ON POSITIVITY PROPERTIES OF FUNDAMENTAL CARDINAL POLYSPLINES¹

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Abstract. Polysplines on strips of order $p$ are natural generalizations of univariate splines. In [3] and [4] interpolation results for cardinal polysplines on strips have been proven. In this paper the following problems will be addressed: (i) positivity of the fundamental polyspline on the strip $[-1,1] \times \mathbb{R}^n$, and (ii) uniqueness of interpolation for polynomially bounded cardinal polysplines.

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1. Introduction

A function $f : U \rightarrow \mathbb{C}$ defined on an open subset $U$ of the euclidean space $\mathbb{R}^{n+1}$ is polyharmonic of order $p$ if it is $2p$ times continuously differentiable

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and \( \Delta^p f(x) = 0 \) for all \( x \in U \), where \( \Delta^p \) is the \( p \)-th iterate of the Laplace operator \( \Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n+1}^2} \). A famous example in the area of interpolation with polyharmonic functions are the so-called thin–plate splines (and more generally, polyharmonic splines) which are linear combinations of translates of the function \( \varphi \) defined by

\[
\varphi(x) = |x|^2 \log |x|; \tag{1}
\]

it is well known that (1) is the fundamental solution of the biharmonic operator \( \Delta^2 \) in \( \mathbb{R}^2 \). Since the appearance of the fundamental work of Duchon [8] such "splines" have been used by numerous authors for interpolation purposes in the multivariate case, see, for example, the papers of W. Madych and S. Nelson [18], K. Jetter [10], and the recent monograph [5]. In all these examples one interpolates data prescribed on a (finite or countable) set of discrete points.

An alternative and completely different "data concept" is provided by the notion of polyspline, introduced by O. Kounchev in [11], and extensively discussed in [12]. Polysplines distinguish from the widely spread data principle and allow to interpolate functions prescribed on surfaces of codimension 1; for a concrete application see [17]. As in [3],[4] and [15] we consider here the case that data functions are prescribed on parallel equidistant hyperplanes. Let us recall that a function \( S : \mathbb{R}^{n+1} \to \mathbb{C} \) is a cardinal polyspline of order \( p \) on strips, when \( S \) is a \( 2p-2 \) times continuously differentiable function on \( \mathbb{R}^{n+1} \) which is polyharmonic of order \( p \) on the strips \((j, j+1) \times \mathbb{R}^n, j \in \mathbb{Z}\), where as usually \((a, b)\) denotes the open interval in \( \mathbb{R} \) with endpoints \( a, b \), and \( \mathbb{Z} \) is the set of all integers. Note that for \( n = 0 \) (with the identification \( \mathbb{R}^0 = \{0\} \) and \( \mathbb{R} \times \{0\} = \mathbb{R} \)) a cardinal polyspline of order \( p \) on strips is just a cardinal spline on the real line \( \mathbb{R} \) of degree \( 2p-1 \) (hence of order \( 2p \)), as discussed by I. Schoenberg in his celebrated monograph [21] (or [22]). In passing, let us remark that in the recent paper [15] it has been proved that the cardinal polysplines on strips occur as a natural limit of polyharmonic splines considered on the lattice \( \mathbb{Z} \times a\mathbb{Z}^n \) when the positive number \( a \to 0 \), and an estimate of the rate of convergence has been given in [16]. A discussion of wavelet analysis of cardinal polysplines can be found in [12] and [13].

In the first section we recall briefly the main results about interpolation with polysplines presented by A. Bejancu, O. Kounchev and the author in [4] (for the case \( p = 2 \) see [3]). An important tool are so-called fundamental cardinal polysplines which can be seen as the multivariate analog of the
fundamental cardinal spline \( L^0 : \mathbb{R} \to \mathbb{R} \) which is by definition the unique cardinal spline which has exponential decay and the interpolation property
\[
L^0(0) = 1 \quad \text{and} \quad L^0(j) = 0 \quad \text{for} \quad j \in \mathbb{Z}, \, j \neq 0.
\] (2)

We call a polyspline \( L_f \) a fundamental cardinal polyspline with respect to the data function \( f : \mathbb{R}^n \to \mathbb{C} \) if
\[
L_f(0, y) = f(y) \quad \text{and} \quad L_f(j, y) = 0 \quad \text{for} \quad j \in \mathbb{Z} \setminus \{0\}, \, y \in \mathbb{R}^n
\] (3)
and if there exists \( C > 0 \) and \( \varepsilon > 0 \) such that \( |L_f(t, y)| \leq Ce^{-\varepsilon|t|} \) for all \( y \in \mathbb{R}^n, \, t \in \mathbb{R} \). The existence of fundamental cardinal polysplines is guaranteed by Theorem 2, and the reader may take formula (9) as a defining formula.

It is a well-known fact that the fundamental cardinal spline \( L^0 \) defined in (2) is non-negative on the unit interval \([-1, 1]\), see [7]. One aim of this paper is to discuss the question whether the fundamental cardinal polyspline \( L_f : \mathbb{R}^{n+1} \to \mathbb{C} \) is non-negative on the strip \([-1, 1] \times \mathbb{R}^n \) for any non-negative integrable function \( f : \mathbb{R}^n \to [0, \infty) \). Unfortunately, we have not been able to give a positive answer to this question, although numerical experiments support this conjecture. However, in the second section we shall prove that the non-negativity of \( L_f \) on \([-1, 1] \times \mathbb{R}^n \) for any non-negative integrable function \( f : \mathbb{R}^n \to [0, \infty) \) is equivalent to the positive definiteness of a certain family of functions \( \xi \mapsto L^\xi(t) \) where \( t \) ranges over \([-1, 1]\). Here \( L^\xi \) is the fundamental cardinal L-spline \( L^\xi : \mathbb{R} \to \mathbb{R} \) (cf. [19] and [3] for definition and details) which can be written as
\[
L^\xi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\left( s^2 + |\xi|^2 \right)^p} S_p(s, \xi) \, ds,
\] (4)
where
\[
S_p(s, \xi) := \sum_{k \in \mathbb{Z}} \frac{1}{(s + 2\pi k)^2 + |\xi|^2}.
\] (5)

In the third section we shall show that for the special, and much simpler, case \( p = 1 \) the fundamental cardinal polyspline \( L_f \) is non-negative on the strip \([-1, 1] \times \mathbb{R}^n \) for any non-negative integrable function \( f : \mathbb{R}^n \to [0, \infty) \). Moreover we give a simplified formula for the fundamental cardinal polyspline \( L_f \) in the case \( p = 1 \).

The last section is devoted to the question under which conditions interpolation with cardinal polysplines on strips is unique. A simple example
shows that even for the case \( p = 1 \) there is no uniqueness if we do not impose some growth conditions. The author believes that for polynomially bounded polysplines interpolation is unique; in the last section it is proved that this is true for the case \( p = 1 \). It is hoped that the results presented here motivate further research on the subject.

Let us recall some terminology and notation: the Fourier transform of an integrable function \( f : \mathbb{R}^n \to \mathbb{C} \) is defined by

\[
\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i(y,\xi)} f(y) \, dy.
\]

By \( B_s(\mathbb{R}^n) \) we denote the set of all measurable functions \( f : \mathbb{R}^n \to \mathbb{C} \) such that the integral

\[
\|f\|_s := \int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right| (1 + |\xi|^s) \, d\xi
\]

is finite (see Definition 10.1.6 in Hörmander [9], vol. 2). By \( S(\mathbb{R}^n) \) we denote the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^n \), see [25, p. 19].

A function \( f : \mathbb{R}^n \to \mathbb{R} \) is radially symmetric if \( f(x) \) depends only on the Euclidean norm \( |x| = \sqrt{x_1^2 + \ldots + x_n^2} \).

2. Interpolation with Polysplines

In this section we recall the interpolation theorem for cardinal polysplines of order \( p \) proved by A. Bejancu, O. Kounchev and the present author. As mentioned above, this result formally includes the theorem of I. Schoenberg about cardinal spline interpolation by setting \( n = 0 \). But it should be emphasized that the proof of Theorem 1 relies on results of Ch. Micchelli in [19] about cardinal interpolation with so-called L-splines which itself is a generalization of Schoenberg’s theorem.

**Theorem 1.** Let \( \gamma \geq 0 \) be fixed. Let integrable functions \( f_j : \mathbb{R}^n \to \mathbb{C} \) be given such that \( f_j \in B_{2p-2}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \), and assume that the following growth condition holds

\[
\|f_j\|_{2p-2} \leq C (1 + |j|^\gamma) \quad \text{for all } j \in \mathbb{Z},
\]

(7)

Then there exists a polyspline \( S \) of order \( p \) on strips satisfying

\[
S(j, y) = f_j(y) \quad \text{for } y \in \mathbb{R}^n, \quad j \in \mathbb{Z},
\]

(8)

as well as the growth estimate

\[
|S(t, y)| \leq D (1 + |t|^\gamma) \quad \text{for all } y \in \mathbb{R}^n.
\]
An important step in the proof of the last theorem is the following:

Theorem 2. Let \( f \in L_1(\mathbb{R}^n) \cap B_{2p-2}(\mathbb{R}^n) \) and define \( L^\xi \) as in (4). Then the function \( L_f \) defined by

\[
L_f(t,y) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\langle y, \xi \rangle} \hat{f}(\xi) L^\xi(t) \, d\xi
\]  

(9)

is a polyspline of order \( p \) such that

\[
\begin{aligned}
L_f(0,y) &= f(y) \quad \text{for } y \in \mathbb{R}^n, \\
L_f(j,y) &= 0 \quad \text{for } y \in \mathbb{R}^n, \quad \text{for all } j \neq 0.
\end{aligned}
\]

There exists a constant \( C > 0 \) and \( \eta > 0 \) such that for every multi-index \( \alpha \in \mathbb{N}_0^{n+1} \) with \( |\alpha| \leq 2(p-1) \), the decay estimate

\[
\left| \frac{\partial^{\alpha}}{\partial x^{\alpha}} L_f(x) \right| \leq C e^{-\eta|t|} \|f\|_{|\alpha|}
\]  

(10)

holds for all \( x = (t,y) \in \mathbb{R}^{n+1} \).

Theorem 3 can be deduced from Theorem 2 by considering the Lagrange-type representation

\[
S(t,y) = \sum_{j=-\infty}^{\infty} L_{f_j}(t-j,y).
\]  

(11)

Details and proofs can be found in [4] and [3]. In this paper we shall make use only of formula (9) which can be taken as a definition for \( L_f \). What we need in this paper is the following fact which also shows that (9) is well-defined.

Theorem 3. There exist constants \( C > 0 \) and \( \eta > 0 \), such that for all \( t \in \mathbb{R}, \xi \in \mathbb{R}^n \) the following estimate holds:

\[
\left| L^\xi(t) \right| \leq C e^{-\eta|t|}.
\]  

(12)

A proof for \( p = 2 \) can be found in [3], and for arbitrary \( p \) in [4].

3. A conjecture about positivity of the fundamental spline

Recall that a function \( g : \mathbb{R}^n \to \mathbb{C} \) is positive definite if for all \( y_1, \ldots, y_N \in \mathbb{R}^n \) and for all complex numbers \( c_1, \ldots, c_N \) the inequality
\[
\sum_{k,l=1}^{N} c_l c_k g(y_k - y_l) \geq 0
\]

holds; for properties of positive definite functions we refer to [23], cf. also
the nice introduction [24]. It is well-known that the product of two positive
definite functions is positive definite. Moreover it is elementary to see that
the Fourier transform \( \hat{g} \) of a non-negative function \( g \in L_1(\mathbb{R}^n) \) is positive
definite. Conversely, if \( g \in L_1(\mathbb{R}^n) \) is positive definite then the Fourier
transform is a non-negative function on \( \mathbb{R}^n \) (Theorem of Mathias).

Properties of the fundamental cardinal spline \( L^0 : \mathbb{R} \rightarrow \mathbb{R} \) have been
investigated by de Boor and Schoenberg in [7]. One particularly nice property
is that \( L^0 \) has an alternating sign on the intervals \((k, k + 1)\) for \( k \in \mathbb{N}_0 \), i.e.,
that

\[ (-1)^k L^0(x + k) \geq 0 \]

for all \( k \in \mathbb{N}_0, x \in (0, 1) \). Numerical experiments have lead us to formulate
the following conjecture:

**Conjecture 4.** Let \( f \in L_1(\mathbb{R}^n) \cap B_{2p-2}(\mathbb{R}^n) \). If \( f \) is non-negative then
the fundamental polyspline \( L_f \) has an alternating sign on the strips \((k, k + 1) \times \mathbb{R}^n\) for \( k \in \mathbb{N}_0 \), i.e.,
that

\[ (-1)^k L_f(t + k, y) \geq 0 \]

for all \( k \in \mathbb{N}_0, t \in (0, 1) \) and \( y \in \mathbb{R}^n \).

Note that for \( k = 0 \) the conjecture implies that \( L_f(t, y) \geq 0 \) for all \((t, y) \in [-1, 1] \times \mathbb{R}^n \). The following result shows that the latter property is
equivalent to the positive definiteness of the function \( \xi \mapsto L^\xi(t) \) for each
\( t \in [-1, 1] \). Note that this formulation is independent of the data function \( f \).

**Theorem 5.** Let \( t \in \mathbb{R} \) be fixed. Then the following statements are
equivalent

(i) The function \( \xi \mapsto L^\xi(t) \) is positive definite.

(ii) For each non-negative \( f \in L_1(\mathbb{R}^n) \) such that \( \hat{f} \in L_1(\mathbb{R}^n) \) the funda-
mental cardinal polyspline \( L_f \) is non-negative on \( \{t\} \times \mathbb{R}^n \).

(iii) For each non-negative, radially symmetric function \( f \in S(\mathbb{R}^n) \) the
function \( L_f \) is non-negative on \( \{t\} \times \mathbb{R}^n \).
Proof. For (i) $\rightarrow$ (ii) let $f \in \mathcal{L}^1(\mathbb{R}^n)$ be non-negative, clearly then $\hat{f}$ is positive definite. By assumption, $\xi \mapsto \mathcal{F}(\xi) L^\xi(t)$ is positive definite. By the above remarks the function $\xi \mapsto \mathcal{F}(\xi) L^\xi(t)$ is positive definite. Since by Theorem 3 the function $\xi \mapsto L^\xi(t)$ is bounded, we know that $\xi \mapsto \hat{f}(\xi) L^\xi(t)$ is integrable. By the theorem of Mathias (see [24, p. 412]) the (inverse) Fourier transform is non-negative, i.e., that for all $y \in \mathbb{R}^n$

$$L_f(t, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(y, \xi)} \hat{f}(\xi) L^\xi(t) \, d\xi \geq 0.$$ 

The implication (ii) $\rightarrow$ (iii) is trivial.

Let us show that (iii) $\rightarrow$ (i). We use arguments from the proof of Bochner’s theorem in [1, p. 196]: Let us define $f_\delta(y) := e^{-\frac{1}{2}\delta|y|^2}$ which is radially symmetric and in the Schwartz class. By Theorem 3 the function $\xi \mapsto L^\xi(t)$ is bounded. Hence $g_\varepsilon$ defined by $g_\varepsilon(\xi) := L^\xi(t) e^{-\varepsilon|\xi|^2}$ is integrable for any $\varepsilon > 0$. Parseval’s identity yields

$$\int_{\mathbb{R}^n} f_\delta(y) \hat{g}_\varepsilon(y) \, dy = \int_{\mathbb{R}^n} \hat{f}_\delta(\xi) g_\varepsilon(\xi) \, d\xi. \quad (13)$$

On the other hand, assumption (iii) implies that

$$L_{f_\varepsilon}(t, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(y, \xi)} e^{-\varepsilon|\xi|^2} L^\xi(t) \, d\xi \geq 0.$$ 

Thus $\hat{g}_\varepsilon(y) = (2\pi)^n L_{f_\varepsilon}(t, -y) \geq 0$ for all $y \in \mathbb{R}^n$. So we obtain from (13) that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\delta|y|^2} \hat{g}_\varepsilon(y) \, dy = \left| \int_{\mathbb{R}^n} e^{-\frac{1}{2}\delta|y|^2} \hat{g}_\varepsilon(y) \, dy \right| \leq M \int_{\mathbb{R}^n} \hat{f}_\delta(\xi) \, d\xi, \quad (14)$$

where $M$ is a constant such that $|g_\varepsilon(\xi)| \leq M$ for all $\xi \in \mathbb{R}^n$ and for all $0 < \varepsilon \leq 1$. Since

$$\int_{\mathbb{R}^n} \hat{f}_\delta(\xi) \, d\xi = (2\pi)^n f_\delta(0) \leq (2\pi)^n$$

we conclude from (14) and Fatou’s lemma that $\hat{g}_\varepsilon$ is integrable. Now the inversion formula

$$g_\varepsilon(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(y, \xi)} \hat{g}_\varepsilon(y) \, dy. \quad (15)$$
shows that \( g_\varepsilon (\xi) := L_\xi (t) e^{-\varepsilon |\xi|^2} \) is positive definite. Then \( L_\xi (t) = \lim_{\varepsilon \to 0} g_\varepsilon (\xi) \) for each \( \xi \in \mathbb{R}^n \) (and fixed \( t \)), and since the pointwise limit of positive definite functions is again positive definite, it follows that \( \xi \mapsto L_\xi (t) \) is positive definite. \( \square \)

4. Positivity of fundamental cardinal polysplines on \([-1, 1] \times \mathbb{R}^n \) for \( p = 1 \).

Recall that a function \( g: \mathbb{R}^n \to \mathbb{C} \) vanishes at infinity if for each \( \varepsilon > 0 \) there exists a compact subset \( K \) of \( \mathbb{R}^n \) such that \( |f (x)| < \varepsilon \) for all \( x \in \mathbb{R}^n \setminus K \). Now we want to prove

**Theorem 6.** Let \( p = 1 \). Let \( f \in L_1 (\mathbb{R}^n) \) such that \( \hat{f} \in L_1 (\mathbb{R}^n) \). If \( f \) is non-negative then \( L_f \) defined in (9) is a non-negative function on \( \mathbb{R}^{n+1} \).

**Proof.** From the definition of \( L_f \) and \( L_\xi \) it follows that

\[
L_f (t, y) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{i \langle y, \xi \rangle} e^{its} \hat{f} (\xi) \frac{\hat{f} (\xi)}{s^2 + |\xi|^2} S_1 (s, \xi) ds d\xi.
\]

Further it can be shown that \( (\xi, s) \mapsto \hat{f} (\xi) / (s^2 + |\xi|^2) S_1 (s, \xi) \) is integrable. The Lemma of Riemann-Lebesgue (see [25, p. 2]) shows that \( L_f : \mathbb{R}^{n+1} \to \mathbb{C} \) vanishes at infinity. Now the next theorem applied to \( L_f \) and \( j \in \mathbb{Z} \), shows that \( L_f \) is a non-negative function. \( \square \)

**Theorem 7.** Let \( S : \mathbb{R}^{n+1} \to \mathbb{C} \) be a cardinal polyspline of order 1 on strips which vanishes at infinity and let \( j \in \mathbb{Z} \). If

\[
S (j, y) \geq 0 \text{ and } S (j + 1, y) \geq 0 \text{ for all } y \in \mathbb{R}^n
\]

then \( S \) is non-negative on \([j, j + 1] \times \mathbb{R}^n \).

**Proof.** Let \( \varepsilon > 0 \) be arbitrary. Since \( S \) vanishes at infinity we can find \( R > 0 \) such that \( |S (t, y)| < \varepsilon \) if \( |t| > R \) or \( |y| > R \). Define \( G_R = [j, j + 1] \times \{ y \in \mathbb{R}^n : |y| \leq R + 1 \} \). Then \( S (t, y) \geq -\varepsilon \) for \( (t, y) \) in the boundary of \( G_R \). Since \( S \) is a harmonic function in the interior of \( G_R \) and continuous on \( G_R \) the minimum principle yields that \( S (t, y) \geq -\varepsilon \) for all \( (t, y) \in G_R \). Hence \( S (t, y) \geq -\varepsilon \) for given \( (t, y) \in G_R \). Since \( \varepsilon > 0 \) is arbitrary we obtain \( S (t, y) \geq 0 \) and the proof is accomplished. \( \square \)
In the rest of this section we want to give an explicit formula for \( L^\xi \) in the case that \( p = 1 \) (see (16)) which clearly leads to a simpler formula for fundamental cardinal polysplines, see formula (17). From formula (16) one can see that \( \xi \mapsto L^\xi(t) \) is positive definite for each \( t \in [-1,1] \), so one obtains with Theorem 5 a second proof that \( L_f \) is non-negative on \([-1,1] \times \mathbb{R}^n \) for a non-negative data function \( f \in L^1(\mathbb{R}^n) \) such that \( \hat{f} \in L^1(\mathbb{R}^n) \). Unfortunately, for \( p \geq 2 \) we do not have simple formulas for \( L^\xi \).

Let us compute \( S_p(s,\xi) \) defined in (5) for \( p = 1 \). An application of Poisson’s summation formula (see [6, p. 204]) shows that

\[
2 \sum_{k \in \mathbb{Z}} \frac{y}{y^2 + (x + 2\pi k)^2} = \sum_{k \in \mathbb{Z}} e^{-|k| y} e^{ikx} = \frac{1 - e^{-2y}}{1 - 2e^{-y} \cos x + e^{-2y}}.
\]

We apply this to \( x := s \) and \( y := |\xi| > 0 \) and obtain for \( S_p \) defined in (5) with \( p = 1 \)

\[
S_1(s,\xi) = \frac{1 - e^{-2|\xi|}}{2|\xi| (1 - 2e^{-|\xi|} \cos s + e^{-2|\xi|})}.
\]

Hence we obtain

\[
L^\xi(t) = \frac{1}{\pi} \frac{|\xi|}{1 - e^{-2|\xi|}} \int_{-\infty}^{\infty} e^{its} \frac{1 - 2e^{-|\xi|} \cos s + e^{-2|\xi|}}{s^2 + |\xi|^2} ds.
\]

Since \( 2e^{its} \cos s = e^{its} (e^{is} + e^{-is}) = e^{i(t+1)s} + e^{is(t-1)} \) we see that \( L^\xi(t) \) is equal to

\[
\frac{|\xi|}{\pi} \frac{1 + e^{-2|\xi|}}{1 - e^{-2|\xi|}} \int_{-\infty}^{\infty} e^{its} \frac{1 - 2e^{-|\xi|} \cos s + e^{-2|\xi|}}{s^2 + |\xi|^2} ds - \frac{|\xi|}{\pi} \frac{e^{-|\xi|}}{1 - e^{-2|\xi|}} \int_{-\infty}^{\infty} e^{i(t+1)s} + e^{is(t-1)} ds.
\]

Since \( \int_{-\infty}^{\infty} e^{its} \frac{1}{s^2 + |\xi|^2} ds = \frac{\pi}{|\xi|} e^{-|t||\xi|} \) a straightforward computation shows that

\[
L^\xi(t) = \frac{1}{e^{\xi(t)} - e^{\xi(-t)}} \left[ (e^{\xi} + e^{-\xi}) e^{-|t||\xi|} - e^{-|t+1||\xi|} - e^{-|t-1||\xi|} \right].
\]

If \( t \geq 1 \) one obtains easily \( L^\xi(t) = 0 \). For \( 0 \leq t \leq 1 \) one has

\[
L^\xi(t) = \frac{e^{\xi(t)} - e^{-(1-t)|\xi|}}{e^{\xi} - e^{-\xi}} = \frac{\sinh ((|\xi| - 1)(1 - t))}{\sinh |\xi|}.
\]

We now summarize the result:
Corollary 8. Let $p = 1$. For $|t| \geq 1$ the function $L^\xi$ vanishes and for $0 \leq t \leq 1$
\[
L^\xi(t) = \frac{\sinh(|\xi|(1-t))}{\sinh|\xi|}.
\] (16)
In case $\xi = 0$ the function $t \mapsto L^0(t)$ is a linear spline and $L^0(t) = 1 - t$ for $0 \leq t \leq 1$.

Now Theorem 2 for $p = 1$ can be read as follows:

Theorem 9. Let $f \in L_1(\mathbb{R}^n)$ such that $\hat{f} \in L_1(\mathbb{R}^n)$. Then there exists a continuous function $L_f : \mathbb{R}^{n+1} \to \mathbb{C}$ which is harmonic in $(-1,0) \times \mathbb{R}^n$ and $(0,1) \times \mathbb{R}^n$ such that
\[
L_f(0,y) = f(y)
\]
for $y \in \mathbb{R}^n$, and it vanishes for all $(t,y) \in \mathbb{R}^{n+1}$ with $|t| \geq 1$. Further for $0 \leq t \leq 1$
\[
L_f(t,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(y,\xi)} \hat{f}(\xi) \frac{\sinh(|\xi|(1-t))}{\sinh|\xi|} d\xi.
\] (17)

The fundamental linear interpolation spline has nice symmetry properties around $x = \frac{1}{2}$. In the following we want to formulate a symmetry property for cardinal polysplines of order 1. Formula (17) suggests that we have to use the addition theorem for sinh:
\[
\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}.
\] (18)

Proposition 10. For $0 \leq s \leq t \leq 1$ the following relation holds
\[
L^\xi(s) = L^\xi(t) + 2L^\xi\left(1 - \frac{t-s}{2}\right) \cosh(\frac{2-s-t}{2} |\xi|).
\] (19)
Proof. Put $x = (1-s) |\xi|$ and $y = (1-t) |\xi|$ in (18): then $x + y = (2-s-t) |\xi|$ and $x - y = (t-s) |\xi|$ and we have
\[
\sinh[(1-s) |\xi|] - \sinh[(1-t) |\xi|] = 2 \cosh(\frac{2-s-t}{2} |\xi|) \sinh(\frac{t-s}{2} |\xi|).
\] (20)
Now divide (20) by sinh $|\xi|$ and use formula (16).

As an illustration put $s = \frac{1}{2} - \delta$ and $t = \frac{1}{2} + \delta$ in (19). Then
\[
L^\xi\left(\frac{1}{2} - \delta\right) - L^\xi\left(\frac{1}{2} + \delta\right) = 2 \cosh\left(\frac{1}{2} |\xi|\right) \cdot L^\xi(1-\delta)
\]
Multiply (19) with $\hat{f}(\xi) e^{i\langle y, \xi \rangle}$ and integrate with respect to $d\xi$. Then (17) implies that for an integrable function $f$ the following formula holds:

$$L_f(\frac{1}{2} - \delta, y) - L_f(\frac{1}{2} + \delta, y) = \frac{2}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle y, \xi \rangle} \hat{f}(\xi) \cosh(\frac{1}{2} |\xi|) L^\xi (1 - \delta) \, d\xi.$$ 

5. **Uniqueness of interpolation for polynomially bounded polysplines for $p = 1$**

In this section we want to prove uniqueness results for interpolation: suppose that $S_1$ and $S_2$ are two polysplines interpolating the same data. It is clear that $S_2 - S_1$ vanishes on $\{j\} \times \mathbb{R}^n$ for all $j \in \mathbb{Z}$. We would like to conclude that $S_2 - S_1 = 0$. The following simple example shows that we have to impose some conditions on the interpolation polysplines even in the case $p = 1$ in order to obtain uniqueness:

**Example 11.** There exists a harmonic function $f$ on $\mathbb{R}^2$ which vanishes on all hyperplanes $\{j\} \times \mathbb{R}$, $j \in \mathbb{Z}$ without being identically zero, namely

$$f(t, y) = \sin \pi t \cdot e^{\pi y}.$$ 

As mentioned in the introduction we believe that interpolation is unique if we assume that $S$ is polynomially bounded, i.e., that there exists a polynomial $p(x)$ such that

$$|S(x)| \leq |p(x)|$$

for all $x \in \mathbb{R}^{n+1}$.

In the following we shall prove this for $p = 1$. In the case that $S_1$ and $S_2$ vanish at infinity we could use Theorem 7 applied to $S_2 - S_1$ and $S_1 - S_2$: then $S_2 - S_1$ and $S_1 - S_2$ are non-negative functions on the whole space, hence $S_2 - S_1 = 0$.

Instead of the minimum principle we will use the Schwarz reflection principle for harmonic functions (see e.g., [2, p. 66]) in order to prove uniqueness. Reflection principles for polyharmonic functions have been investigated by several authors and we refer to [20] for a nice introduction. However, it seems that the latter results can not be used for a proof of uniqueness of interpolation for polysplines of order $p > 1$.

**Proposition 12.** Suppose that $S : \mathbb{R}^{n+1} \to \mathbb{C}$ is a cardinal polyspline of order 1 on strips with $S(j, y) = 0$ for all $j \in \mathbb{Z}$ and $y \in \mathbb{R}^n$. Then there
exists a harmonic function $h : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ such that
\[
\begin{align*}
h(t, y) &= S(t, y) \quad \text{for } t \in (0, 1) \text{ and } y \in \mathbb{R}^n, \quad (21) \\
h(j, y) &= 0 \quad \text{for } j \in \mathbb{Z} \text{ and } y \in \mathbb{R}^n, \quad (22)
\end{align*}
\]
and for each natural number $N$
\[
\max_{|y| \leq N, t \in \mathbb{R}} |h(t, y)| \leq \max_{|y| \leq N, 0 \leq t \leq 1} |S(t, y)|. \tag{23}
\]

Proof. Clearly $S$ is a harmonic function on the strip $(0, 1) \times \mathbb{R}^n$, and it is continuous on the closure of the strip. By the Schwarz reflection principle, $S$ can be extended to a continuous function $S_1$ on $[-1, 1] \times \mathbb{R}^n$ by defining
\[
S_1(-t, y) = -S(t, -y) \quad \text{for } t \in [-1, 0]
\]
which is harmonic on $(-1, 1) \times \mathbb{R}^n$. Further $S_1(-1, y) = -S(1, -y) = 0$ for all $y \in \mathbb{R}^n$, so $S_1$ vanishes on the boundary of the new strip $[-1, 0] \times \mathbb{R}^n$ and clearly the maximum of $|h|$ on $\{(t, y) : |y| \leq N, -1 \leq t \leq 0\}$ can be estimated by
\[
\max_{|y| \leq N, -1 \leq t \leq 0} |S_1(t, y)| \leq \max_{|y| \leq N, 0 \leq t \leq 1} |S(t, y)|.
\]
Now apply the same procedure to $S_1 : [-1, 0] \times \mathbb{R}^n$ at the hyperplane $\{-1\} \times \mathbb{R}^n$, obtaining an extension $S_2$ on $[-2, 0] \times \mathbb{R}^n$ of $S_1$ with
\[
\max_{|y| \leq N, -2 \leq t \leq -1} |S_2(t, y)| \leq \max_{|y| \leq N, -1 \leq t \leq 0} |S_1(t, y)| \leq \max_{|y| \leq N, 0 \leq t \leq 1} |S(t, y)|.
\]
Proceed in this way for negative $j \in \mathbb{Z}$, then for positive $j \in \mathbb{Z}$ and we arrive at a harmonic function $h : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ with the desired properties. \qed

Theorem 13. Let $S : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ be a cardinal polyspline of order 1 on strips vanishing on the affine hyperplanes $\{j\} \times \mathbb{R}^n$, $j \in \mathbb{Z}$. If $S$ is polynomially bounded then $S$ is identically zero.

Proof. By Proposition 12 there exists a harmonic function $h : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ with (21), (22) and (23). Since $S$ is polynomially bounded, (23) implies that $h$ is polynomially bounded. It follows that $h$ is a harmonic polynomial, see [2, p. 41]. A polynomial $h(t, y)$ which vanishes on the hyperplanes $\{j\} \times \mathbb{R}^{n+1}$ for all $j \in \mathbb{Z}$ is identically zero: the equation $h(0, y) = 0$ for all $y \in \mathbb{R}^n$ implies that the (finite) Taylor expansion of $h(t, y)$ contains only non-trivial summands where the variable $t$ occurs. Hence $h(t, y) = t \cdot h_1(t, y)$
with a polynomial $h_1$. Similarly, $h_1(1, y) = 0$ for all $y \in \mathbb{R}^n$ implies that $h_1(t, y) = (t - 1) h_2(t, y)$. Hence we can write

$$h(t, y) = t(t - 1) \ldots (t - m) h_m(t, y).$$

If $m$ is bigger than the total degree of $h$ we obtain a contradiction, showing that $h$ must be zero. By (21) we conclude that $S$ must be zero on $(0, 1) \times \mathbb{R}^n$. In order to show that $S$ is zero on $\mathbb{R}^{n+1}$ consider the polyspline $S_j$ defined by $S_j(t, y) = S(t - j, y)$ for $(t, y) \in \mathbb{R}^{n+1}$, $j \in \mathbb{Z}$. By the above, $S_j$ is zero in $(0, 1) \times \mathbb{R}^n$. Hence $S$ must be zero on $(j, j + 1) \times \mathbb{R}^n$. \qed

\textbf{Corollary 14.} Interpolation with polynomially bounded cardinal polysplines of order 1 on strips is unique.

\textbf{REFERENCES}


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