\]

where \( D = -i \frac{d}{dx} \) and \( P(x) = \begin{bmatrix} 0 & 1 \\ -x^2 & -2x^2 \end{bmatrix} \). Hence, \( P(x) \) is Petrovskii correct (cf. [6] and [14] for the notion). In the subsequent theorem, we consider the \( C \)-wellposedness of the equation (20); the obtained results are better compared with those clarified in [17, Example 1.2.9(vi)].

**Theorem 2.5.** Let \( r > \frac{1}{2} \). Then the operator \( \overline{P(A)} \) generates an exponentially bounded, analytic \((1 + |A|^2)^{-r}\)-regularized semigroup \((T_r(t))_{t \geq 0}\) of angle \( \frac{\pi}{2} \). Furthermore, the semigroup \((T_r(t))_{t \in \Sigma_2^0}\) can be extended to \( \Sigma_2^1 \),

\[
||T_r(t)|| \leq K(1 + |t|)^{\frac{r}{2}} e^{t \sin(|\text{arg}(t)|)}, \quad t \in \Sigma_2^1 \setminus \{0\} \text{ and the following holds:}
\]

(a) The mapping \( t \mapsto T_r(t), \ t \in \Sigma_2^1 \) is continuous.

(b) \((T_r(it))_{t \in \mathbb{R}}\) is an exponentially bounded, norm continuous \((1 + |A|^2)^{-r}\)-regularized group generated by \( i\overline{P(A)} \).

**Proof.** We exploit the following elementary fact: Suppose \( a, b, c, d \in \mathbb{C} \), \( \alpha, \beta \) are the eigenvalues of \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( \alpha \neq \beta \) and \( n \in \mathbb{N} \). Then

\[
\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]^n = \frac{1}{\alpha - \beta} \left[ \begin{array}{cc} (\alpha - d)\alpha^n - (\beta - d)\beta^n & b(\alpha^n - \beta^n) \\ c(\alpha^n - \beta^n) & (\alpha - a)\alpha^n - (\beta - a)\beta^n \end{array} \right].
\]

Let \( \sqrt{r} \) be taken as the principle branch. Put \( \alpha(x) := -x^2 + \sqrt{x^4 - x^2}, x \in \mathbb{R}, \beta(x) := -x^2 - \sqrt{x^4 - x^2}, x \in \mathbb{R} \) and \( n(x) := \frac{2x}{\sqrt{x^4 - x^2}}, x \in \mathbb{R} \setminus \{-1, 0, 1\} \).

Fix, for a moment, \( \gamma \in [-\pi/2, \pi/2] \). Noticing that \( e^{i\gamma} \alpha(x), e^{i\gamma} \beta(x) \) are the eigenvalues of \( e^{i\gamma} P(x), \ x \in \mathbb{R} \), we easily infer that:

\[
e^{i\gamma} P(x) = \frac{1}{2\sqrt{x^4 - x^2}} \begin{bmatrix} A_{11}(t, x, \gamma) & A_{12}(t, x, \gamma) \\ A_{21}(t, x, \gamma) & A_{22}(t, x, \gamma) \end{bmatrix}, \quad t \geq 0, \ x \in \mathbb{R} \setminus \{-1, 0, 1\},
\]

where:

\[
A_{11}(t, x, \gamma) = -\beta(x)e^{t\gamma} \alpha(x) + \alpha(x)e^{t\gamma} \beta(x), \quad A_{12}(t, x, \gamma) = e^{t\gamma} \alpha(x) - e^{t\gamma} \beta(x),
\]

\[
A_{21}(t, x, \gamma) = -x^2 A_{12}(t, x, \gamma) \quad \text{and} \quad A_{22}(t, x, \gamma) = \alpha(x)e^{t\gamma} \alpha(x) - \beta(x)e^{t\gamma} \beta(x).
\]
Clearly, \( \text{Re}(e^{i\gamma(-x^2 \pm \sqrt{x^4-x^2}))} \leq 0 \) provided \( x \in (-\infty, -1] \cup [1, \infty) \), and \( \text{Re}(e^{i\gamma(-x^2 \pm \sqrt{x^4-x^2}))} \leq -x^2 \cos \gamma + |x|\sqrt{1-x^2} \sin \gamma \leq \frac{1}{2} \sin |\gamma| \) provided \( x \in (-1, 1) \). Applying the Darboux inequality, one gets:

\[
|e^{t e^{i\gamma}a(x)} - e^{t e^{i\gamma}b(x)}| \\
\leq 2|\sqrt{x^4-x^2}|t \max_{\xi \in [e^{i\gamma}a(x), e^{i\gamma}b(x)]} e^{t \text{Re}\xi} \\
\leq 2|\sqrt{x^4-x^2}|te^{\frac{1}{2}t\sin|\gamma|}, \ t \geq 0, \ x \in \mathbb{R}.
\] (21)

One obtains similarly that the following estimate holds:

\[
|e^{t e^{i\gamma}a(x)} + e^{t e^{i\gamma}b(x)}| \leq 2(1 + e^\frac{1}{2}t\sin|\gamma|), \ t \geq 0, \ x \in \mathbb{R}.
\] (22)

Using (21)-(22), it follows that:

\[
\|e^{t e^{i\gamma}P(x)}\| \leq K(1 + t)e^{\frac{1}{2}t\sin|\gamma|}, \ t \geq 0, \ x \in \mathbb{R}.
\] (23)

Taking into account (23) and the Wilcox identity \( \frac{d}{dx}e^{tP(x)} = \int_0^t e^{(t-s)P(x)} \frac{d}{dx}P(x)e^{sP(x)}ds, \ t \geq 0, x \in \mathbb{R} \), we get that:

\[
\left\| \frac{d}{dx}e^{t e^{i\gamma}P(x)} \right\| \leq K(1 + t)^2(1 + |x|)e^{\frac{1}{2}t\sin|\gamma|}, \ t \geq 0, \ x \in \mathbb{R}.
\] (24)

By (23)-(24) and [9, Theorem 2.7(2)], one gets that, for every \( t \geq 0 \) and \( i, j \in \{1, 2\} \), the mapping \( x \mapsto (1 + x^2)^{-r}A_{ij}(t, x, \gamma)/2\sqrt{x^4-x^2}, \ x \in \mathbb{R} \) belongs to the space \( \mathcal{A} \) and that:

\[
\left\| (1 + x^2)^{-r}A_{ij}(t, x, \gamma)/2\sqrt{x^4-x^2}(A) \right\| \leq K(1 + t)^\frac{3}{2}e^{\frac{1}{2}t\sin|\gamma|}, \ t \geq 0.
\] (25)

Set, for every \( t \geq 0 \), \( T_{r, \gamma}(t) := (e^{t e^{i\gamma}P(x)}(1 + x^2)^{-r})(A) \). Then \( T_{r, \gamma}(t+s)(1 + |A|^2)^{-r} = T_{r, \gamma}(t)T_{r, \gamma}(s), \ t, s \geq 0, \ T_{r, \gamma}(0) = (1 + |A|^2)^{-r} \) and \( T_{r, \gamma}(t)(1 + |A|^2)^{-s} = T_{r+s, \gamma}(t), \ s > 0, \ t \geq 0 \). Then the proof of [38, Theorem 1.2] combined with the above equalities and [17, Corollary 2.4.11] indicates that, for a sufficiently large \( s > 0 \), \( (T_{r, \gamma}(t)(1 + |A|^2)^{-s})_{t \geq 0} \) is an exponentially bounded \((1 + |A|^2)^{-r-s}\)-regularized semigroup generated by \( e^{i\gamma\overline{P}(A)} \) \((\gamma \in (-\pi/2, \pi/2))\). With (25) in view, we obtain that, for every \( \gamma \in (-\pi/2, \pi/2) \), \((T_{r, \gamma}(t))_{t \geq 0}\) is an exponentially bounded, norm continuous \((1 + |A|^2)^{-r}\)-regularized semigroup generated by \( e^{i\gamma\overline{P}(A)} \). An application of [16, Theorem 2.3] gives that \( \overline{P}(A) \) is the integral generator of an exponentially bounded, analytic \((1 + |A|^2)^{-r}\)-regularized semigroup.
\[ (T_r(\mathcal{t}) \equiv (e^{tP(x)}(1 + x^2)^{\sigma-\gamma})(A))_{\mathcal{t} \in \Sigma_{\pi}}. \]

By the foregoing, \((T_r(\mathcal{t}))_{\mathcal{t} \in \Sigma_{\pi}}\) can be extended to \(\Sigma_{\pi}^2\) and \(\|T_r(\mathcal{t})\| \leq K(1 + |t|)^{\frac{3}{2}}e^{\frac{1}{2}t\sin(|\arg(t)|)}, t \in \Sigma_{\pi}^2 \backslash \{0\}\). Assume \(t, s \in \Sigma_{\pi}^2\) and \(ts \neq 0\). Denote by \(\mathcal{F}^{-1}\) the inverse Fourier transform. Then \([5, (12.2)]\) implies that there exists \(M > 0\) such that:

\[
\|T_r(\mathcal{t}) - T_r(s)\| \leq M \sum_{1 \leq i,j \leq 2} \left\| \mathcal{F}^{-1}\left( \frac{1}{\sqrt{x^4 + x^2}} (A_{ij}(|t|, x, \arg(t)) - A_{ij}(|s|, x, \arg(s))(1 + x^2)^{-\gamma}) \right) \right\|_{L^1(\mathbb{R})},
\]

which immediately implies (a) by \([35, \text{Lemma 5.2, p. 20}]\). The proof of (b) is simple and therefore omitted. \(\square\)

Before proceeding further, we would like to observe that the linearized FitzHugh-Nagumo equations (cf. \([31, \text{p. 39}]\)), which describe the propagation of nerve impulse, can be also treated with \(C\)-regularized semigroups since the corresponding matrix \(P(x) = \begin{bmatrix} -x^2 + a & -1 \\ b & 0 \end{bmatrix} (a, b \in \mathbb{C}, x \in \mathbb{R})\) is Petrovskii correct. The isothermal motion of a one-dimensional body with small viscosity and capillarity (\([3], [10], [38]\)) is described, in the simplest situation, by the system:

\[
\begin{align*}
u_t &= 2au_{xx} + bv_x - cv_{xxx}, \\
v_t &= u_x, \\
u(0) &= u_0, \quad v(0) = v_0,
\end{align*}
\]

where \(a, b, c > 0\). The associated polynomial matrix

\[
P(x) = \begin{bmatrix} -2ax^2 & ibx + icx^3 \\ ix & 0 \end{bmatrix}
\]

is Shilov 2-parabolic. It is well known (\([10]\)) that \(P(D)\) does not generate a strongly continuous semigroup in \(L^1(\mathbb{R}) \times L^1(\mathbb{R})\). Set

\[
e^{t\sigma P(x)} := \begin{bmatrix} B_{11}(t, x, \gamma) & B_{12}(t, x, \gamma) \\ B_{21}(t, x, \gamma) & B_{22}(t, x, \gamma) \end{bmatrix}, \quad \gamma \in (0, \frac{\pi}{2}), \quad t \geq 0, \quad x \in \mathbb{R},
\]

and, for every \(x \in \mathbb{R} \backslash \{0, \pm \sqrt{\frac{b}{\alpha^2 - c}}\}\), \(\alpha(x) := -ax^2 + \sqrt{(a^2 - c)x^4 - bx^2}, \beta(x) := -ax^2 - \sqrt{(a^2 - c)x^4 - bx^2}\) and \(\chi(x) := \frac{4x^3(a^2 - c) - 2bx}{\alpha(x) - \beta(x)}\). Then

\[
\begin{bmatrix} B_{11}(t, x, \gamma) & B_{12}(t, x, \gamma) \\ B_{21}(t, x, \gamma) & B_{22}(t, x, \gamma) \end{bmatrix} = \frac{1}{\alpha(x) - \beta(x)} \begin{bmatrix} A_{11}(t, x, \gamma) & A_{12}(t, x, \gamma) \\ A_{21}(t, x, \gamma) & A_{22}(t, x, \gamma) \end{bmatrix},
\]

where
where, for every $t \geq 0$, $\gamma \in (0, \frac{\pi}{2}]$ and $x \in \mathbb{R} \setminus \{0, \pm \sqrt{\frac{b}{a^2-c}}\}$:

\[
A_{11}(t, x, \gamma) = \alpha(x)e^{te^{i\gamma}\alpha(x)} + (\alpha(x) + 2ax^2)e^{te^{i\gamma}\beta(x)},
\]
\[
A_{12}(t, x, \gamma) = (ibx + icx^3)(e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)}),
\]
\[
A_{21}(t, x, \gamma) = ix(e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)})
\]
and
\[
A_{22}(t, x, \gamma) = (\alpha(x) + 2ax^2)(e^{te^{i\gamma}\alpha(x)} + e^{te^{i\gamma}\beta(x)}).
\]

In the case $a^2 \neq c$, the following theorem improves the results of Q. Zheng, Y. Li [36] and Q. Zheng [37]-[38].

**Theorem 2.6.**

(a) Let $a^2 - c < 0$ and $r' > \frac{1}{2}$. Then $P(A)$ is the integral generator of an analytic $(1 + |A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t \geq 0}$ of angle $\theta := \arctan \frac{a}{\sqrt{c-a^2}}$ and there exists a function $p : (-\theta, \theta) \to (0, \infty)$ such that:

\[
\|T_{r'}(z)\| \leq K(1 + |z|)\frac{3}{2}e^{p(|\arg(z)|)\sin(|\arg(z)|)|z|}, \quad z \in \Sigma_{\theta}.
\]

Furthermore, the mapping $z \mapsto T_{r'}(z)$, $z \in \Sigma_{\theta_0}$ is continuous for every $\theta_0 \in (0, \theta)$.

(b) Let $a^2 - c > 0$ and $r' > \frac{1}{2}$. Then $P(A)$ generates an analytic $(1 + |A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t \geq 0}$ of angle $\frac{\pi}{2}$, and the following holds:

\[
\|T_{r'}(z)\| \leq K(1 + |z|)\frac{3}{2}\sqrt{b}\sin(|\arg(z)|)|z|, \quad z \in \Sigma_{\frac{\pi}{2}}.
\]

Furthermore, the mapping $z \mapsto T_{r'}(z)$, $z \in \Sigma_{\theta_0}$ is continuous for every $\theta_0 \in (0, \frac{\pi}{2})$.

(c) Let $a^2 - c > 0$ and $r' > \frac{3}{4}$. Then $P(A)$ generates an exponentially bounded, norm continuous $(1 + |A|^2)^{-r'}$-regularized cosine function $(C_{r'}(t))_{t \geq 0}$.

**Proof.** Notice that, for every $t \geq 0$, $x \in \mathbb{R} \setminus \{0, \pm \sqrt{\frac{b}{a^2-c}}\}$ and
\[
\gamma \in (-\pi, \pi) : \\
2 \frac{d}{dx} B_{11}(t, x, \gamma) = \frac{1}{2}(\alpha(x) - \beta(x))(-2ax + \chi(x)) - \alpha(x)\chi(x) e^{t e^{i\gamma} \alpha(x)}
\]
\[
+ \frac{2\alpha(x)}{\alpha(x) - \beta(x)} e^{t e^{i\gamma} (-2ax + \chi(x))} e^{t e^{i\gamma} \alpha(x)}
\]
\[
+ \frac{1}{2}(\alpha(x) - \beta(x))(2ax + \chi(x)) - (\alpha(x) + 2ax^2)\chi(x) e^{t e^{i\gamma} \beta(x)}
\]
\[
- \frac{2\alpha(x) + 4ax^2}{\alpha(x) - \beta(x)} e^{t e^{i\gamma} (2ax + \chi(x))} e^{t e^{i\gamma} \beta(x)},
\]

\[
2 \frac{d}{dx} B_{12}(t, x, \gamma) = \frac{1}{2} (ib + 3icx^2)(\alpha(x) - \beta(x)) - (ibx + icx^3)\chi(x) (e^{t e^{i\gamma} \alpha(x)} - e^{t e^{i\gamma} \beta(x)})
\]
\[
+ \frac{2(ibx + icx^3)}{\alpha(x) - \beta(x)} e^{t e^{i\gamma} [-2ax + \chi(x)]} e^{t e^{i\gamma} \alpha(x)} + (2ax + \chi(x)) e^{t e^{i\gamma} \beta(x)}],
\]

\[
2 \frac{d}{dx} B_{21}(t, x, \gamma) = \frac{1}{2} i(\alpha(x) - \beta(x)) - ix\chi(x) (e^{t e^{i\gamma} \alpha(x)} - e^{t e^{i\gamma} \beta(x)})
\]
\[
+ \frac{2i x te^{i\gamma}}{\alpha(x) - \beta(x)} [(-2ax + \chi(x)) e^{t e^{i\gamma} \alpha(x)} + (2ax + \chi(x)) e^{t e^{i\gamma} \beta(x)}],
\]

and
\[
2 \frac{d}{dx} B_{22}(t, x, \gamma) = 2 \frac{d}{dx} B_{11}(t, x, \gamma) + 2a \frac{4x(\alpha(x) - \beta(x)) - x^2\chi(x)}{(a^2 - c)x^4 - bx^2}.
\]

We first consider the case \(a^2 - c < 0\). If \(\gamma \in (-\theta, \theta)\), then there exist \(p(\gamma) > 0\) and \(q(\gamma) > 0\) such that:

\[
\max(\Re(e^{i\gamma} \alpha(x)), \Re(e^{i\gamma} \beta(x))) \leq -ax^2 \cos \gamma + b|x| \sqrt{b - (a^2 - c) \sin |\gamma|} \leq -q(\gamma)x^2 + p(\gamma) \sin |\gamma|.
\]
By (3), we obtain that, for every $t \geq 0$ and $x \in \mathbb{R}$:

$$\| e^{tx\gamma}P(x) \| \leq K(1+t+|x|^3)e^{-tq(\gamma)x^2+p(\gamma)t\sin|\gamma|} \leq K(1+t)(1+|x|)e^{tp(\gamma)\sin|\gamma|}. \tag{26}$$

Let $L \geq 1$ be sufficiently large. Then the inequality $\max(|txe^{-tq(\gamma)x^2}|, |tx^2e^{-tq(\gamma)x^2}|) \leq K$, $t \geq 0$, $|x| \geq L$ and the concrete representation of $\frac{d}{dx}e^{xP(x)}$, $z \in \mathbb{C}$ imply that:

$$\left\| \frac{d}{dx}e^{tx\gamma}P(x) \right\| \leq K, \ t \geq 0, \ |x| \geq L. \tag{27}$$

By (26), we easily infer that:

$$\left\| \frac{d}{dx}e^{tx\gamma}P(x) \right\| \leq K(1+t)^2e^{tp(\gamma)\sin|\gamma|}, \ t \geq 0, \ |x| \leq L. \tag{28}$$

Using (26)-(28) and the product rule, it follows that:

$$\left\| \frac{d}{dx}(e^{tx\gamma}P(x)(1+x^2)^{-r'}) \right\| \leq K(1+t)^2 e^{p(\gamma)t\sin|\gamma|}(1+|x|)^{-2r'}, \ t \geq 0, \ x \in \mathbb{R}.$$

Now one can apply [9, Theorem 2.7(2)] in order to see that:

$$\left\| (e^{tx\gamma}P(x)(1+x^2)^{-r'})(A) \right\| \leq K(1+t)^2 e^{p(\gamma)t\sin|\gamma|}, \ t \geq 0. \tag{29}$$

Put $T_{r',\gamma}(t) := (e^{tx\gamma}P(x)(1+x^2)^{-r'})(A), \ t \geq 0, \ \gamma \in (-\theta, \theta)$ and $T_{r'}(z) := (e^{zP(x)}(1+x^2)^{-r'})(A), \ z \in \Sigma_{\theta}$. By (29) and the proof of [38, Theorem 1.1], one gets that, for every $\gamma \in (-\theta, \theta)$, $(T_{r',\gamma}(t))_{t \geq 0}$ is a global $(1+|A|^2)^{-r'}$-regularized semigroup generated by $e^{t\gamma}P(A)$: By [16, Theorem 2.3], we have that $\frac{d}{dt}P(A)$ generates an analytic $(1+|A|^2)^{-r'}$-regularized semigroup of angle $\theta$. The inequality stated in (a) is a consequence of (29), while the continuity of mapping $z \mapsto T_{r'}(z), \ z \in \Sigma_{\theta_0}$ ($\theta_0 \in (0, \theta)$) follows from the estimate

$$\left\| T_{r'}(t_1e^{i\gamma_1}) - T_{r'}(t_2e^{i\gamma_2}) \right\| \leq K \sum_{1 \leq i,j \leq 2} \left\| F^{-1}((B_{ij}(t_1,x,\gamma_1) - B_{ij}(t_2,x,\gamma_2))(1+x^2)^{-r'}) \right\|_{L^1(\mathbb{R})},$$

for any $t_1, t_2 \geq 0, |\gamma_1|, |\gamma_2| \leq \theta_0$, and the proof of [35, Lemma 1.2]. In order to prove (b), one can use a similar argumentation and the estimate $\max(\Re(e^{i\gamma}\alpha(x)), \Re(e^{i\gamma}\beta(x))) \leq (\sqrt{a^2-c}-a)x^2 \cos \gamma + \frac{b}{\sqrt{a^2-c}} \sin |\gamma|, \ \gamma \in$
It is not clear how one can prove the assertion of Theorem 2.6 in the case \( a^2 > c \). To prove (c), set \( f(t, x) := \sum_{n=0}^{\infty} \frac{t^{2n}P(x)^n}{(2n)!}, \ t \geq 0, \ x \in \mathbb{R} \). Then
\[
f(t, x) = \frac{1}{\alpha(x) - \beta(x)} \begin{bmatrix} C_{11}(t, x) & C_{12}(t, x) \\ C_{21}(t, x) & C_{22}(t, x) \end{bmatrix},
\]
where, for any \( t \geq 0 \) and \( x \in \mathbb{R} \setminus \{0, \pm \sqrt{\frac{b}{a^2 - 1}}\} : 
\[
C_{11}(t, x) = \alpha(x) \cosh(t\sqrt{\alpha(x)}) + (\alpha(x) + 2ax^2) \cosh(t\sqrt{\beta(x)}),
\]
\[
C_{12}(t, x) = (ibx + icx^3)(\cosh(t\sqrt{\alpha(x)}) - \cosh(t\sqrt{\beta(x)})),
\]
\[
C_{21}(t, x) = ix(\cosh(t\sqrt{\alpha(x)}) - \cosh(t\sqrt{\beta(x)})) \text{ and }
\]
\[
C_{22}(t, x) = (\alpha(x) + 2ax^2)(\cosh(t\sqrt{\alpha(x)}) + \cosh(t\sqrt{\beta(x)})).
\]

Then \( \max(|\cosh(t\sqrt{\alpha(x)})|, |\cosh(t\sqrt{\beta(x)})|) \leq 1, \ t \geq 0, \ |x| \geq L \). Using this estimate, the product rule as well as the representation formulae for \( f(t, x) \) and \( \frac{d}{dx} f(t, x) \), it follows that there exist two exponentially bounded positive functions \( f_0(t) \) and \( f_1(t) \) such that \( ||f(t, x)(1 + x^2)^{-r'}|| \leq Kf_0(t)(1 + |x|)^{-2r'}, \ t \geq 0, \ x \in \mathbb{R} \) and \( ||\frac{d}{dx}(f(t, x)(1 + x^2)^{-r'})|| \leq Kf_0(t) + f_1(t)(1 + |x|)^{-2r'}, \ t \geq 0, \ x \in \mathbb{R} \). Define \( C_r(t) := (f(t, x)(1 + x^2)^{-r'})(A), \ t \geq 0 \). Then it is obvious that \( (C_r(t))_{t \geq 0} \) is an exponentially bounded, norm continuous \( (1 + |A|^2)^{-r'} \)-regularized cosine function generated by \( \overline{P}(A) \). \( \square \)

Remark 2.7.

(a) It is not clear how one can prove the assertion of Theorem 2.6 in the case \( a^2 = c \). The main problem is that, for every fixed \( t \geq 0 \) and \( \gamma \in [0, \frac{\pi}{2}] \), the function \( x \mapsto \frac{d}{dx} B_{12}(t, x, \gamma) \), \( x \in \mathbb{R} \) is no longer bounded as \( |x| \to \infty \). Nevertheless, the operator \( \overline{P}(A) \) generates an exponentially bounded, analytic \( (1 + |A|^2)^{-3/4} \)-regularized semigroup \( (T_{3/4}(t))_{t \geq 0} \) of angle \( \frac{\pi}{2} \); as before, one can additionally refine the results obtained in Theorem 2.5-Theorem 2.6, provided that \( X = L^p(\mathbb{R}) \) \( (1 < p < \infty) \).

(b) With the exception of the continuity of mapping \( z \mapsto T_r(z), \ z \in \Sigma_{\alpha} \), the assertions of Theorem 2.6(a)-(b) continue to hold in the case \( r' = \frac{1}{2} \) (cf. [17, Example 1.2.9(vii.1)-(vii.2)] and Remark 2.4(a)). Furthermore, Theorem 2.6(c) is a proper extension of the result stated in [17, Example 1.2.9(vii.3)].
(c) Let $a^2 = c$, $r' = \frac{1}{2}$ and $r'' = \frac{3}{4}$. Then $\lim_{t \to 0^+} (B_{12}(t, x, 0))(A)g = 0$ and $\lim_{t \to 0^+} (B_{22}(t, x, 0))(A)g = g$ for all $g \in R((1 + |A|^2)^{-r''})$. Keeping in mind the proof of Theorem 2.6, we get that $\| (B_{11}(t, x, 0)(1 + x^2)^{-\frac{1}{2}})(A) \| \leq K(1 + t)^{\frac{3}{2}}$ and $\| (B_{21}(t, x, 0)(1 + x^2)^{-\frac{3}{2}})(A) \| \leq K(1 + t)^{\frac{3}{2}}$, $t \geq 0$, $x \in \mathbb{R}$. By the denseness of $E$ in $X$, the above implies $\lim_{t \to 0^+} (B_{11}(t, x, 0))(A)f = f$ for all $f \in R((1 + |A|^2)^{-r'})$. On the other hand, the representation formulae for $B_{21}(t, x, 0)$ and $\frac{d}{dx} B_{21}(t, x, 0)$, as well as the proofs of Theorem 2.1 and Theorem 2.6, taken together imply $\lim_{t \to 0^+} \| (B_{21}(t, x, 0))(A) \| = 0$. Hence, $R((1 + |A|^2)^{-r'}) \times R((1 + |A|^2)^{-r'\prime}) \subseteq \Omega(T_0)$.

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